Geometry of the L_q-centroid bodies of an isotropic log-concave measure

Apostolos Giannopoulos, Pantelis Stavrakakis, Antonis Tsolomitis
and Beatrice-Helen Vritsiou

Abstract

We study some geometric properties of the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave measure μ on $\mathbb{R}^n$. For any $2 \leqslant q \leqslant \sqrt{n}$ and for $\varepsilon \in (\varepsilon_0(q,n), 1)$ we determine the inradius of a random $(1-\varepsilon)n$-dimensional projection of $Z_q(\mu)$ up to a constant depending polynomially on ε. Using this fact we obtain estimates for the covering numbers $N(\sqrt{q}B_2^n, tZ_q(\mu))$, $t \geqslant 1$, thus showing that $Z_q(\mu)$ is a β-regular convex body. As a consequence, we also get an upper bound for $M(Z_q(\mu))$.

MSC: Primary 52A21; Secondary 46B07, 52A40, 60D05.

1 Introduction

Given a convex body K of volume 1 or a log-concave probability measure μ on $\mathbb{R}^n$, we define the L_q-centroid bodies $Z_q(K)$ or $Z_q(\mu)$, $q \in (0, +\infty)$, through their support function $h_{Z_q(K)}$ or $h_{Z_q(\mu)}$, which is defined as follows: for every $y \in \mathbb{R}^n$,

$$(1.1) \quad h_{Z_q(K)}(y) := \|\langle \cdot, y \rangle\|_{L_q(K)} = \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q},$$

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L_q(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}.$$

These bodies then incorporate information about the distribution of linear functionals with respect to the uniform measure on K or with respect to the probability measure μ. The L_q-centroid bodies were introduced, under a different normalization, by Lutwak, Yang and Zhang in [25], while in [30] for the first time, and in [31] later on, Paouris used geometric properties of them to acquire detailed information about the distribution of the Euclidean norm with respect to the uniform measure on isotropic convex bodies. An asymptotic theory for the L_q-centroid bodies has since been developed in the context of isotropic measures and it seems to advance in parallel with all recent developments in the area.

Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$(1.2) \qquad \int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. Similarly, a log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if its barycenter is at the origin and if its inertia matrix is the identity matrix; in that case the isotropic constant of the measure is defined as

$$(1.3) \qquad L_\mu := \sup_{x \in \mathbb{R}^n} \left(f_\mu(x) \right)^{1/n},$$

where f_μ is the density of μ with respect to the Lebesgue measure. One very well-known open question in the theory of isotropic measures is the hyperplane conjecture, which asks if there exists an absolute constant $C > 0$ such that

$$(1.4) \qquad L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain proved in [4] that $L_n \leqslant c\sqrt[4]{n}\log n$, while Klartag [18] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [19].

A motivation for this paper is a recent reduction [16] of the hyperplane conjecture to the study of geometric properties of the L_q-centroid bodies, and in particular to the study of the parameter

$$(1.5) \qquad I_1\left(K, Z_q^\circ(K)\right) := \int_K \|x\|_{Z_q^\circ(K)}(x)dx = \int_K h_{Z_q(K)}(x)dx.$$

The main result of [16] is, in a sense, a continuation of Bourgain's approach to the problem and, roughly speaking, can be formulated as follows: Given $q \geqslant 2$ and $\frac{1}{2} \leqslant s \leqslant 1$, an upper bound of the form $I_1(K, Z_q^\circ(K)) \leqslant C_1 q^s \sqrt{n} L_K^2$ for all isotropic convex bodies K in $\mathbb{R}^n$ leads to the estimate

$$(1.6) \qquad L_n \leqslant \frac{C_2 \sqrt[4]{n} \log n}{q^{\frac{1-s}{2}}}.$$

Bourgain's estimate may be recovered by choosing $q = 2$, however, clarifying the behaviour of $I_1(K, Z_q^\circ(K))$ might allow one to use much larger values of q. This behaviour is most naturally related to the geometry of the bodies $Z_q(K)$, and especially how this geometry is affected by or affects the geometry of the body K. This is not yet fully understood and, in view of (1.6), we believe that its deeper study would be very useful.

In Section 3 we give an account of some basic known results for the bodies $Z_q(K)$ and, more generally, the bodies $Z_q(\mu)$ where μ is an isotropic log-concave measure on $\mathbb{R}^n$. In the range $2 \leqslant q \leqslant \sqrt{n}$, for example, some of their global parameters are completely determined: the volume radius and the mean width of $Z_q(\mu)$ are of the same order $\sqrt{q}$. The purpose of this work is to provide new information on the local structure of $Z_q(\mu)$, which in turn has some interesting consequences. Our first main result concerns proportional projections of the centroid bodies.

2

Theorem 1.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Fix $1 \leqslant \alpha < 2$. For every $0 < \varepsilon < 1$ and any $q \leqslant \sqrt{\varepsilon n}$ there are $k \geqslant (1 - \varepsilon)n$ and $F \in G_{n,k}$ such that*

$$(1.7) \qquad P_F\big(Z_q(\mu)\big) \supseteq c(2 - \alpha)\varepsilon^{\frac{1}{2} + \frac{2}{\alpha}} \sqrt{q}\, B_F,$$

where $c > 0$ is an absolute constant (independent of α, ε, the measure μ, q or n). Moreover, for any $2 \leqslant q \leqslant \varepsilon n$ there are $k \geqslant (1 - \varepsilon)n$ and $F \in G_{n,k}$ such that

$$(1.8) \qquad P_F\big(Z_q(\mu)\big) \supseteq \frac{c_1(2 - \alpha)\varepsilon^{\frac{1}{2} + \frac{2}{\alpha}}}{L_{\varepsilon n}} \sqrt{q}\, B_F \supseteq \frac{c_2(2 - \alpha)\varepsilon^{\frac{1}{4} + \frac{2}{\alpha}}}{\sqrt[4]{n}} \sqrt{q}\, B_F,$$

where $c_1, c_2 > 0$ are absolute constants.

The proof of Theorem 1.1 is given in Section 5. We use Pisier's theorem on the existence of α-regular ellipsoids for symmetric convex bodies; we combine this with available information on the L_q-centroid bodies as well as results from [11] concerning the circumradius of proportional sections of α-regular convex bodies, the proofs of which are outlined in Section 4. Let us mention that the dual result is a direct consequence of the low M^*-estimate, since the mean width of $Z_q(\mu)$ is known to be of the order of $\sqrt{q}$: if $2 \leqslant q \leqslant \sqrt{n}$ and if $\varepsilon \in (0, 1)$ and $k = (1 - \varepsilon)n$, then a subspace $F \in G_{n,k}$ satisfies

$$(1.9) \qquad P_F(Z_q^\circ(\mu)) \supseteq \frac{c_1\sqrt{\varepsilon}}{\sqrt{q}} B_F$$

with probability greater than $1 - \exp(-c_2\varepsilon n)$, where $c_1, c_2 > 0$ are absolute constants.

In Section 6 we discuss bounds for the covering numbers of a Euclidean ball by $Z_q(\mu)$. It was proved in [14] and [15] that if μ is an isotropic log-concave measure on $\mathbb{R}^n$ then, for any $1 \leqslant q \leqslant n$ and $t \geqslant 1$,

$$(1.10) \qquad \log N\big(Z_q(\mu), c_1 t\sqrt{q} B_2^n\big) \leqslant c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{qn}}{t},$$

where $c_1, c_2, c_3 > 0$ are absolute constants. Using Theorem 1.1 and an entropy extension result from [24] we obtain regular entropy estimates for the dual covering numbers.

Theorem 1.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume $1 \leqslant \alpha < 2$. Then, for any $q \leqslant \sqrt{n}$ and any*

$$1 \leqslant t \leqslant \min\big\{\sqrt{q},\ c_1(2 - \alpha)^{-1}(n/q^2)^{\frac{\alpha+4}{2\alpha}}\big\}$$

we have

$$(1.11) \qquad \log N\big(\sqrt{q} B_2^n, t Z_q(\mu)\big) \leqslant c(\alpha) \frac{n}{t^{\frac{2\alpha}{\alpha+4}}} \max\Big\{\log \frac{\sqrt{2q}}{t},\ \log \frac{1}{(2 - \alpha)t}\Big\},$$

3

where $c(\alpha) \leqslant C(2 - \alpha)^{-2/3}$ and c_1, C are absolute constants. Moreover, for any $2 \leqslant q \leqslant n$ and any

$$1 \leqslant t \leqslant \min\left\{\sqrt{q}, \ c_2(2 - \alpha)^{-1} L_n\left(\frac{n}{q}\right)^{\frac{\alpha+4}{2\alpha}}\right\}$$

we have

(1.12) $\qquad \log N\left(\sqrt{q} B_2^n, t Z_q(\mu)\right) \leqslant c(\alpha) L_n^{\frac{2\alpha}{\alpha+4}} \dfrac{n}{t^{\frac{2\alpha}{\alpha+4}}} \max\left\{\log \dfrac{2q}{t^2}, \ \log \dfrac{L_n}{(2-\alpha)t}\right\},$

where $c(\alpha)$ is as above and c_2 is an absolute constant.

Note that, since $Z_q(\mu) \supseteq B_2^n$, we are interested in bounds for the above covering numbers when t is in the interval $[1, \sqrt{q}]$. An analysis of the restrictions in Theorem 1.2 shows that, given any $q \leqslant n^{3/7}$, (1.11) holds true with any t in the "interesting" interval, while the same is true for (1.12) as long as $q \leqslant \sqrt{L_n} n^{3/4}$. Although all these estimates are most probably not optimal, we can still conclude that $Z_q(\mu)$, with $q \leqslant n^{3/7}$ or $q \leqslant \sqrt{L_n} n^{3/4}$, is a β-regular convex body in the sense of Pisier's theorem (for some concrete positive value of β). As a consequence of this fact we get an upper bound for the parameter

$$M\left(Z_q(\mu)\right) = \int_{S^{n-1}} \|x\|_{Z_q(\mu)} \, d\sigma(x).$$

Recall that the dual Sudakov inequality of Pajor and Tomczak-Jaegermann (see e.g. [33]) provides 2-regular entropy estimates for the numbers $N(B_2^n, tC)$ in terms of $M(C)$, namely it shows that

$$\log N(B_2^n, tC) \leqslant cn \left(\frac{M(C)}{t}\right)^2$$

for every $t \geqslant 1$. In Section 7 we use in a converse manner the entropy estimates of Theorem 1.2 to obtain non-trivial upper bounds for $M(Z_q(\mu))$.

Theorem 1.3. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $1 \leqslant q \leqslant n^{3/7}$,*

(1.13) $\qquad\qquad\qquad M\left(Z_q(\mu)\right) \leqslant C \dfrac{(\log q)^{5/6}}{\sqrt[6]{q}}.$

Moreover, for every q such that $L_n^2 \log^2 q \leqslant q \leqslant \sqrt{L_n}\, n^{3/4}$,

(1.14) $\qquad\qquad\qquad M\left(Z_q(\mu)\right) \leqslant C \dfrac{\sqrt[3]{L_n}(\log q)^{5/6}}{\sqrt[6]{q}}.$

4

Observe now that, if K is an isotropic convex body in $\mathbb{R}^n$ with isotropic constant L_K, then the measure μ_K with density $f_{\mu_K}(x) := L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic and, for every $q > 0$, it holds that $Z_q(K) = L_K Z_q(\mu_K)$. Using also the fact that $M(K) \leqslant M(Z_q(K))$ for every symmetric convex body K and every $q > 0$, we can use the above bounds for $M(Z_q(\mu_K))$ to obtain an upper bound for $M(K)$ in the isotropic case.

Theorem 1.4. *Let K be a symmetric isotropic convex body in $\mathbb{R}^n$. Then,*

$$M(K) \leqslant C \frac{\sqrt[4]{L_n}(\log n)^{5/6}}{L_K \sqrt[8]{n}}.$$

This is a question that until recently had not attracted much attention. Valettas, using a slightly different approach [35], has shown that

$$M(K) \leqslant \frac{C(\log n)^{1/3}}{\sqrt[12]{n} L_K}$$

for every symmetric isotropic convex body K in $\mathbb{R}^n$, where $C > 0$ is an absolute constant. Note that, on the other hand, there are many approaches concerning the corresponding question about the mean width that give the best currently known estimate:

$$w(K) \leqslant C n^{3/4} L_K$$

for every isotropic convex body K in $\mathbb{R}^n$. Nevertheless, this problem as well remains open (for a discussion about it, see [15] and the references therein).

We close this paper with some additional observations on the geometry of the centroid bodies $Z_q(\mu)$ and their polars. We first provide lower bounds for the radius of their sections; actually, they hold true for every $1 \leqslant k < n$ and any $F \in G_{n,k}$. By duality, these estimates (combined with e.g. Proposition 5.3) determine the inradius of their random projections. We also provide upper bounds for the parameters $M_{-k}(Z_q(\mu))$ and $I_{-k}(\overline{Z_q}(\mu))$ (see Section 8 for the precise definitions). These imply small ball probability estimates for the Euclidean norm on $Z_q(\mu)$. All the results are based on the main estimates from Section 5 and Section 6. Although they are not optimal, we describe our approach and sketch their proofs; we expect that further progress can be achieved along the same lines.

2   Notation and preliminaries

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $\| \cdot \|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $| \cdot |$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

Convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with nonempty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if the barycenter of C is at the origin, i.e. $\int_C \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The volume radius of C is the quantity

$$(2.1) \qquad \mathrm{vrad}(C) = \left(\frac{|C|}{|B_2^n|} \right)^{1/n}.$$

The support function of a convex body C is defined by

$$(2.2) \qquad h_C(y) := \max\{ \langle x, y \rangle : x \in C \}$$

and characterizes C uniquely. The mean width of C is

$$(2.3) \qquad w(C) := \int_{S^{n-1}} h_C(\theta) \sigma(d\theta)$$

and the radius of C is the quantity $R(C) := \max\{ \|x\|_2 : x \in C \}$. Also, if the origin is an interior point of C, the polar body C° of C is defined as follows:

$$(2.4) \qquad C^\circ := \big\{ y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in C \big\}.$$

Let $C \subset \mathbb{R}^n$ be a symmetric convex body. We write $\| \cdot \|_C$ for the norm $\|x\|_C = \min\{ t \geqslant 0 : x \in tC \}$ induced to $\mathbb{R}^n$ by C, and we define

$$(2.5) \qquad M(C) := \int_{S^{n-1}} \|\theta\|_C d\sigma(\theta).$$

Note that $M(C) = w(C^\circ)$ and that

$$(2.6) \qquad M(C)^{-1} \leqslant \mathrm{vrad}(C) \leqslant w(C) = M(C^\circ);$$

the left hand side inequality is easily checked if we express the volume of C as an integral in polar coordinates and use Hölder's and Jensen's inequalities, while the right hand side inequality is the classical Urysohn's inequality. We also need Milman's low M^*-estimate (see [29] or [33] for precise references): if C is a symmetric convex body in $\mathbb{R}^n$, then a subspace $F \in G_{n,k}$ satisfies

$$(2.7) \qquad R(C \cap F) \leqslant c_1 \sqrt{\frac{n}{n-k}} \, w(C)$$

with probability greater than $1 - \exp(-c_2(n-k))$, where $c_1, c_2 > 0$ are absolute constants.

6

Furthermore, if C is a symmetric convex body in $\mathbb{R}^n$, we define $k_*(C)$ to be the largest positive integer $k \leqslant n$ with the property that the measure $\nu_{n,k}$ of $F \in G_{n,k}$ for which we have $\frac{1}{2}w(C)B_F \subseteq P_F(C) \subseteq 2w(C)B_F$ is greater than $\frac{n}{n+k}$. It is known that this parameter is completely determined by the dimension, the mean width and the radius of C:

$$(2.8) \qquad c_3 n \frac{w(C)^2}{R(C)^2} \leqslant k_*(C) \leqslant c_4 n \frac{w(C)^2}{R(C)^2}.$$

Recall that the covering number $N(A, B)$ of a body A by a second body B is the least integer N for which there exist N translates of B whose union covers A. Milman (see e.g. [26]) proved that there exists an absolute constant $\beta > 0$ such that every centered convex body K in $\mathbb{R}^n$ has a linear image $\tilde{K}$ which satisfies $|\tilde{K}| = |B_2^n|$ and

$$(2.9) \qquad \max\{N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ)\} \leqslant \exp(\beta n).$$

We say that a convex body K which satisfies this estimate is in M-position with constant β.

Pisier [32] has proposed a different approach to this result, which allows one to find a whole family of M-ellipsoids and to give more detailed information on the behaviour of the corresponding covering numbers. The precise statement is as follows.

Theorem 2.1 (Pisier). *For every $0 < \alpha < 2$ and every symmetric convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}$ of K such that*

$$\max\{N(\tilde{K}, tB_2^n), N(B_2^n, t\tilde{K}), N(\tilde{K}^\circ, tB_2^n), N(B_2^n, t\tilde{K}^\circ)\} \leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t \geqslant 1$, where $c(\alpha)$ depends only on α, and $c(\alpha) = O\big((2 - \alpha)^{-\alpha/2}\big)$ as $\alpha \to 2$.

For basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces, whose unit balls are various symmetric convex bodies appearing in this paper, we refer to the books [34], [29] and [33].

Log-concave probability measures. We denote by $\mathcal{P}_n$ the class of all Borel probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_n$ is denoted by f_μ. We say that $\mu \in \mathcal{P}_n$ is centered and we write $\mathrm{bar}(\mu) = 0$ if, for all $\theta \in S^{n-1}$,

$$(2.10) \qquad \int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \theta \rangle f_\mu(x) dx = 0.$$

A measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1 - \lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction of

7

log f to it is concave. It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_n$ and its density f_μ is log-concave. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$(2.11) \qquad L_\mu := \left(\frac{\sup_{x \in \mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx} \right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$(2.12) \qquad \mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if $\mathrm{bar}(\mu) = 0$ and $\mathrm{Cov}(\mu)$ is the identity matrix and we write $\mathcal{IL}_n$ for the class of isotropic log-concave probability measures on $\mathbb{R}^n$. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic, i.e. it satisfies (1.2), if and only if the log-concave probability measure μ_K with density $x \mapsto L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic.

Let $\mu \in \mathcal{P}_n$. For every $1 \leqslant k \leqslant n - 1$ and every $E \in G_{n,k}$, the marginal of μ with respect to E is the probability measure with density

$$(2.13) \qquad f_{\pi_E \mu}(x) = \int_{x+E^\perp} f_\mu(y)dy.$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E \mu$ is also centered, isotropic or log-concave, respectively.

For more information on isotropic convex bodies and log-concave measures see [28], [2] and [9].

3 L_q-centroid bodies: basic facts

Recall that, if μ is a log-concave probability measure on $\mathbb{R}^n$, the L_q-centroid body $Z_q(\mu)$, $q \geqslant 1$, of μ is the centrally symmetric convex body with support function

$$(3.1) \qquad h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}.$$

Observe that μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. From Hölder's inequality it follows that $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leqslant p \leqslant q < \infty$. Conversely, using Borell's lemma (see [29, Appendix III]), one can check that

$$(3.2) \qquad Z_q(\mu) \subseteq c\frac{q}{p} Z_p(\mu)$$

for all $1 \leqslant p < q$. In particular, if μ is isotropic, then $R(Z_q(\mu)) \leqslant cq$.

8

As we saw in the previous section, if K is a convex body in $\mathbb{R}^n$ of volume 1, then the measure with density $x \mapsto \mathbf{1}_K(x)$ is a log-concave probability measure on $\mathbb{R}^n$ and the L_q-centroid bodies $Z_q(K)$ of K can be defined as above. By Hölder's inequality we again have $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq \operatorname{conv}\{K, -K\}$ for all $1 \leqslant p \leqslant q < \infty$, so, if K is symmetric, then $Z_p(K) \subseteq K$ for every $p < \infty$.

Using Fubini's theorem we see that, for every $1 \leqslant k \leqslant n-1$ and every $F \in G_{n,k}$ and $q \geqslant 1$,

$$(3.3) \qquad P_F\big(Z_q(\mu)\big) = Z_q\big(\pi_F(\mu)\big).$$

In [30] Paouris shows that the moments

$$(3.4) \qquad I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q \, dx \right)^{1/q}, \quad q \in (-n, +\infty) \setminus \{0\},$$

of the Euclidean norm with respect to an isotropic log-concave probability measure μ on $\mathbb{R}^n$ remain comparable to $I_2(\mu) = \sqrt{n}$ when $q \geqslant 2$ does not exceed a parameter $q_*(\mu)$ of the measure that he defines as follows:

$$(3.5) \qquad q_*(\mu) := \max\big\{ q \leqslant n : k_*\big(Z_q(\mu)\big) \geqslant q \big\}.$$

He proves this by showing that

$$I_q(\mu) \simeq \sqrt{\frac{n}{q}} \, w\big(Z_q(\mu)\big)$$

and that

$$(3.6) \qquad w\big(Z_q(\mu)\big) \simeq \sqrt{q}$$

for every $q \leqslant q_*(\mu)$. He establishes that

$$(3.7) \qquad q_*(\mu) \geqslant c_1 \sqrt{n}$$

for every isotropic measure μ on $\mathbb{R}^n$, where $c_1 > 0$ is an absolute constant, and it is also shown that there are isotropic measures μ on $\mathbb{R}^n$ such that $q_*(\mu) \simeq \sqrt{n}$. Furthemore, he proves that

$$(3.8) \qquad I_q(\mu) \simeq R\big(Z_q(\mu)\big)$$

for every $q \in [q_*(\mu), n]$ and he gives an upper bound for the volume radius of the L_q-centroid bodies:

$$(3.9) \qquad |Z_q(\mu)|^{1/n} \leqslant c_2 \sqrt{q/n}$$

for all $1 \leqslant q \leqslant n$, where c_2 is an absolute constant.

In [31] Paouris extends his approach to describe the behaviour of the negative moments of the Euclidean norm with respect to an isotropic measure μ on $\mathbb{R}^n$. He shows that

$$(3.10) \qquad I_{-q}(\mu) \simeq I_2(\mu) = \sqrt{n} \quad \text{for all } 0 < q \leqslant q_*(\mu).$$

However, unlike the positive moments $I_q(\mu)$ that, as we saw, do not remain comparable to $I_2(\mu)$ once q gets larger than $q_*(\mu)$, the behaviour of the corresponding negative moments is not known and, in fact, (3.10) may hold with any positive q up to $n-1$. This question actually is equivalent to the hyperplane conjecture, as Dafnis and Paouris proved in [8] by introducing another parameter, that for each $\delta \geqslant 1$ is given by

$$(3.11) \qquad q_{-c}(\mu, \delta) := \max\big\{1 \leqslant q \leqslant n-1 : I_{-q}(\mu) \geqslant \delta^{-1} I_2(\mu) = \delta^{-1}\sqrt{n}\big\},$$

namely measures how large the range of (3.10) is if we allow the implied constants to depend on δ. Dafnis and Paouris established that

$$(3.12) \qquad L_n \leqslant C\delta \sup_{\mu \in \mathcal{IL}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu,\delta)}}\ \log\Big(\frac{en}{q_{-c}(\mu,\delta)}\Big)$$

for every $\delta \geqslant 1$, and they also showed that, if the hyperplane conjecture is correct, that is, if (1.4) holds true, then we will have

$$(3.13) \qquad q_{-c}\big(\mu, \delta_0\big) = n-1$$

for some $\delta_0 \simeq 1$, for every isotropic log-concave measure μ on $\mathbb{R}^n$. Note that, by (3.10), we already know that

$$(3.14) \qquad q_{-c}\big(\mu, \delta_1\big) \geqslant q_*(\mu) \geqslant c_1\sqrt{n},$$

where $\delta_1 \geqslant 1$ and $c_1 > 0$ are absolute constants.

Next we turn to lower bounds for the volume radius of the L_q-centroid bodies of measures $\mu \in \mathcal{IL}_{[n]}$. From [25] we know that, for every $1 \leqslant q \leqslant n$,

$$(3.15) \qquad \big|Z_q(\mu)\big|^{1/n} \geqslant c_2\sqrt{q/n}\, L_\mu^{-1}$$

for some absolute constant $c_2 > 0$. In [19] Klartag and Milman define a "hereditary" variant of $q_*(\mu)$ as follows:

$$(3.16) \qquad q_*^H(\mu) := n \inf_k \inf_{E \in G_{n,k}} \frac{q_*(\pi_E \mu)}{k},$$

where $\pi_E\mu$ is the marginal of μ with respect to E, and then, for every $q \leqslant q_*^H(\mu)$, they give a lower bound for the volume radius of the bodies $Z_q(\mu)$ matching the upper bound in (3.9):

$$(3.17) \qquad \big|Z_q(\mu)\big|^{1/n} \geqslant c_3\sqrt{q/n}$$

where $c_3 > 0$ is an absolute constant. Recall that, if μ is an isotropic log-concave measure, then so are all its marginals, thus, for every subspace $E \in G_{n,k}$, we have by (3.7) that $q_*(\pi_E\mu) \geqslant c_1\sqrt{k}$. This implies that, for all measures $\mu \in \mathcal{IL}_{[n]}$, $q_*^H(\mu) \geqslant c_1\sqrt{n}$. Note also that, for those measures for which we have $q_*(\mu) \simeq \sqrt{n}$,

10

the parameter $q_*^H(\mu)$ as well does not exceed a constant multiple of $\sqrt{n}$. However, the bound (3.17) might hold for larger $q \in [1, n]$ even in the latter case, as was shown by the fourth named author in [37]. There a hereditary variant of $q_{-c}(\mu, \delta)$ is introduced as follows:

$$q_{-c}^H(\mu, \delta) := n \inf_k \inf_{E \in G_{n,k}} \frac{q_{-c}(\pi_E \mu, \delta)}{k}$$

for every $\delta \geqslant 1$, and in a similar way as for the bound (3.17) it is established that

$$(3.18) \qquad \left| Z_q(\mu) \right|^{1/n} \geqslant c_4 \delta^{-1} \sqrt{q/n}$$

for every $q \leqslant q_{-c}^H(\mu, \delta)$. Of course, we can again see that

$$(3.19) \qquad q_{-c}^H(\mu, \delta_1) \geqslant c_1 \sqrt{n}$$

for some $\delta_1 \simeq 1$ using (3.14) and the definition of $q_{-c}^H(\mu, \delta_1)$, but obviously this estimate might be improved if the latter bound were also; such an improvement may be possible but is not trivial, since it would result in better bounds for the isotropic constant problem too, as one can see from (3.12).

To conclude this Section, we should stress that in the subsequent proofs we will be using the lower bounds for the volume radius of the L_q-centroid bodies, however the only concrete estimate which we currently have for the parameters $q_*^H(\mu)$ or $q_{-c}^H(\mu, \delta)$, and which we can insert in our later estimates, is that all of them are at least of the order of $\sqrt{n}$ when μ is an isotropic log-concave measure on $\mathbb{R}^n$. For more information on all the results of this Section, see [7].

4   Diameter of sections of α-regular bodies

We say that a body $\tilde{K}$ which satisfies the conclusion of Theorem 2.1 is an α-regular body. A strong form of the reverse Brunn-Minkowski inequality can be proved for α-regular bodies; actually, we only need the regularity of the covering numbers $N(K, tB_2^n)$.

Lemma 4.1. *Let $\gamma \geqslant 1$, $\alpha > 0$ and let $K_1, \ldots, K_m$ be symmetric convex bodies in $\mathbb{R}^n$ which satisfy*

$$N(K_j, tB_2^n) \leqslant \exp\left(\frac{\gamma n}{t^\alpha}\right)$$

for all $1 \leqslant j \leqslant m$ and all $t \geqslant 1$. Then,

$$(4.1) \qquad |K_1 + \cdots + K_m|^{1/n} \leqslant C \gamma^{\frac{1}{\alpha}} m^{1 + \frac{1}{\alpha}} |B_2^n|^{1/n}.$$

Proof. We include the very simple proof of this fact. Observe that

$$N(K_1 + \cdots + K_m, tmB_2^n) = N(K_1 + \cdots + K_m, tB_2^n + \cdots + tB_2^n)$$

$$\leqslant \prod_{j=1}^m N(K_j, tB_2^n) \leqslant \exp(\gamma nm/t^\alpha)$$

for all $t \geqslant 1$. It follows that

$$|K_1 + \cdots + K_m|^{1/n} \leqslant tm \, \exp(\gamma m/t^\alpha)|B_2^n|^{1/n}.$$

Choosing $t = (\gamma m)^{1/\alpha}$ we get the result. $\qquad\square$

We also need to recall the precise probabilistic form of the low M^*-estimate (which can be found in [17] and [27]): If A is a symmetric convex body in $\mathbb{R}^n$ and if $\varepsilon, \delta \in (0,1)$, then we have

$$R(A \cap F) \leq \frac{w(A)}{(1-\delta)\sqrt{\varepsilon}}$$

for all F in a subset $L_{n,k}$ of $G_{n,k}$ of measure $\nu_{n,k}(L_{n,k}) \geqslant 1 - c_1 \exp(-c_2 \delta^2 \varepsilon n)$, where $k = \lfloor (1-\varepsilon)n \rfloor$ and $c_1, c_2 > 0$ are absolute constants. Then, a well-known application of the low M^*-estimate (see e.g. [10, Theorem 2.1]) states that if $r > 0$ is the solution of the equation

$$(4.2) \qquad\qquad \frac{w(A \cap rB_2^n)}{r} = \frac{1}{2}\sqrt{\varepsilon},$$

then a typical $\lfloor (1-\varepsilon)n \rfloor$-dimensional central section of A has radius smaller than r (with probability greater than $1 - \exp(-c\varepsilon n)$).

The main observation of this Section, which essentially appears in [11], is the following: if A is a symmetric convex body in $\mathbb{R}^n$ whose covering numbers $N(A, tB_2^n)$ are α-regular for some $\alpha > 0$, then one can get an upper bound for the diameter of random proportional sections of A.

Theorem 4.2. *Let $\gamma \geqslant 1$, $\alpha > 0$ and let A be a symmetric convex body in $\mathbb{R}^n$ which satisfies*

$$N(A, tB_2^n) \leqslant \exp\left(\frac{\gamma n}{t^\alpha}\right)$$

for all $t \geqslant 1$. Then, for every $\varepsilon \in (0,1)$ a subspace $F \in G_{n,\lfloor (1-\varepsilon)n \rfloor}$ satisfies

$$R(A \cap F) \leqslant C\gamma^{\frac{1}{\alpha}}/\varepsilon^{\frac{1}{2}+\frac{1}{\alpha}},$$

with probability greater than $1 - c_1 \exp(-c_2 \varepsilon n)$, where $c_1, c_2, C > 0$ are absolute constants.

Proof. Let $\varepsilon \in (0,1)$ and set $k = \lfloor (1-\varepsilon)n \rfloor$. We define $r > 0$ by the equation

$$w(A \cap rB_2^n) = \frac{1}{2}\sqrt{\varepsilon}\, r.$$

From the precise probabilistic form of the low M^*-estimate (see above), we know that there exists a subset $L_{n,k}$ of $G_{n,k}$ with measure $\nu_{n,k}(L_{n,k}) \geqslant 1 - c_1 \exp(-c_2 \varepsilon n)$, such that

$$w(A \cap F) \leqslant R(A \cap F) \leqslant r$$

for every $F \in L_{n,k}$. We use the following fact from [6]: if $X = (\mathbb{R}^n, \|\cdot\|)$ is an n-dimensional normed space with unit ball W, and if $M = \int_{S^{n-1}} \|x\| \, d\sigma(x)$ and b is the smallest positive constant for which $\|x\| \leqslant b\|x\|_2$ for all $x \in \mathbb{R}^n$, then there exist an integer $s \leqslant C(b/M)^2$ and s orthogonal transformations $U_1, \ldots, U_s \in O(n)$ such that

$$\frac{M}{2} B_2^n \subseteq \frac{1}{s} \sum_{i=1}^{s} U_i(W^\circ) \subseteq 2M \, B_2^n.$$

We apply this result for the body $W := (A \cap rB_2^n)^\circ$. Note that $b = r$ and $M(W) = w(A \cap rB_2^n) = \sqrt{\varepsilon} r/2$, and hence we can find $s \leqslant \frac{c_3}{\varepsilon}$ and orthogonal transformations $U_1, \ldots, U_s$, satisfying

(4.3) $$\frac{1}{4}\sqrt{\varepsilon} \, rB_2^n \subseteq \frac{1}{s} \sum_{i=1}^{s} U_i(A \cap rB_2^n) \subseteq \sqrt{\varepsilon} \, rB_2^n.$$

Set $A_1 = \frac{1}{s} \sum U_i(A \cap rB_2^n)$. Now we can give an upper bound for r using Lemma 4.1. Clearly, the bodies $U_i(A \cap rB_2^n)$ satisfy

$$N\big(U_i(A \cap rB_2^n), tB_2^n\big) \leqslant N(A, tB_2^n) \leqslant \exp\left(\frac{\gamma n}{t^\alpha}\right)$$

for all $t \geqslant 1$, therefore

$$\frac{1}{4}\sqrt{\varepsilon} \, r \leqslant \left(\frac{|A_1|}{|B_2^n|}\right)^{\frac{1}{n}} \leqslant c_4 \gamma^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}}.$$

This shows that

$$R(A \cap F) \leqslant r \leqslant c_5 \gamma^{\frac{1}{\alpha}} / \varepsilon^{\frac{1}{2} + \frac{1}{\alpha}}$$

with probability greater than $1 - c_1 \exp(-c_2 \varepsilon n)$. $\qquad\qquad$ □

5 Projections of L_q-centroid bodies

Our aim is to obtain lower bounds for the inradius of proportional projections of $Z_q(\mu)$ and $Z_q^\circ(\mu)$. Let $1 \leqslant k \leqslant n-1$ and consider a random subspace $F \in G_{n,k}$. An upper bound for the radius of $Z_q(\mu) \cap F$, and hence a lower bound for the inradius of $P_F(Z_q^\circ(\mu))$, follows from the low M^*-estimate (2.7) and (3.6).

Proposition 5.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $2 \leqslant q \leqslant q_*(\mu)$ and if $\varepsilon \in (0,1)$ and $k = \lfloor(1-\varepsilon)n\rfloor$, then a subspace $F \in G_{n,k}$ satisfies*

(5.1) $$R\big(Z_q(\mu) \cap F\big) \leqslant \frac{c_1\sqrt{q}}{\sqrt{\varepsilon}} \quad \text{or equivalently} \quad P_F\big(Z_q^\circ(\mu)\big) \supseteq \frac{c_2\sqrt{\varepsilon}}{\sqrt{q}} \, B_F$$

with probability greater than $1 - c_3 \exp(-c_4 \varepsilon n)$, where c_i are absolute constants. □

We provide analogous upper bounds for $R(Z_q^\circ(\mu) \cap F)$, $F \in G_{n,k}$. The idea of the proof comes from [21] (see the concluding remarks of this section). We start with the following immediate consequence of Theorem 4.2.

13

Corollary 5.2. *Let A be a symmetric convex body in $\mathbb{R}^m$. Assume that $\gamma \geqslant 1$, $\alpha > 0$, and E is an ellipsoid in $\mathbb{R}^m$ such that*

$$N(A, tE) \leqslant \exp\left(\frac{\gamma m}{t^\alpha}\right)$$

for all $t \geqslant 1$. Then, for every $\varepsilon \in (0, 1)$ there exists $F \in G_{m, \lfloor (1-\varepsilon)m \rfloor}$ such that

$$A \cap F \subseteq c\gamma^{\frac{1}{\alpha}} \varepsilon^{-(\frac{1}{2} + \frac{1}{\alpha})} E \cap F,$$

where $c > 0$ is an absolute constant.

Proposition 5.3 (version for "small" q). *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Let $1 \leqslant \alpha < 2$. For every $0 < \varepsilon < 1$ and any $q \leqslant \sqrt{\varepsilon n}$ there exist $k \geqslant (1-\varepsilon)n$ and $F \in G_{n,k}$ such that*

$$P_F\big(Z_q(\mu)\big) \supseteq c(2 - \alpha)\varepsilon^{\frac{1}{2} + \frac{2}{\alpha}} \sqrt{q}\, B_F.$$

Proof. Recall from (3.3) that for every $1 \leqslant m \leqslant n$ and any $H \in G_{n,m}$ we have

$$P_H\big(Z_q(\mu)\big) = Z_q\big(\pi_H(\mu)\big).$$

In Section 3 we saw that, if ν is an isotropic log-concave measure on $\mathbb{R}^m$, then

$$|Z_q(\nu)|^{1/m} \geqslant c\sqrt{q/m}$$

for all $q \leqslant q_{-c}^H(\nu, \delta_0)$, where $\delta_0 \geqslant 1$ is an absolute constant sufficiently large so that $q_{-c}^H(\nu, \delta_0) \geqslant c\sqrt{m}$. It follows that

$$(5.2) \qquad \left|P_H\big(Z_q(\mu)\big)\right|^{1/m} \geqslant c_1 \sqrt{q/m}$$

for all $H \in G_{n,m}$ and all $q \leqslant \sqrt{m}$. We fix $1 \leqslant \alpha < 2$ and consider an α-regular M-ellipsoid E of $Z_q(\mu)$, namely an ellipsoid such that

$$\max\left\{N\big(Z_q(\mu), tE\big), N\big(E, tZ_q(\mu)\big)\right\} \leqslant e^{c(\alpha)n/t^\alpha}$$

for all $t \geqslant 1$, where $c(\alpha) \leqslant C(2 - \alpha)^{-\alpha/2}$.

Let $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the axes of E, and let $\{u_1, \ldots, u_n\}$ be an orthonormal basis which corresponds to the λ_j. For every $1 \leqslant m, s \leqslant n$ we set

$$H_m := \mathrm{span}\{u_1, \ldots, u_m\} \quad \text{and} \quad F_s = \mathrm{span}\{u_{s+1}, \ldots, u_n\}.$$

Since $E \cap H_m = P_{H_m}(E)$, we have

$$N\Big(P_{H_m}\big(Z_q(\mu)\big), t(E \cap H_m)\Big) \leqslant N\big(Z_q(\mu), tE\big) \leqslant e^{c(\alpha)n/t^\alpha},$$

and hence

$$(5.3) \qquad \left|P_{H_m}\big(Z_q(\mu)\big)\right|^{1/m} \leqslant e^{c(\alpha)n/(t^\alpha m)}|B_{H_m}|^{1/m}(t\lambda_m).$$

Choose $t = (c(\alpha)n/m)^{1/\alpha}$. Assuming that $q \leqslant \sqrt{m}$, from (5.2) and (5.3) we get

$$(5.4) \qquad\qquad \lambda_m \geqslant \left(\frac{m}{c(\alpha)n}\right)^{1/\alpha} \sqrt{q}.$$

Next, let $0 < \varepsilon < 1$ and set $s = \lfloor \frac{\varepsilon n}{2} \rfloor$. We have

$$N\Big(E \cap F_s, tP_{F_s}\big(Z_q(\mu)\big)\Big) \leqslant N\big(E, tZ_q(\mu)\big) \leqslant e^{c(\alpha)n/t^\alpha} \leqslant e^{2c(\alpha)(n-s)/t^\alpha},$$

for every $t \geqslant 1$.

We now use the duality of entropy theorem of Artstein-Avidan, Milman and Szarek [1]: There exist two absolute constants a and $b > 0$ such that for any dimension n and any symmetric convex body A in $\mathbb{R}^n$ one has

$$N(B_2^n, a^{-1}A^\circ)^{1/b} \leqslant N(A, B_2^n) \leqslant N(B_2^n, aA^\circ)^b.$$

It follows that

$$N(Z_q^\circ(\mu) \cap F_s, tE^\circ \cap F_s) \leqslant N\Big(E \cap F_s, atP_{F_s}\big(Z_q(\mu)\big)\Big)^b \leqslant e^{c_1(\alpha)(n-s)/t^\alpha}.$$

We apply Corollary 5.2 with the body $Z_q^\circ(\mu) \cap F_s$ (and $\gamma = c_1(\alpha)$) to find a subspace F of F_s, of dimension $k \geqslant (1 - \varepsilon/2)(n - s) \geqslant (1 - \varepsilon)n$, such that

$$Z_q^\circ(\mu) \cap F \subseteq \frac{C}{\sqrt{2-\alpha}\, \varepsilon^{\frac{1}{2}+\frac{1}{\alpha}}}\, E^\circ \cap F,$$

and hence

$$(5.5) \qquad\qquad P_F\big(Z_q(\mu)\big) \supseteq c\sqrt{2-\alpha}\, \varepsilon^{\frac{1}{2}+\frac{1}{\alpha}} P_F(E).$$

From (5.4) we have

$$E \cap F_s \supseteq \lambda_{s+1} B_{F_s} \supseteq c\sqrt{2-\alpha}\, \varepsilon^{1/\alpha} \sqrt{q}\, B_{F_s},$$

provided that $q \leqslant \sqrt{\varepsilon n}$. Then,

$$P_F(E) = P_F\big(P_{F_s}(E)\big) = P_F(E \cap F_s) \supseteq c\sqrt{2-\alpha}\, \varepsilon^{1/\alpha} \sqrt{q} P_F(B_{F_s})$$
$$= c\sqrt{2-\alpha}\, \varepsilon^{1/\alpha} \sqrt{q} B_F.$$

Combining this fact with (5.5) we conclude the proof. $\qquad\square$

Proposition 5.4 (version for "large" q). *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Let $1 \leqslant \alpha < 2$. For every $0 < \varepsilon < 1$ and any $2 \leqslant q \leqslant \varepsilon n$ there exist $k \geqslant (1 - \varepsilon)n$ and $F \in G_{n,k}$ such that*

$$P_F\big(Z_q(\mu)\big) \supseteq \frac{c_1(2-\alpha)\varepsilon^{\frac{1}{2}+\frac{2}{\alpha}}}{L_{\varepsilon n}} \sqrt{q}\, B_F \supseteq \frac{c_2(2-\alpha)\varepsilon^{\frac{1}{4}+\frac{2}{\alpha}}}{\sqrt[4]{n}} \sqrt{q}\, B_F.$$

Proof. We apply the same argument more or less, only instead of the lower bounds (3.17), (3.18) for the volume radius of the L_q-centroid bodies we use (3.15). It follows that

$$(5.6) \qquad \left| P_H\big(Z_q(\mu)\big) \right|^{1/m} \geqslant \frac{c_1}{L_m} \sqrt{q/m}$$

for all $H \in G_{n,m}$ and all $q \leqslant m$. We define H_m, F_s as in the proof of Proposition 5.3 and we consider an α-regular M-ellipsoid E of $Z_q(\mu)$. This time, assuming that $q \leqslant m$, from (5.6) and (5.3) we get

$$\lambda_m \geqslant \frac{1}{L_m} \left(\frac{m}{c(\alpha)n} \right)^{1/\alpha} \sqrt{q}.$$

Next, fix some $\varepsilon \in (0,1)$, set $s = \lfloor \frac{\varepsilon n}{2} \rfloor$ and consider any $q \leqslant \varepsilon n/2$. As previously, we find a subspace F of F_s, of dimension $k \geqslant (1-\varepsilon/2)(n-s) \geqslant (1-\varepsilon)n$, such that

$$P_F\big(Z_q(\mu)\big) \supseteq c\sqrt{2-\alpha}\, \varepsilon^{\frac{1}{2}+\frac{1}{\alpha}} P_F(E)$$

and

$$P_F(E) = P_F\big(P_{F_s}(E)\big) \supseteq P_F(\lambda_{s+1}B_{F_s}) \supseteq \frac{c\sqrt{2-\alpha}}{L_s}\varepsilon^{1/\alpha} \sqrt{q}\, B_F.$$

Since $s \simeq \varepsilon n$ and $L_{\varepsilon n} \leqslant C \sqrt[4]{\varepsilon n}$, the result follows. $\qquad\square$

As we saw in Section 3, if K is an isotropic symmetric convex body in $\mathbb{R}^n$, then

$$P_F(K) \supseteq P_F\big(Z_q(K)\big)$$

for all $q > 0$. Recall also that the measure μ_K with density $L_K^n \mathbf{1}_{K/L_K}$ is isotropic and $Z_q(K) = L_K Z_q(\mu_K)$. Choosing $q = \varepsilon n$ and applying Proposition 5.4 with $\mu = \mu_K$ we get:

Corollary 5.5. *Let K be a symmetric isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant \alpha < 2$ and $0 < \varepsilon < 1$ there exist $k \geqslant (1-\varepsilon)n$ and $F \in G_{n,k}$ such that*

$$P_F(K) \supseteq L_K P_F\big(Z_q(\mu_K)\big) \supseteq c(2-\alpha)\, \varepsilon^{\frac{3}{4}+\frac{2}{\alpha}} \sqrt[4]{n} L_K\, B_F.$$

Remark 5.6. Some variants of Corollary 5.5 have appeared in the literature before. In [35] a stronger estimate is obtained with a different method: if K is a symmetric isotropic convex body in $\mathbb{R}^n$ then, for any $0 < \varepsilon < 1$, there exists a subspace F of $\mathbb{R}^n$ with $\dim F \geqslant (1-\varepsilon)n$ such that

$$P_F(K) \supseteq c\varepsilon^{3/2} \frac{\sqrt[4]{n}}{\log n} L_K B_F,$$

where $c > 0$ is an absolute constant. A similar result with cubic dependence on ε appears in [3]. Our argument is very much related to the one in [21] where,

under the additional assumption that $L_n \leqslant C$ for all $n \geqslant 1$, the existence of some $F \in G_{n,\lfloor(1-\varepsilon)n\rfloor}$ so that
$$P_F(K) \supseteq c\varepsilon^3 \sqrt{n} B_F$$
is established for all isotropic convex bodies K in $\mathbb{R}^n$ and all $0 < \varepsilon < 1$. Under this assumption, our argument would result in the estimate $P_F(K) \supseteq c(2-\alpha)\varepsilon^{\frac{\alpha+2}{\alpha}} \sqrt{n} B_F$ for all $1 \leqslant \alpha < 2$.

Remark 5.7. Proposition 5.3 and Proposition 5.4 guarantee the existence of *one* $\lfloor(1-\varepsilon)n\rfloor$-dimensional projection of $Z_q(\mu)$ with "large" inradius. However, it is proved in [13] that, for every fixed proportion $\mu \in (0,1)$ and every $0 < s < 1/(2-\mu)$, the maximal inradius of $\lfloor\mu n\rfloor$-dimensional projections and the random inradius of $\lfloor s\mu n\rfloor$-dimensional projections of a symmetric convex body K in $\mathbb{R}^n$ are comparable up to a constant depending on μ and s. More precisely, if $a(\lambda, K)$ denotes the maximal (and if $b(\lambda, K)$ denotes the "random") inradius of a $\lfloor\lambda n\rfloor$-dimensional projection of K then

$$\left(\frac{c\mu\big(1 - s(2 - \mu)\big)}{1 - s\mu} \sqrt{1 - \mu} \right) a(\mu, K) \leqslant b(s\mu, K)$$

for every $n \geqslant n_0(\mu, s)$. Using this fact one can obtain versions of the results of this Section concerning random proportional projections of $Z_q(\mu)$. Since an estimate for the maximal inradius is sufficient for our subsequent work in this paper, we do not present the precise statements.

6 Covering numbers

Using Proposition 5.3, for any isotropic log-concave measure μ on $\mathbb{R}^n$ we can get some estimates for the covering numbers $N(\sqrt{q}B_2^n, tZ_q(\mu))$. These will follow from an entropy extension result from [24]:

Lemma 6.1. *Let K, L be symmetric convex bodies in $\mathbb{R}^n$ and assume that $L \subseteq RK$. Let F be a subspace of $\mathbb{R}^n$ with $\dim F = n - m$ and let $0 < r < t < R$. Then, we have*

(6.1) $$N(L, tK) \leqslant 2^m \left(\frac{2R+t}{t-r} \right)^m N\left(P_F(L), \frac{r}{2} P_F(K) \right).$$

Remark 6.2. Alternatively, one might use an analogous result, due to Vershynin and Rudelson (see [36, Lemma 5.2]): If K is a symmetric convex body in $\mathbb{R}^n$ such that $K \supseteq \delta B_n$ and if $P_F(K) \supseteq B_F$ for some $F \in G_{n,k}$, $k \geqslant (1 - \varepsilon)n$, then

$$N(B_2^n, 4K) \leqslant (C/\delta)^{2\varepsilon n}.$$

The reader may wish to check that applying this fact instead of Lemma 6.1 leads to the same estimate in Proposition 6.3 below.

Proposition 6.3. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume that $q \leqslant \sqrt{n}$. Then, for any $1 \leqslant \alpha < 2$ and any*

$$1 \leqslant t \leqslant \min\big\{\sqrt{q},\ c_2(2-\alpha)^{-1}(n/q^2)^{\frac{\alpha+4}{2\alpha}}\big\}$$

we have

$$\max\Big\{\log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big), \log N\big(\sqrt{q}Z_q^\circ(\mu), tB_2^n\big)\Big\}$$

$$(6.2) \qquad\qquad\qquad\qquad \leqslant c(\alpha)\frac{n}{t^{\frac{2\alpha}{\alpha+4}}} \max\left\{\log\frac{\sqrt{2q}}{t},\ \log\frac{1}{(2-\alpha)t}\right\},$$

where $c(\alpha) \leqslant C(2-\alpha)^{-2/3}$.

Proof. Note that, since $B_2^n \subseteq Z_q(\mu)$, the interesting range for t is up to $\sqrt{q}$. Given some $\varepsilon \in (0,1)$, let $k = (1-\varepsilon)n$ and $F \in G_{n,k}$. Applying Lemma 6.1 for the bodies $\sqrt{q}B_2^n$ and $Z_q(\mu)$ with $R = \sqrt{q}$ and $r = t/2$ we see that, for every $1 \leqslant t < \sqrt{q}$,

$$(6.3) \qquad N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \leqslant \left(\frac{c_1\sqrt{q}}{t}\right)^{\varepsilon n} N\left(\sqrt{q}B_F, \frac{t}{4}P_F\big(Z_q(\mu)\big)\right).$$

If $q^2 \leqslant \varepsilon n$ then Proposition 5.3 shows that, for every $1 \leqslant \alpha < 2$, there exists $F \in G_{n,k}$, $k = (1-\varepsilon)n$, such that $P_F(Z_q(\mu)) \supseteq c_2(2-\alpha)\varepsilon^{\frac{1}{2}+\frac{2}{\alpha}}\sqrt{q}B_F$. Thus, we arrive at

$$(6.4) \qquad N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \leqslant \left(\frac{c_3\sqrt{q}}{t}\right)^{\varepsilon n} N\left(B_F, c_4 t(2-\alpha)\varepsilon^{\frac{1}{2}+\frac{2}{\alpha}}B_F\right).$$

In the end we choose $\varepsilon \simeq [(2-\alpha)t]^{-\frac{2\alpha}{\alpha+4}}$ (the restriction $[(2-\alpha)t]^{\frac{2\alpha}{\alpha+4}} \leqslant cn/q^2$ is needed at this point—note that, if e.g. $q \leqslant n^{3/7}$, then this allows us to consider any t up to $\sqrt{q}$). With this choice of ε, we get from (6.4) that

$$(6.5) \qquad\qquad \log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \leqslant c(\alpha)\frac{n\log(2q/t^2)}{t^{\frac{2\alpha}{\alpha+4}}}.$$

This proves the upper bound for the first covering number in (6.2) provided that $t \geqslant c_1(2-\alpha)^{-1}$ (since $(2-\alpha)t \simeq \varepsilon^{\frac{\alpha+4}{2\alpha}}$ must be less than 1).

When $1 \leq t \leq c(2-\alpha)^{-1}$ we use the inequality

$$N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \leqslant N\big(\sqrt{q}B_2^n, c_1(2-\alpha)^{-1}Z_q(\mu)\big)N\big(Z_q(\mu), c_1^{-1}(2-\alpha)tZ_q(\mu)\big).$$

Noticing that the latter covering number is less than $\big(1+c(2-\alpha)^{-1}t^{-1}\big)^n$ completes the proof for the first covering number in (6.2) after elementary calculations.

The bound for the second covering number in (6.2) follows from the duality of entropy theorem. $\qquad\square$

Using Proposition 5.4 instead of Proposition 5.3 we get the following:

Proposition 6.4. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume that $2 \leqslant q \leqslant n$. Then, for any $1 \leqslant \alpha < 2$ and any*

$$1 \leqslant t \leqslant \min\left\{\sqrt{q},\ c_2(2-\alpha)^{-1}L_n\left(\frac{n}{q}\right)^{\frac{\alpha+4}{2\alpha}}\right\}$$

we have

$$\max\left\{\log N\big(\sqrt{q}B_2^n,\ tZ_q(\mu)\big),\ \log N\big(\sqrt{q}Z_q^\circ(\mu),\ tB_2^n\big)\right\}$$

(6.6)
$$\leqslant c(\alpha)L_n^{\frac{2\alpha}{\alpha+4}}\ \frac{n}{t^{\frac{2\alpha}{\alpha+4}}}\ \max\left\{\log\frac{\sqrt{2q}}{t},\ \log\frac{L_n}{(2-\alpha)t}\right\},$$

where $c(\alpha) \leqslant C(2-\alpha)^{-2/3}$ and $c_1, c_2, C > 0$ are absolute constants.

Proof. We proceed along the same lines as before, and the only other thing that we need to take into account is the fact that, for every $\varepsilon \in (0,1)$, $L_{\varepsilon n} \leqslant cL_n$ for some absolute constant c. Note that here we are allowed to consider any t up to $\sqrt{q}$ if we restrict ourselves to those q that do not exceed $\sqrt{L_n}n^{3/4}$. $\qquad\square$

Remark 6.5. If we do not use the monotonicity of L_n but we rather use the bound $L_{\varepsilon n} \leqslant \sqrt[4]{\varepsilon n}$, we will end up with an upper bound of the form

(6.7)
$$\frac{C}{(2-\alpha)^{\frac{4\alpha}{\alpha+8}}}\frac{n^{\frac{2\alpha+8}{\alpha+8}}\log q}{t^{\frac{4\alpha}{\alpha+8}}}$$

for $\max\{\log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big), \log N\big(\sqrt{q}Z_q^\circ(\mu), tB_2^n\big)\}$. We thus get a better exponent of t for each α, but the restrictions in the proof force t to be in the range

$$c(2-\alpha)^{-1}\sqrt[4]{n} \leqslant t \leqslant c(2-\alpha)^{-1}\frac{n^{\frac{2\alpha+8}{4\alpha}}}{q^{\frac{\alpha+8}{4\alpha}}}$$

(again t can vary up to $\sqrt{q}$ if, for example, $q \leqslant n^{6/7}$). In this range, the latter bound for the covering numbers is more efficient only if L_n depends "badly" on n.

7 Upper bound for $M(Z_q(\mu))$

To make use of the covering estimates we have just obtained so as to give an upper bound for $M(Z_q(\mu))$, we employ the Dudley–Fernique decomposition (see e.g. [9, §2.5.2]). We consider the symmetric convex body $K := \sqrt{q}Z_q^\circ(\mu)$ and, for any $1 \leqslant j \leqslant \log q$, we consider the entropy number $N(K, 2^{-j}RB_2^n)$, where $R = R(K) \leqslant \sqrt{q}$. There exists $N_j \subseteq K$ with $|N_j| = N(K, 2^{-j}RB_2^n)$ such that, for any $x \in K$, there exists $v \in N_j$ satisfying $\|x - v\|_2 \leqslant 2^{-j}R$. We set $N_0 = \{0\}$ and $Z_j = N_j - N_{j-1}$. Then, we have:

Lemma 7.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. For any $m \in \mathbb{N}$ and any $x \in K$ there exist $z_1, \ldots, z_m, w_m$ with $z_j \in Z_j \cap \frac{3R}{2^j} B_2^n$ and $w_m \in \frac{R}{2^m} B_2^n$ such that*

$$(7.1) \qquad x = z_1 + \cdots + z_m + w_m,$$

where $R = R(K)$ is the radius of K.

Theorem 7.2 (version for small q). *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $1 \leqslant \alpha < 2$, for every*

$$(7.2) \qquad 1 \leqslant q \leqslant c(2-\alpha)^{-1/6} n^{\frac{\alpha+4}{3\alpha+8}},$$

we have

$$(7.3) \qquad M\big(Z_q(\mu)\big) \leqslant C(2-\alpha)^{-1/3} \frac{\sqrt{\max\{\log q, \ \log(2-\alpha)^{-1}\}}}{q^{\frac{\alpha}{2\alpha+8}}},$$

where $C > 0$ is an absolute constant. In particular, for every $1 \leqslant q \leqslant n^{3/7}$,

$$(7.4) \qquad M\big(Z_q(\mu)\big) \leqslant C \frac{(\log q)^{5/6}}{\sqrt[6]{q}}.$$

Proof. We set $K = \sqrt{q} Z_q^\circ(\mu)$. Using Lemma 7.1, for any $m \in \mathbb{N}$ and any $x \in K$ we can find $(z_j)_{j \leqslant m} \subset Z_j \cap \frac{3R}{2^j} B_2^n$ and $w_m \in \frac{R}{2^m} B_2^n$ such that $x = z_1 + \cdots + z_m + w_m$. For any $\theta \in S^{n-1}$ one has

$$(7.5) \qquad |\langle x, \theta \rangle| \leqslant \sum_{j=1}^m |\langle z_j, \theta \rangle| + |\langle w_m, \theta \rangle|.$$

We write $\bar{z} = z/\|z\|_2$ for all $z \neq 0$. We have

$$
\begin{aligned}
(7.6) \quad w(K) &= \int_{S^{n-1}} \max_{x \in K} |\langle x, \theta \rangle| \, d\sigma(\theta) \\
&\leqslant \sum_{j=1}^m \int_{S^{n-1}} \max_{z \in Z_j} |\langle \theta, z \rangle| \, d\sigma(\theta) + \int_{S^{n-1}} \max_{w \in 2^{-m} R B_2^n} |\langle w, \theta \rangle| \, d\sigma(\theta) \\
&\leqslant \sum_{j=1}^m \frac{3R}{2^j} \int_{S^{n-1}} \max_{z \in Z_j} |\langle \theta, \bar{z} \rangle| \, d\sigma(\theta) + \frac{R}{2^m} \\
&\leqslant \sum_{j=1}^m \frac{c_3 R}{2^j} \frac{\sqrt{\log |Z_j|}}{\sqrt{n}} + \frac{R}{2^m},
\end{aligned}
$$

where we have used the following:

Fact. For any $u_1, \ldots, u_N \in S^{n-1}$ we have

$$(7.7) \qquad \int_{S^{n-1}} \max_{j \leqslant N} |\langle \theta, u_j \rangle| \, d\sigma(\theta) \leqslant c_3 \frac{\sqrt{\log N}}{\sqrt{n}}.$$

20

By the definition of Z_j and Proposition 6.3 we obtain

$$(7.8) \quad \log|Z_j| \leqslant \log|N_j| + \log|N_{j-1}| \leqslant c(\alpha)n \left(\frac{2^j}{R}\right)^{\frac{2\alpha}{\alpha+4}} \max\{\log q, \ \log(2-\alpha)^{-1}\},$$

where we assume that $R/2^m \geqslant 1$ and $c(\alpha) \leqslant C(2-\alpha)^{-\frac{2\alpha}{\alpha+4}}$. Plugging this into (7.6) we conclude that

$$(7.9) \quad w(K) \leqslant \frac{c_5}{(2-\alpha)^{\frac{\alpha}{\alpha+4}}} R^{\frac{4}{\alpha+4}} \sqrt{\max\left\{\log q, \ \log\frac{1}{2-\alpha}\right\}} \sum_{j\leqslant m} \frac{1}{2^{\frac{4j}{\alpha+4}}} + \frac{R}{2^m}$$

$$\leqslant \frac{C}{(2-\alpha)^{\frac{\alpha}{\alpha+4}}} q^{\frac{2}{\alpha+4}} \sqrt{\max\left\{\log q, \ \log\frac{1}{2-\alpha}\right\}},$$

if m is large enough so that $R/2^m \simeq 1$. It remains to observe that

$$w(K) = \sqrt{q}\, w\big(Z_q^\circ(\mu)\big) = \sqrt{q}\, M\big(Z_q(\mu)\big),$$

and hence

$$M\big(Z_q(\mu)\big) \leqslant \frac{C}{(2-\alpha)^{\frac{\alpha}{\alpha+4}}} \frac{\sqrt{\max\{\log q, \ \log(2-\alpha)^{-1}\}}}{q^{\frac{\alpha}{2\alpha+8}}}.$$

Finally, to obtain (7.4), we set $\alpha = 2 - \frac{1}{\log q}$. $\qquad\square$

Given a symmetric isotropic convex body K in $\mathbb{R}^n$ we apply Theorem 7.2 with the isotropic measure μ_K, that has density $L_K^n \mathbf{1}_{K/L_K}$, and with $q = n^{3/7}$ (which is the optimal choice for this purpose), and we get:

Theorem 7.3. *Let $K \subset \mathbb{R}^n$ be isotropic and symmetric. Then,*

$$M(K) \leqslant C \frac{(\log n)^{5/6}}{L_K \sqrt[14]{n}}.$$

Using Proposition 6.4 instead of Proposition 6.3, we also get:

Theorem 7.4. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $1 \leqslant \alpha < 2$, for every*

$$(7.10) \qquad\qquad 1 \leqslant q \leqslant c_1(2-\alpha)^{-1/3} L_n^{\frac{2\alpha}{2\alpha+4}} n^{\frac{\alpha+4}{2\alpha+4}},$$

we have

$$(7.11) \qquad M\big(Z_q(\mu)\big) \leqslant C(2-\alpha)^{-1/3} L_n^{\frac{\alpha}{\alpha+4}} \frac{\sqrt{\max\{\log q, \ \log L_n(2-\alpha)^{-1}\}}}{q^{\frac{\alpha}{2\alpha+8}}},$$

where c_1, $C > 0$ are absolute constants. In particular, for every q such that $L_n \log q \leqslant q \leqslant \sqrt{L_n}\, n^{3/4}$,

$$(7.12) \qquad M\big(Z_q(\mu)\big) \leqslant C \frac{\sqrt[3]{L_n}(\log q)^{5/6}}{\sqrt[6]{q}}$$

and, for every symmetric isotropic convex body K in $\mathbb{R}^n$,

$$M(K) \leqslant C \frac{\sqrt[4]{L_n}(\log n)^{5/6}}{L_K\,\sqrt[8]{n}}.$$

Remark 7.5. Similarly, using Remark 6.5 instead of Proposition 6.3, we see that, for every $1 \leqslant \alpha < 2$, for every

$$(7.13) \qquad c_1(2-\alpha)^{-2}\sqrt{n} \leqslant q \leqslant c_2(2-\alpha)^{-1/3} n^{\frac{2\alpha+8}{3\alpha+8}},$$

we have

$$(7.14) \qquad M\big(Z_q(\mu)\big) \leqslant C(2-\alpha)^{-2/5}\, \sqrt[4]{n}^{\frac{2\alpha}{\alpha+8}}\, \frac{\sqrt{\log q}}{q^{\frac{\alpha}{\alpha+8}}},$$

where c_1, c_2 and $C > 0$ are absolute constants, and, for every $\sqrt{n}\log n \leqslant q \leqslant n^{6/7}$,

$$(7.15) \qquad M\big(Z_q(\mu)\big) \leqslant C \frac{\sqrt[10]{n}(\log n)^{9/10}}{\sqrt[5]{q}}.$$

Therefore, for every isotropic symmetric convex body K in $\mathbb{R}^n$,

$$M(K) \leqslant C \frac{(\log n)^{9/10}}{L_K\,\sqrt[14]{n}}.$$

8 Further observations

In this last Section we collect a number of additional observations on the geometry of the centroid bodies $Z_q(\mu)$.

1. Inradius of projections. We first provide lower bounds for $R(Z_q(\mu) \cap F)$ and $R(Z_q^\circ(\mu) \cap F)$; actually, they hold true for every $1 \leqslant k < n$ and any $F \in G_{n,k}$. By duality, these estimates (combined with e.g. Proposition 5.3) determine the inradius of $P_F(Z_q^\circ(\mu))$ and $P_F(Z_q(\mu))$. Our starting point is the next proposition, which can be essentially found in [11].

Proposition 8.1. *Let A be a symmetric convex body in $\mathbb{R}^n$. Assume that there exists $\gamma \geqslant 1$ such that*

$$N(B_2^n, tA) \leqslant \exp\left(\frac{\gamma n}{t^p}\right)$$

for every $t \geqslant 1$. For every $\delta \in (0,1)$ and every $F \in G_{n,\lfloor \delta n \rfloor}$ we have

$$w(A \cap F) \geqslant \frac{c\delta^{1/p}}{\gamma^{1/p}}.$$

Proof. Let $k = \lfloor \delta n \rfloor$ and consider any $F \in G_{n,k}$. Using the assumption and the duality of entropy theorem we see that the projection $P_F(A^\circ)$ of A° onto F satisfies

$$N\big(P_F(A^\circ), tB_F\big) \leqslant N(A^\circ, tB_2^n) \leqslant \exp\left(\frac{\gamma k}{\delta t^p}\right),$$

for every $t \geqslant 1$. We apply Theorem 4.2 with $W = P_F(A^\circ)$, $n = k$ and $\varepsilon = 1/2$. There exists $H \in G_{k,\lfloor k/2 \rfloor}(F)$ for which

$$P_F(A^\circ) \cap H \subseteq \frac{c\gamma^{1/p}}{\delta^{1/p}} B_H.$$

Taking polars in H we see that $P_H(A \cap F) \supseteq \frac{c\delta^{1/p}}{\gamma^{1/p}} B_H$.

Recall now that, given a symmetric convex body C in $\mathbb{R}^m$ and an s-dimensional subspace L of $\mathbb{R}^m$, one has $M(C \cap L) \lesssim \sqrt{m/s}\, M(C)$ (see [12, Section 4.2]). Therefore, setting $C = (A \cap F)^\circ$ and $L = H$, we obtain

$$w(A \cap F) = M\big((A \cap F)^\circ\big) \geqslant \frac{1}{\sqrt{2}} M\big((A \cap F)^\circ \cap H\big) = \frac{c}{\sqrt{2}} w\big(P_H(A \cap F)\big)$$

$$\geqslant \frac{c'\delta^{1/p}}{\gamma^{1/p}},$$

as claimed. $\qquad\square$

Theorem 8.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume that $q \leqslant \sqrt{n}$. Then, for any $1 \leqslant \alpha < 2$, $0 < \delta < 1$ and any $F \in G_{n,\lfloor \delta n \rfloor}$ we have*

$$R\big(Z_q(\mu) \cap F\big) \geqslant w\big(Z_q(\mu) \cap F\big) \geqslant c \frac{(2-\alpha)\delta^{\frac{\alpha+4}{2\alpha}}}{\left(\log \frac{1}{2-\alpha}\right)^{\frac{\alpha+4}{2\alpha}}} \sqrt{q}.$$

Proof. From Proposition 6.3 we know that

$$\log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \leqslant c(\alpha) \frac{n}{t^{\frac{2\alpha}{\alpha+4}}} \max\left\{\log \frac{\sqrt{2q}}{t}, \, \log \frac{1}{(2-\alpha)t}\right\},$$

where $c(\alpha) \leqslant C(2-\alpha)^{-2/3}$. Thus, we may apply Proposition 8.1 with

$$\gamma = c(\alpha) \frac{\max\{\log q, \log \frac{1}{2-\alpha}\}}{\sqrt{q}^p}$$

and $p = \frac{2\alpha}{\alpha+4}$. $\qquad\square$

A similar argument applies to $Z_q^\circ(\mu)$. Since $M(Z_q^\circ(\mu)) = w(Z_q(\mu)) \simeq \sqrt{q}$ for all $q \leqslant \sqrt{n}$, from the dual Sudakov inequality we have

$$\log N\big(B_2^n, t\sqrt{q}Z_q^\circ(\mu)\big) \leqslant \frac{cn}{t^2}$$

for all $t \geqslant 1$. Applying Proposition 8.1 with $\gamma = cq$ and $p = 2$ we get:

Theorem 8.3. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume that $q \leqslant \sqrt{n}$. Then, for any $1 \leqslant \alpha < 2$, $0 < \delta < 1$ and any $F \in G_{n,\lfloor \delta n \rfloor}$ we have*

$$R\big(Z_q^\circ(\mu) \cap F\big) \geqslant w\big(Z_q^\circ(\mu) \cap F\big) \geqslant \frac{c\sqrt{\delta}}{\sqrt{q}}.$$

2. Upper bound for $M_{-k}(Z_q(\mu))$. Let C be a symmetric convex body in $\mathbb{R}^n$. For every $p \neq 0$, one can define

$$(8.1) \qquad\qquad M_p(C) := \left(\int_{S^{n-1}} \|\theta\|_C^p \, d\sigma(\theta) \right)^{1/p}.$$

Litvak, Milman and Schechtman proved in [23] that if b is the smallest constant for which $\|x\| \leqslant b\|x\|_2$ holds true for every $x \in \mathbb{R}^n$, then

$$\max\left\{ M(C), c_1 \frac{b\sqrt{q}}{\sqrt{n}} \right\} \leqslant M_p(C) \leqslant \max\left\{ 2M(C), c_2 \frac{b\sqrt{q}}{\sqrt{n}} \right\}$$

for all $p \in [1, n]$, where $c_1, c_2 > 0$ are absolute constants. In particular,

$$(8.2) \qquad\qquad M_p(C) \simeq M(C)$$

as long as $p \leqslant k(C) := k_*(C^\circ)$. Klartag and Vershynin defined in [22] the parameter

$$d(C) = \min\left\{ -\log\sigma\left(\left\{ x \in S^{n-1} : \|x\| \leqslant \frac{M(C)}{2} \right\} \right), n \right\}$$

and they observed that $d(C)$ is always larger than $k(C)$. Their main result is an analogue of (8.2) for negative values of p: one has

$$(8.3) \qquad\qquad M_{-p}(C) \simeq M(C)$$

as long as $0 < p \leqslant d(C)$.

Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Since $M_{-p}(Z_q(\mu))$ is clearly smaller than $M(Z_q(\mu))$, our next aim is to provide upper bounds for these quantities and to compare them to the ones from Section 7. We will use a formula for $M_{-k}(C)$ which appears in [31].

Lemma 8.4. *Let C be a symmetric convex body in $\mathbb{R}^n$. For every integer $1 \leqslant k < n$,*

$$(8.4) \qquad M_{-k}(C) \simeq \left(\int_{G_{n,k}} \mathrm{vrad}\big(P_F(C^\circ)\big)^{-k} d\nu_{n,k}(F) \right)^{-1/k}.$$

The proof is simple: using the Blaschke-Santaló and the reverse Santaló inequality (see [5]) one can write

$$(8.5) \qquad M_{-k}^{-1}(C) = \left(\int_{S^{n-1}} \frac{1}{\|x\|_C^k} d\sigma(x) \right)^{1/k}$$

$$= \left(\frac{1}{\omega_k} \int_{G_{n,k}} \omega_k \int_{S_F} \frac{1}{\|x\|_{C \cap F}^k} d\sigma(x) d\nu_{n,k}(F) \right)^{1/k}$$

$$= \left(\int_{G_{n,k}} \frac{|C \cap F|}{|B_2^k|} d\nu_{n,k}(F) \right)^{1/k}$$

$$\simeq \left(\int_{G_{n,k}} \frac{|B_2^k|}{|P_F(C^\circ)|} d\nu_{n,k}(F) \right)^{1/k}.$$

Proposition 8.5. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $2 \leqslant q \leqslant \sqrt{n}$ and $k \geqslant q^2 \log^2 q$ one has*

$$(8.6) \qquad M_{-k}\big(Z_q(\mu)\big) \leqslant \frac{c \log^3 q}{\sqrt{q}} \left(\frac{n}{k} \right)^{3/2}.$$

Proof. We choose $\alpha = 2 - \frac{1}{\log q}$. From the proof of Proposition 6.3 we see that if $t \leqslant \sqrt{q}$ and

$$\frac{q^2}{(\log q)^{2/3}} t^{2/3} \leqslant n$$

then

$$N\big(\sqrt{q} Z_q^\circ(\mu),\ t B_2^n\big) \leqslant \exp\left(c_1 \frac{n(\log q)^{2/3}}{t^{2/3}} \log \frac{2q}{t^2} \right).$$

where c_1 is an absolute constant. Thus, for any integer $1 \leqslant k < n$ and any $F \in G_{n,k}$ we have

$$\sqrt{q} \big| P_F\big(Z_q^\circ(\mu)\big) \big|^{1/k} \leqslant t |B_2^k|^{1/k} N\left(\sqrt{q} P_F\big(Z_q^\circ(\mu)\big), t B_F \right)^{1/k}$$

$$\leqslant t |B_2^k|^{1/k} \exp\left(c_1 \frac{n(\log q)^{2/3}}{k t^{2/3}} \log \frac{2q}{t^2} \right).$$

Choosing $t \simeq (n/k)^{3/2} \log^3 q$ we conclude that

$$\left(\frac{\big| P_F\big(Z_q^\circ(\mu)\big) \big|}{|B_2^k|} \right)^{1/k} \leqslant \frac{c \log^3 q}{\sqrt{q}} \left(\frac{n}{k} \right)^{3/2},$$

and the result follows from Lemma 8.4. $\qquad\square$

Note. Since $Z_q(\mu) \supseteq B_2^n$, one has the obvious upper bound $M_{-k}(Z_q(\mu)) \leqslant 1$ for all $1 \leqslant k < n$. So, the estimate of Proposition 8.5 is non-trivial provided that $k \geqslant n \log^2 q / q^{1/3}$.

Remark 8.6. Similarly, starting from Proposition 6.4 and following the proof of Proposition 8.5 we obtain the following.

Proposition 8.7. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $2 \leqslant q \leqslant n$ and $k \geqslant L_n^{2/3} q$ one has*

$$(8.7) \qquad M_{-k}\big(Z_q(\mu)\big) \leqslant \frac{cL_n}{\sqrt{q}} \left(\frac{n}{k}\right)^{3/2}.$$

In view of (8.3) a natural question is to give lower bounds for $d(Z_q(\mu))$; because, if $d(Z_q(\mu))$ is large enough so that we can use Proposition 8.5 or Proposition 8.7, then we would have an alternative source of, possibly better, information on $M(Z_q(\mu))$ as well. What we know is a simple lower bound for $k(Z_q(\mu))$ and $k(Z_q^\circ(\mu))$ in the range $2 \leqslant q \leqslant q_*(\mu)$. For every $2 \leqslant q \leqslant q_*(\mu)$ one has

$$(8.8) \qquad \min\Big\{k\big(Z_q(\mu)\big), k\big(Z_q^\circ(\mu)\big)\Big\} \geqslant \frac{c_1 n}{q},$$

where c_1 is an absolute constant.

To see this, recall that $B_2^n \subseteq Z_q(\mu) \subseteq c_2 q B_2^n$, and hence

$$R\big(Z_q(\mu)\big) \leqslant c_2 q \quad \text{and} \quad R\big(Z_q^\circ(\mu)\big) \leqslant 1.$$

Then, using (3.6) we get

$$k_*\big(Z_q(\mu)\big) \geqslant c_3 n \frac{w^2\big(Z_q(\mu)\big)}{R^2\big(Z_q(\mu)\big)} \geqslant c_4 n \frac{q}{q^2} = \frac{c_4 n}{q}.$$

Also, using the fact that $w(C^\circ)w(C) \geqslant 1$ for every symmetric convex body C, and taking into account (3.6), we see that $w(Z_q^\circ(\mu)) \geqslant c/\sqrt{q}$ for every $2 \leqslant q \leqslant q_*(\mu)$, and thus

$$k_*\big(Z_q^\circ(\mu)\big) \geqslant c_5 n \frac{w^2\big(Z_q^\circ(\mu)\big)}{R^2\big(Z_q^\circ(\mu)\big)} \geqslant \frac{c_6 n}{q}.$$

3. Small ball probability estimates. Given a symmetric convex body C in $\mathbb{R}^n$ we set $\overline{C} = |C|^{-1/n}C$. In this last subsection we describe an approach which can lead to small ball probability estimates for the centroid bodies. It is convenient to normalize the volume, and consider $\overline{Z_q}(\mu)$ instead of $Z_q(\mu)$. Recall that if $q \leqslant \sqrt{n}$ then $|Z_q(\mu)|^{1/n} \simeq \sqrt{q/n}$, and hence

$$\overline{Z_q}(\mu) \simeq \sqrt{n/q}\, Z_q(\mu).$$

Then, $w(\overline{Z_q}(\mu)) \simeq \sqrt{n/q}\, w(Z_q(\mu)) \simeq \sqrt{n}$, which implies

$$\log N\big(\overline{Z_q}(\mu), sB_2^n\big) \leqslant c_1 n \left(\frac{w\big(\overline{Z_q}(\mu)\big)}{s}\right)^2 \leqslant \frac{c_2 n^2}{s^2}.$$

We use the following fact ([8], Lemma 5.6).

Lemma 8.8. *Let C be a centered convex body of volume 1 in $\mathbb{R}^n$. Assume that, for some $s > 0$,*

$$(8.9) \qquad\qquad r_s := \log N(K, sB_2^n) < n.$$

Then,

$$I_{-r_s}(K) \leqslant cs.$$

We apply Lemma 8.8 for $\overline{Z_q}(\mu)$ to get:

Proposition 8.9. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $2 \leqslant q \leqslant \sqrt{n}$ then*

$$I_{-r}\big(\overline{Z_q}(\mu)\big) \leqslant \frac{c_3 n}{\sqrt{r}}$$

for all $1 \leqslant r \leqslant cn$.

From Markov's inequality we have

$$\left| \left\{ x \in \overline{Z_q}(\mu) : \|x\|_2 \leqslant \varepsilon I_{-r}\big(\overline{Z_q}(\mu)\big) \right\} \right| \leqslant \varepsilon^r.$$

Since $I_{-n/q}(\overline{Z_q}(\mu)) \simeq I_2(\overline{Z_q}(\mu)) \simeq \sqrt{n}$, we obtain the following small ball probability estimate.

Proposition 8.10. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. If $2 \leqslant q \leqslant \sqrt{n}$ then*

$$\left| \{ x \in \overline{Z_q}(\mu) : \|x\|_2 \leqslant c\varepsilon\sqrt{n} \} \right| \leqslant \varepsilon^{n/q}$$

for all $\varepsilon \in (0,1)$.

References

[1] S. Artstein, V. Milman and S. Szarek, *Duality of metric entropy*, Annals of Math. **159** (2004), 1313–1328.

[2] K. M. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. **88** (1988), 69–84.

[3] J. Bastero, *Upper bounds for the volume and diameter of m-dimensional sections of convex bodies*, Proc. Amer. Math. Soc. **135** (2007), 1851–1859.

[4] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1469** (1991), 127–137.

[5] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$*, Invent. Math. **88** (1987), 319–340.

[6] J. Bourgain, J. Lindenstrauss and V. D. Milman, *Minkowski sums and symmetriza-tions*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1317** (1988), 44–74.

[7] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Notes on isotropic convex bodies*, Book in preparation, available at `http://users.uoa.gr/~apgiannop/`.

[8] N. Dafnis and G. Paouris, *Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture*, J. Funct. Anal. **258** (2010), 1933–1964.

[9] A. Giannopoulos, *Notes on isotropic convex bodies*, Lecture Notes, Warsaw 2003, available at `http://users.uoa.gr/~apgiannop/`.

[10] A. A. Giannopoulos and V. D. Milman, *On the diameter of proportional sections of a symmetric convex body*, International Mathematics Research Notices (1997) No. 1, 5–19.

[11] A. A. Giannopoulos and V. D. Milman, *Mean width and diameter of proportional sections of a symmetric convex body*, J. Reine Angew. Math. **497** (1998), 113–139.

[12] A. Giannopoulos and V. D. Milman, *Euclidean structure in finite-dimensional normed spaces*, Handbook of the Geometry of Banach Spaces (Johnson-Lindenstrauss eds.), Vol. 1 (2001), 707–779.

[13] A. Giannopoulos, V. D. Milman and A. Tsolomitis, *Asymptotic formulas for the diameter of sections of symmetric convex bodies*, Journal of Functional Analysis **223** (2005), 86–108.

[14] A. Giannopoulos, G. Paouris and P. Valettas, *On the existence of subgaussian direc-tions for log-concave measures*, Contemporary Mathematics **545** (2011), 103–122.

[15] A. Giannopoulos, G. Paouris and P. Valettas, *On the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math. **2050** (2012), 227–253.

[16] A. Giannopoulos, G. Paouris and B-H. Vritsiou, *A remark on the slicing problem*, Journal of Functional Analysis 262 (2012), 1062–1086.

[17] Y. Gordon, *On Milman's inequality and random subspaces which escape through a mesh in $\mathbf{R}^n$*, Lecture Notes in Mathematics **1317** (1988), 84–106.

[18] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.

[19] B. Klartag and E. Milman, *Centroid Bodies and the Logarithmic Laplace Transform—A Unified Approach*, J. Funct. Anal. **262** (2012), 10–34.

[20] B. Klartag and E. Milman, *Inner regularization of log-concave measures and small-ball estimates*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math. **2050** (2012), 267–278.

[21] B. Klartag and V. D. Milman, *Rapid Steiner Symmetrization of most of a convex body and the slicing problem*, Combin. Probab. Comput. **14**, no. 5–6 (2005) 829–843.

[22] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky theorem*, Israel J. Math. **157** (2007), 193–207.

[23] A. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), 95–124.

[24] A. Litvak, V. D. Milman, A. Pajor and N. Tomczak-Jeagermann, *Entropy extension*, Funct. Anal. Appl. **40** (2006), 298–303.

[25] E. Lutwak, D. Yang and G. Zhang, L^p *affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[26] V. D. Milman, *Isomorphic symmetrization and geometric inequalities*, Lecture Notes in Mathematics **1317** (1988), 107–131.

[27] V. D. Milman, *Some applications of duality relations*, Lecture Notes in Mathematics **1469** (1991), 13–40.

[28] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Lecture Notes in Mathematics **1376**, Springer, Berlin (1989), 64–104.

[29] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics **1200** (1986), Springer, Berlin.

[30] G. Paouris, *Concentration of mass in convex bodies*, Geometric and Functional Analysis **16** (2006), 1021–1049.

[31] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287–308.

[32] G. Pisier, *A new approach to several results of V. Milman*, J. Reine Angew. Math. **393** (1989), 115–131.

[33] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[34] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, Cambridge (1993).

[35] P. Valettas, *Upper bound for the ℓ-norm in the isotropic position*, Private communication.

[36] R. Vershynin, *Isoperimetry of waists and local versus global asymptotic convex geometries* (with an appendix by M. Rudelson and R. Vershynin), Duke Mathematical Journal **131** (2006), 1–16.

[37] B-H. Vritsiou, *Further unifying two approaches to the hyperplane conjecture*, Int. Math. Res. Not. (2012), DOI: 10.1093/imrn/rns263.

APOSTOLOS GIANNOPOULOS: Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece.
E-mail: `apgiannop@math.uoa.gr`

PANTELIS STAVRAKAKIS: Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece.
E-mail: `pantstav@yahoo.gr`

ANTONIS TSOLOMITIS: Department of Mathematics, University of the Aegean, Karlovassi 832 00, Samos, Greece.
E-mail: `antonis.tsolomitis@gmail.com`

BEATRICE-HELEN VRITSIOU: Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece.
E-mail: `bevritsi@math.uoa.gr`