Μάθημα 23

$$\int \frac{x+2}{(x+1)^2(x^2+1)}\,dx \qquad (1)$$

$$\frac{x+2}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{\Gamma x+\Delta}{x^2+1} \iff \qquad (2)$$

$$\iff x+2 = A(x+1)(x^2+1) + B(x^2+1) + (\Gamma x+\Delta)(x+1)^2 \qquad (3)$$

$$= A(x^3+x+x^2+1) + Bx^2+B + (\Gamma x+\Delta)(x^2+2x+1) \qquad (4)$$

$$= Ax^3+Ax+Ax^2+A + Bx^2+B + \Gamma x^3+2\Gamma x^2+\Gamma x+\Delta x^2+2\Delta x+\Delta \qquad (5)$$

$$= \underline{Ax^3} + Ax + \underline{\underline{Ax^2}} + A + \underline{\underline{Bx^2}} + B + \underline{\Gamma x^3} + \underline{\underline{2\Gamma x^2}} + \Gamma x + \underline{\underline{\Delta x^2}} + 2\Delta x + \Delta \qquad (6)$$

$$= (A+\Gamma)x^3 + (A+B+2\Gamma+\Delta)x^2 + (A+\Gamma+2\Delta)x + (A+B+\Delta) \qquad (7)$$

(8)

Άρα $A+\Gamma=0$ (9)

$A+B+2\Gamma+\Delta=0 \rightarrow B+\Delta+\Gamma=0$ (10)

$A+\Gamma+2\Delta=1 \Rightarrow 2\Delta=1 \Rightarrow \boxed{\Delta=1/2}$ $\rightarrow 2\Gamma=-2 \Rightarrow \boxed{\Gamma=-1}$ (11)

$A+B+\Delta=2 \rightarrow B+\Delta-\Gamma=2$ $\boxed{A=1}$ (12)

$B=-\Gamma-\Delta=1-1/2=1/2$ (13)

Οπότε $A=1 \quad B=1/2 \quad \Gamma=-1 \quad \Delta=1/2$ (14)

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + c, \quad \int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx = \frac{(x+1)^{-2+1}}{-2+1} + c \qquad (15)$$

$$\int \frac{-x+1/2}{x^2+1}\,dx = -\int \frac{x-1/2}{x^2+1}\,dx = -\frac{1}{2}\int \frac{2x-1}{x^2+1}\,dx = \qquad (16)$$

$$= -\frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx = -\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\arctan(x) + c \qquad (17)$$

$I=\int \frac{5x^2+12x+1}{x^3+3x^2-4}\,dx$. Πιθανές ρίζες του παρονομαστή (1)

$\pm 1,\ \pm 2,\ \pm 4$ (2)

Η $x=1$ είναι ρίζα οπότε (3)

$$\begin{array}{l|l} x^3+3x^2-4 & x-1 \\ \hline -x^3+x^2 & x^2+4x+4 \\ \quad 4x^2-4 & \\ \quad -4x^2+4x & \\ \qquad 4x-4 & \\ \qquad -4x+4 & \\ \qquad\quad 0 & \end{array}$$

Άρα $x^3+3x^2-4=(x-1)(x+2)^2$ (4)

$$\frac{5x^2+12x+1}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{\Gamma}{(x+2)^2}\implies$$ (5)

$$\implies 5x^2+12x+1=A(x+2)^2+B(x-1)(x+2)+\Gamma(x-1)$$ (6)

$$=Ax^2+4Ax+4A+Bx^2+2Bx-Bx-2B+\Gamma x-\Gamma$$ (7)

$$=(A+B)x^2+(4A+2B+\Gamma)x+4A-2B-\Gamma$$ (8)

Άρα $A+B=5$, $4A+2B+\Gamma=12$, $4A-2B-\Gamma=1$ (9)

Λύνουμε το σύστημα (10)

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 4 & 1 & 1 & 12 \\ 4 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -3 & 1 & -8 \\ 0 & -3 & -2 & -11 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -3 & 1 & -8 \\ 0 & 0 & -3 & -3 \end{array}\right)$$

$\boxed{\Gamma=1}$ (11)

$-3B+\Gamma=-8 \implies -3B+1=-8$ (12)

$\implies \boxed{B=3}$ (13)

$A+B=5 \implies \boxed{A=2}$ (14)

Άρα

$$I=\int\frac{2}{x-1}\,dx+\int\frac{3}{x+2}\,dx+\int\frac{1}{(x+2)^2}\,dx$$ (15)

$$=2\ln|x-1|+3\ln|x+2|-\frac{1}{x+2}+c$$ (16)

Ρητές συναρτήσεις των $\cos x$ και $\sin x$ (1)

$$\int R(\cos x, \sin x)\, dx \quad (2)$$

Θέτουμε $u = \tan\frac{x}{2}$ και παρατηρούμε ότι (3)

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right) \quad (4)$$

$$= \frac{1}{1+\tan^2\frac{x}{2}} \cdot \left(1 - \tan^2\frac{x}{2}\right) = \frac{1-u^2}{1+u^2} \quad (5)$$

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2} = 2\cos^2\frac{x}{2}\,\tan\frac{x}{2} = \frac{2u}{1+u^2} \quad (6)$$

$\frac{x}{2} = \arctan u \Rightarrow dx = \frac{2}{1+u^2}\,du$

Έτσι αναγόμαστε στο $\int R\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right) \cdot \frac{2}{1+u^2}\,du$ (7)

όπου R ρητή συνάρτηση

Π.χ. $\int \frac{1+\sin x}{1-\cos x}\,dx \overset{u=\tan\frac{x}{2}}{=\!=\!=} \int \frac{1+\frac{2u}{1+u^2}}{1-\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du$ (8)

$$= \int \frac{\frac{1+u^2+2u}{1+u^2}}{\frac{1+u^2-1+u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du = \quad (9)$$

$$= \int \frac{(1+u)^2}{u^2(1+u^2)}\,du = \int \frac{1+u^2+2u}{u^2(1+u^2)}\,du = \quad (10)$$

$$= \int \frac{1+u^2}{u^2(1+u^2)}\,du + \int \frac{2u}{u^2(1+u^2)}\,du = \quad (11)$$

$$= \int \frac{1}{u^2}\,du + \int \frac{2\cancel{u}}{u(1+u^2)}\,du = \cancel{-\frac{1}{u} + \ln(1+u^2) + c} = \quad (12)$$

$$= -\frac{1}{u} + \int\left(\frac{2}{u} - \frac{2}{1+u^2}\right)du = -\frac{1}{u} + 2\ln|u| - 2\arctan(u)$$

$$= -\frac{1}{\tan\frac{x}{2}} \cancel{-2}\ln\left(1+\tan^2\frac{x}{2}\right) + c + 2\ln\left|\tan\frac{x}{2}\right| \quad (1)$$

$$\int R(x, \sqrt{1-x^2})\,dx, \quad \int R(x, \sqrt{x^2-1}), \quad \int R(x, \sqrt{1+x^2}) \quad (2)$$

$x = \sin t$ (8)

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$$\begin{cases} u = \dfrac{\sqrt{1-x^2}-1}{x} & (10) \\ \sqrt{1-x^2} = \dfrac{1-u^2}{1+u^2} & \\ x = -\dfrac{2u}{u^2+1} & (11) \\ dx = 2\dfrac{u^2-1}{(u^2+1)^2}\,du & (12) \end{cases}$$

$x = \frac{1}{\cos t}$ (9)

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$u = x + \sqrt{x^2-1}$ (13)

$$\Longrightarrow \begin{cases} x = (u^2+1)/2u & (14) \\ \sqrt{x^2-1} = \dfrac{u^2-1}{2u} & (15) \\ dx = \dfrac{u^2-1}{2u^2}\,du & (16) \end{cases}$$

$x = \tan t$ (3)

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$u = x + \sqrt{x^2+1}$ (4)

$$\Longrightarrow \begin{cases} x = \dfrac{u^2-1}{2u} & (5) \\ \sqrt{x^2+1} = \dfrac{u^2+1}{2u} & (6) \\ dx = \dfrac{u^2+1}{2u^2}\,du & (7) \end{cases}$$

Πχ

$$\int \sqrt{x^2-1}\,dx = \int \frac{u^2-1}{2u}\,\frac{u^2-1}{2u^2}\,du = \quad (17)$$

$$= \int \frac{(u^2-1)^2}{4u^3}\,du = \frac{1}{4}\int \frac{u^4-2u^2+1}{u^3}\,du = \quad (18)$$

$$= \frac{1}{4}\left(\int u\,du - 2\int \frac{1}{u}\,du + \int \frac{1}{u^3}\,du\right) = \quad (19)$$

$$= \frac{1}{4}\left(\frac{u^2}{2} + 2\ln|u| + \frac{u^{-3+1}}{-3+1}\right) + c = \quad (20)$$

$$= \frac{(x+\sqrt{x^2-1})^2}{8} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| - \frac{1}{8}\,\frac{1}{(x+\sqrt{x^2-1})^2} + c \quad (21)$$

$$\int \frac{1}{(u^2+1)^3}\,du = \int \frac{u^2+1-u^2}{(u^2+1)^3}\,du = \int \frac{1}{(u^2+1)^2} - \int \frac{u^2}{(u^2+1)^3}\,du \qquad (1)$$

Το $\int \frac{1}{(u^2+1)^2}\,du$ υπολογίστηκε πριν (2)

$$\int \frac{u^2}{(u^2+1)^3}\,du = \int \frac{1}{4} u \underbrace{\left(-\frac{1}{(u^2+1)^2}\right)'}_{=2(u^2+1)^{-3}\,2u}\,du = \qquad (3)\ (4)$$

$$= -\frac{1}{4}\frac{u}{(u^2+1)^2} + \frac{1}{4}\int \frac{1}{(u^2+1)^2}\,du \qquad (5)$$

$$\text{Άρα } \int \frac{1}{(u^2+1)^3}\,du = \int \frac{1}{(u^2+1)^2}\,du + \frac{1}{4}\frac{u}{(u^2+1)^2} - \frac{1}{4}\int \frac{1}{(u^2+1)^2}\,du \qquad (6)$$

$$= \frac{1}{4}\frac{u}{(u^2+1)^2} + \frac{3}{4}\int \frac{1}{(u^2+1)^2}\,du \qquad (7)$$

$$= \frac{1}{4}\frac{u}{(u^2+1)^2} + \frac{3}{4}\left(\frac{1}{2}\arctan u + \frac{1}{2}\frac{u}{u^2+1}\right) + c \qquad (8)$$

Επιστρέφουμε όλα τα ολοκληρώματα στην ⊛ (9)

και επιστρέφουμε τη μεταβλητή u σε x. (10)

(o-3)5

$$\int \sqrt{1-x^2}\,dx = \int \frac{1-u^2}{1+u^2} \; 2\,\frac{u^2-1}{(u^2+1)^2}\,du = \qquad (1)$$

$$= -2 \int \frac{(u^2-1)^2}{(u^2+1)^3}\,du = -2 \int \frac{\left((u^2+1)-2\right)^2}{(u^2+1)^3}\,du \qquad (2)$$

$$= -2 \int \frac{(u^2+1)^2 - 4(u^2+1) + 4}{(u^2+1)^3}\,du = \qquad (3)$$

$$= -2 \left[\int \frac{1}{u^2+1}\,du - 4 \int \frac{1}{(u^2+1)^2}\,du + 4 \int \frac{1}{(u^2+1)^3}\,du \right] \qquad (4)$$

☆

$$\int \frac{1}{u^2+1}\,du = \arctan u + c \qquad (5)$$

$$\int \frac{1}{(u^2+1)^2}\,du = \int \frac{u^2+1-u^2}{(u^2+1)^2}\,du = \int \frac{1}{u^2+1} - \int \frac{u^2}{(u^2+1)^2}\,du \qquad (6)$$

$$= \arctan(u) - \int \frac{u^2}{(u^2+1)^2}\,du \qquad (7)$$

$$= \arctan(u) - \int \frac{1}{2} u \underbrace{\left(-\frac{1}{u^2+1}\right)'}_{=\frac{2u}{(u^2+1)^2}} du \qquad (8)$$

(9)

$$= \arctan u - \frac{1}{2}\left[u\left(-\frac{1}{u^2+1}\right) - \int u' \left(-\frac{1}{u^2+1}\right) du \right] \qquad (10)$$

$$= \arctan u + \frac{1}{2}\,\frac{u}{u^2+1} - \frac{1}{2} \int \frac{1}{u^2+1}\,du = \qquad (11)$$

$$= \arctan u + \frac{1}{2}\,\frac{u}{u^2+1} - \frac{1}{2} \arctan u + c = \qquad (12)$$

$$= \frac{1}{2} \arctan u + \frac{1}{2}\,\frac{u}{u^2+1} + c \qquad (13)$$