

Μάθημα 25

(0211)

A 6 (i) $\int \sin(\log x) dx$ (Θεσε $t = \log x, x = e^t$) (1)

A 7 (ii) $\int \frac{x e^x}{(x+1)^2} dx = \int \frac{(x+1)e^x - e^x}{(x+1)^2} dx$ (2)

$= \int \left(\frac{e^x}{x+1}\right)' dx = \frac{e^x}{x+1} + c$ (3)

A 8 (ii) $\int \frac{\log(\tan x)}{\cos^2 x} dx$ $\frac{u = \tan x}{du = \frac{1}{\cos^2 x} dx}$ (4)

$= \int \log u du = \int u' \log u du = u \log u - \int u (\log u)' du$ (5)

$= u \log u - u + c = (\tan x) \log(\tan x) - (\tan x) + c$ (6)

A 9 (iv) $\int_0^{\pi/4} x \tan^2 x dx = \int_0^{\pi/4} x (\tan^2 x + 1 - 1) dx$ (7)

$= \int_0^{\pi/4} \frac{x}{\cos^2 x} dx - \int_0^{\pi/4} x dx = \int_0^{\pi/4} x (\tan x)' dx - \int_0^{\pi/4} x dx$ (8)

$= x \tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x dx - \frac{x^2}{2} \Big|_0^{\pi/4} =$ (9)

$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin x}{\cos x} dx - \frac{\pi^2}{32}$ $\frac{u = \cos x}{du = -\sin x dx}$ $x=0 \quad u=1$ $x=\frac{\pi}{4} \quad u=\frac{\sqrt{2}}{2}$ (10)

$= \frac{\pi}{4} - \frac{\pi^2}{32} + \int_1^{\sqrt{2}/2} \frac{1}{u} du = \frac{\pi}{4} - \frac{\pi^2}{32} + \ln \frac{\sqrt{2}}{2} - \ln 1 =$ (11)

$= \frac{\pi}{4} - \frac{\pi^2}{32} + \ln \frac{\sqrt{2}}{2}$ (12)

B 11 (v) | $I = \int \frac{1}{(1+x^2)^2} dx$ $\frac{\left(\frac{1}{1+x^2}\right)'}{(1+x^2)^2} \cdot 2x$ (1)

$= - \int \left(\frac{1}{1+x^2}\right)' \frac{1}{2x} dx = - \frac{1}{1+x^2} \frac{1}{2x} + \int \frac{1}{1+x^2} \left(-\frac{1}{2x^2}\right) dx$ (2)

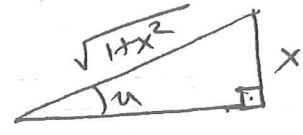
$= \frac{-1}{2x(x^2+1)} - \frac{1}{2} \int \frac{1}{x^2(x^2+1)} dx$ (3)

$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \Rightarrow \dots \Rightarrow$ (4)

$\Rightarrow \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$ (5)

(3) $= - \frac{1}{2x(x^2+1)} - \frac{1}{2} \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx$ (6)

$= - \frac{1}{2x(x^2+1)} + \frac{1}{2} \frac{1}{x} + \frac{1}{2} \arctan(x) + C$ (7)

Answer dengan $x = \tan u$  $dx = \frac{1}{\cos^2 u} du$ (8)

Apakah $I = \int \frac{1}{(1+\tan^2 u)^2} \frac{1}{\cos^2 u} du = \int \frac{1}{\left(\frac{1}{\cos^2 u}\right)^2} \frac{1}{\cos^2 u} du$ (9)

$= \int \frac{\cos^4 u}{\cos^2 u} du = \int \cos^2 u du = \int \frac{1 + \cos(2u)}{2} du$ (10)

$= \frac{1}{2} u + \frac{1}{4} \sin(2u) + C = \frac{1}{2} u + \frac{1}{2} \sin u \cos u + C$ (11)

$= \frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{\sqrt{1+x^2}} \frac{1}{\sqrt{1+x^2}} + C$ (12)

B12 | $I = \int_0^{\pi} \frac{x \sin x}{(1+\cos x)^2} dx$ (όχι $1+\cos^2 x$, αλλά $(1+\cos x)^2$)
 (αλλάς του υπονομούσαν)

(2)

Βρίσκουμε την αρχικοπαύση

$$\int \frac{x \sin x}{(1+\cos x)^2} dx = \int x \left(\frac{1}{1+\cos x} \right)' dx = \quad (3)$$

$$= \frac{x}{1+\cos x} - \int \frac{1}{1+\cos x} dx = \frac{x}{1+\cos x} - \int \frac{1}{2 \cos^2 \frac{x}{2}} dx \quad (4)$$

(5)

(διότι $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow 2 \cos^2 x - 1 = \cos 2x$
 $\Rightarrow 1 + \cos 2x = 2 \cos^2 x$)

(6)

$$\frac{x}{2} = u \quad \frac{dx}{2} = du \quad \frac{x}{1+\cos x} - \int \frac{1}{2 \cos^2 u} du =$$

(7)

$$= \frac{x}{1+\cos x} - \tan u + c = \frac{x}{1+\cos x} - \tan \frac{x}{2} + c \quad (8)$$

Άρα $I = \left(\frac{x}{1+\cos x} - \tan \frac{x}{2} \right) \Big|_0^{\pi}$ Στο $x=0$ και 0 (9)

αλλά στο $x=\pi$ έχουμε απροσδιόριστο $\frac{0}{0}$ (10)

δοκίμηση ~~πάλι~~ εννοείται ως $I = \lim_{x \rightarrow \pi^-} \left(\frac{x}{1+\cos x} - \tan \frac{x}{2} \right)$ (11)

και αναφέρεται Γενικευμένο οδοκίμηση

$$I = \lim_{x \rightarrow \pi^-} \left(\frac{x}{2 \cos^2 \frac{x}{2}} - \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) = \lim_{x \rightarrow \pi^-} \frac{x - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \quad (12)$$

$$= \lim_{x \rightarrow \pi^-} \frac{x - \sin x}{2 \cos^2 \frac{x}{2}} = +\infty \quad (13)$$

B 16(i) $I = \int_0^\infty x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx$

$\int_0^t x e^{-x^2} dx \xrightarrow{\substack{x^2 = u \\ 2x dx = du}} \int_0^{t^2} \frac{1}{2} e^{-u} du =$ (15)

$x=0 \quad u=0$
 $x=t \quad u=t^2$

$= -\frac{1}{2} e^{-u} \Big|_0^{t^2} = -\frac{1}{2} e^{-t^2} + \frac{1}{2}$ (16)

Ans $I = \lim_{t \rightarrow \infty} \frac{1}{2} (1 - e^{-t^2}) = \frac{1}{2}$ (17)

B 18(i) $\lim_{x \rightarrow \infty} x^3 e^{-x^6} \int_0^{x^3} e^{t^2} dt$ (18)

$x^3 e^{-x^6} \xrightarrow{x \rightarrow \infty} 0$ (check for L'Hospital) (19)

$\int_0^{x^3} e^{t^2} dt \geq \int_0^{x^3} 1 dt = x^3 \xrightarrow{x \rightarrow \infty} \infty$ (20)

Ans $\lim_{x \rightarrow \infty} \frac{\int_0^{x^3} e^{t^2} dt}{x^{-3} e^{x^6}} \xrightarrow{L'H} \lim_{x \rightarrow \infty} \frac{e^{x^6} \cdot 3x^2}{-3x^{-4} e^{x^6} + x^{-3} \cdot 6x^5 e^{x^6}}$ (21)

$= \lim_{x \rightarrow \infty} \frac{3x^2}{-3x^{-4} + 6x^2} = \lim_{x \rightarrow \infty} \frac{3x^2}{6x^2 - 3} = 0$ (22)

$= \frac{3}{6} = \frac{1}{2}$

Ντομας ΟΕ3 79, 1.39(8)

$$\int x \operatorname{arcsin}(x^2) dx = \int \left(\frac{x^2}{2}\right)' \operatorname{arcsin}(x^2) dx = \quad (1)$$

$$= \frac{x^2}{2} \operatorname{arcsin}(x^2) - \int \frac{x^2}{2} (\operatorname{arcsin} x^2)' dx = \quad (2)$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) - \frac{1}{2} \int x^2 \frac{1}{\sqrt{1-x^4}} 2x dx = \quad (3)$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) - \int \frac{x^3}{\sqrt{1-x^4}} dx \quad \begin{array}{l} u = 1-x^4 \\ du = -4x^3 dx \\ x^3 dx = -\frac{1}{4} du \end{array} \quad (4)$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) - \int \frac{1}{\sqrt{u}} \left(-\frac{1}{4}\right) du = \quad (5)$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) + \frac{1}{2} \int \frac{1}{2\sqrt{u}} du = \quad (6)$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) + \frac{1}{2} \sqrt{u} + c =$$

$$= \frac{1}{2} x^2 \operatorname{arcsin}(x^2) + \frac{1}{2} \sqrt{1-x^4} + c, \quad (8)$$

N. ΤΟΥΓΙΑΣ σελ 75, 1.36 (ε)

$$\int \frac{dx}{\sqrt{e^{2x} + 4e^x + 1}} \quad \begin{array}{l} \text{F}_{u=e^x} \\ du = e^x dx \\ \Rightarrow \frac{du}{u} = dx \end{array} \quad \int \frac{1}{\sqrt{u^2 + 4u + 1}} \frac{du}{u} \quad (1)$$

$$= \int \frac{1}{u \sqrt{u^2 + 2 \cdot 2 \cdot u + 2^2 - 4 + 1}} du = \int \frac{1}{u \sqrt{(u+2)^2 - 3}} du \quad (2)$$

$$= \int \frac{1}{\sqrt{3} u \sqrt{\left(\frac{u+2}{\sqrt{3}}\right)^2 - 1}} du \quad \begin{array}{l} \frac{u+2}{\sqrt{3}} = \frac{1}{\cos t} \\ u = \frac{\sqrt{3}}{\cos t} - 2 \\ du = \frac{\sqrt{3} \sin t}{\cos^2 t} dt \end{array} \quad \int \frac{1}{\sqrt{3} \left(\frac{\sqrt{3}}{\cos t} - 2\right) \sqrt{\frac{1}{\cos^2 t} - 1}} \quad (3)$$

$$\cdot \frac{\sqrt{3} \sin t}{\cos^2 t} dt = \int \frac{1}{\frac{\sqrt{3} - 2 \cos t}{\cos t}} \cdot \frac{\sin t}{\cos t} dt \quad (4)$$

$$= \int \frac{1}{\sqrt{3} - 2 \cos t} dt \quad \begin{array}{l} \tan \frac{t}{2} = y \\ dt = \frac{2 dy}{1+y^2} \\ \cos t = \frac{1-y^2}{1+y^2} \end{array} \quad \int \frac{1}{\sqrt{3} - 2 \frac{1-y^2}{1+y^2}} \frac{2}{1+y^2} dy \quad (5)$$

$$= \int \frac{2}{\sqrt{3} + \sqrt{3} y^2 - 2 + 2 y^2} dy = \int \frac{2}{(\sqrt{3} + 2) y^2 - (2 - \sqrt{3})} dy \quad (6)$$

$$= \int \frac{2}{(\sqrt{3} + 2 y)^2 - (\sqrt{2 - \sqrt{3}})^2} dy = \int \frac{2}{(\sqrt{3} + 2 y - \sqrt{2 - \sqrt{3}})(\sqrt{3} + 2 y + \sqrt{2 - \sqrt{3}})} dy \quad (7)$$

$$= \frac{2}{2 \sqrt{2 - \sqrt{3}}} \int \left(\frac{1}{\sqrt{3} + 2 y - \sqrt{2 - \sqrt{3}}} - \frac{1}{\sqrt{3} + 2 y + \sqrt{2 - \sqrt{3}}} \right) dy \quad (8)$$

$$= \frac{1}{\sqrt{2 - \sqrt{3}} \sqrt{2 + \sqrt{3}}} \ln \left| \sqrt{3} + 2 y - \sqrt{2 - \sqrt{3}} \right| - \frac{1}{\sqrt{2 - \sqrt{3}} \sqrt{2 + \sqrt{3}}} \ln \left| \sqrt{3} + 2 y + \sqrt{2 - \sqrt{3}} \right| + C \quad (9)$$

$$= \frac{1}{\sqrt{4-3}} \ln \left| \frac{\sqrt{2+\sqrt{3}} y - \sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}} y + \sqrt{2-\sqrt{3}}} \right| + C \quad (1)$$

$$= \ln \left| \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} \right| + C \quad (2)$$

$$= \ln \left| \frac{y - (2-\sqrt{3})}{y + (2-\sqrt{3})} \right| + C = \quad (3)$$

$$= \ln \left| \frac{\tan \frac{t}{2} - (2-\sqrt{3})}{\tan \frac{t}{2} + (2-\sqrt{3})} \right| + C = \quad (4)$$

$$\begin{aligned} \cos t &= \frac{\sqrt{3}}{2+u} \\ t &= \arccos \frac{\sqrt{3}}{2+u} \end{aligned} \quad \ln \left| \frac{\tan \left(\arccos \frac{\sqrt{3}}{2+u} \right) - 2 + \sqrt{3}}{\tan \left(\arccos \frac{\sqrt{3}}{2+u} \right) + 2 - \sqrt{3}} \right| + C \quad (5)$$

$$= \ln \left| \frac{\tan \left(\arccos \frac{\sqrt{3}}{2+e^x} \right) - 2 + \sqrt{3}}{\tan \left(\arccos \frac{\sqrt{3}}{2+e^x} \right) + 2 - \sqrt{3}} \right| + C = (6)$$