Μάθημα 25

A 6(i) $\int \sin(\log x)\, dx \qquad (\text{Θέσε } t = \log x,\ x = e^t)$ (1)

A 7(ii) $\displaystyle\int \frac{x e^x}{(x+1)^2}\, dx = \int \frac{(x+1)e^x - e^x}{(x+1)^2}\, dx$ (2)

$\displaystyle = \int \left(\frac{e^x}{x+1}\right)' dx = \frac{e^x}{x+1} + c$ (3)

A 8(ii) $\displaystyle\int \frac{\log(\tan x)}{\cos^2 x}\, dx \quad \frac{u = \tan x}{du = \frac{1}{\cos^2 x}\, dx}$ (4)

$\displaystyle = \int \log u\, du = \int u' \log u\, du = u \log u - \int u (\log u)' du$ (5)

$\displaystyle = u \log u - u + c = (\tan x)\log(\tan x) - (\tan x) + c$ (6)

A 9(iv) $\displaystyle\int_0^{\pi/4} x \tan^2 x\, dx = \int_0^{\pi/4} x(\tan^2 x + 1 - 1)\, dx$ (7)

$\displaystyle = \int_0^{\pi/4} \frac{x}{\cos^2 x}\, dx - \int_0^{\pi/4} x\, dx = \int_0^{\pi/4} x (\tan x)'\, dx - \int_0^{\pi/4} x\, dx$ (8)

$\displaystyle = x \tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx - \frac{x^2}{2}\Big|_0^{\pi/4} =$ (9)

$\displaystyle = \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx - \frac{\pi^2}{32} \quad \frac{u = \cos x}{du = -\sin x}$ (10)

$x = 0 \quad u = 1$

$x = \frac{\pi}{4} \quad u = \sqrt{2}/2$

$\displaystyle = \frac{\pi}{4} - \frac{\pi^2}{32} + \int_1^{\sqrt{2}/2} \frac{1}{u}\, du = \frac{\pi}{4} - \frac{\pi^2}{32} + \ln\frac{\sqrt{2}}{2} - \ln 1 =$ (11)

$\displaystyle = \frac{\pi}{4} - \frac{\pi^2}{32} + \ln\frac{\sqrt{2}}{2}$ $\blacksquare$ (12)

$$\text{B 11 (ν)} \quad I = \int \frac{1}{(1+x^2)^2}\,dx \qquad \left(\frac{1}{1+x^2}\right)' = -\frac{1}{(1+x^2)^2}\cdot 2x \qquad (1)$$

$$= -\int \left(\frac{1}{1+x^2}\right)' \frac{1}{2x}\,dx = -\frac{1}{1+x^2}\,\frac{1}{2x} + \int \frac{1}{1+x^2}\left(-\frac{1}{2x^2}\right)dx \qquad (2)$$

$$= \frac{-1}{2x(x^2+1)} - \frac{1}{2}\int \frac{1}{x^2(x^2+1)}\,dx \quad \circledast \qquad (3)$$

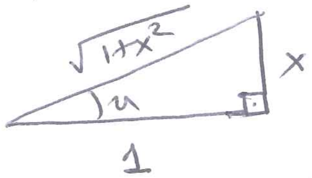
$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{\Gamma x+\Delta}{x^2+1} \Rightarrow \cdots \Rightarrow \qquad (4)$$

$$\Rightarrow \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1} \qquad (5)$$

$$\circledast = -\frac{1}{2x(x^2+1)} - \frac{1}{2}\int\left(\frac{1}{x^2} - \frac{1}{x^2+1}\right)dx \qquad (6)$$

$$= -\frac{1}{2x(x^2+1)} + \frac{1}{2}\,\frac{1}{x} + \frac{1}{2}\arctan(x) + c \qquad (7)$$

<u>Αλλιώς</u> θέτουμε $x = \tan u$



$$dx = \frac{1}{\cos^2 u}\,du \qquad (8)$$

Άρα $$I = \int \frac{1}{(1+\tan^2 u)^2}\,\frac{1}{\cos^2 u}\,du = \int \frac{1}{\left(\frac{1}{\cos^2 u}\right)^2}\,\frac{1}{\cos^2 u}\,du \qquad (9)$$

$$= \int \frac{\cos^4 u}{\cos^2 u}\,du = \int \cos^2 u\,du = \int \frac{1+\cos(2u)}{2}\,du \qquad (10)$$

$$= \frac{1}{2}u + \frac{1}{4}\sin(2u) + c = \frac{1}{2}u + \frac{1}{2}\sin u\cos u + c \qquad (11)$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\,\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}} + c \qquad (12)$$

B 12 $\quad I = \int_0^{\pi} \frac{x \sin x}{(1+\cos x)^2} dx$ (όχι $1+\cos^2 x$, αλλά $(1+\cos x)^2$ αλλιώς δεν υπολογίζεται) (1)

Βρίσκουμε την αντιπαράγωγο (2)

$$\int \frac{x\sin x}{(1+\cos x)^2} dx = \int x \left(\frac{1}{1+\cos x}\right)' dx = \quad (3)$$

$$= \frac{x}{1+\cos x} - \int \frac{1}{1+\cos x} dx = \frac{x}{1+\cos x} - \int \frac{1}{2\cos^2 \frac{x}{2}} dx \quad (4)$$

(διότι $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow 2\cos^2 x - 1 = \cos 2x$ (5)

$\Rightarrow 1 + \cos 2x = 2\cos^2 x$) (6)

$$\underset{dx = 2du}{\overset{\frac{x}{2}=u}{=\!=}} \frac{x}{1+\cos x} - \int \frac{1}{2\cos^2 u} 2du = \quad (7)$$

$$= \frac{x}{1+\cos x} - \tan u + c = \frac{x}{1+\cos x} - \tan\frac{x}{2} + c \quad (8)$$

Άρα $I = \left(\frac{x}{1+\cos x} - \tan\frac{x}{2}\right)\Big|_0^{\pi}$. Στο $x=0$ κάνει 0 (9)

αλλά στο $x=\pi$ έχουμε απροσδιοριστία άρα το (10)

ολοκλήρωμα ~~[illegible]~~ εννοείται ως $I = \lim\limits_{x\to\pi^-} \left(\frac{x}{1+\cos x} - \tan\frac{x}{2}\right)$ (11)

και ονομάζεται **γενικευμένο ολοκλήρωμα**

$$I = \lim_{x\to\pi^-}\left(\frac{x}{2\cos^2\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \lim_{x\to\pi^-} \frac{x - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \quad (12)$$

$$= \lim_{x\to\pi^-} \frac{x - \sin x}{2\cos^2\frac{x}{2}} = +\infty. \quad (13)$$

B16(i) $I=\int_0^{\infty} x e^{-x^2}dx = \lim_{t\to+\infty}\int_0^t x e^{-x^2}dx$ (14)

• $\int_0^t x e^{-x^2}dx \xrightarrow[2xdx=du]{x^2=u} \int_0^{t^2}\frac{1}{2}e^{-u}\,du =$ (15)

$x=0 \quad u=0$

$x=t \quad u=t^2$

$= -\frac{1}{2}e^{-u}\Big|_0^{t^2} = -\frac{1}{2}e^{-t^2}+\frac{1}{2}$ (16)

Άρα $I=\lim_{t\to\infty}\frac{1}{2}\left(1-e^{-t^2}\right)=\frac{1}{2}$ (17)

B 18(i) Βρείτε το $\lim_{x\to+\infty} x^3 e^{-x^6}\int_0^{x^3} e^{t^2}dt$ (18)

$x^3 e^{-x^6}\xrightarrow[x\to\infty]{} 0$ (άσκηση με L'Hospital) (19)

$\int_0^{x^3} e^{t^2}dt \geq \int_0^{x^3} 1\,dt = x^3 \xrightarrow[x\to+\infty]{} +\infty$ (20)

Άρα $\lim_{x\to\infty}\frac{\int_0^{x^3} e^{t^2}dt}{x^{-3}e^{x^6}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{e^{x^6}}{-3x^{-4}e^{x^6}+x^{-3}6x^5e^{x^6}}$ (21)

$= \lim_{x\to\infty}\frac{1}{-3x^{-4}+6x^2} = \lim_{x\to+\infty}\frac{x^4}{6x^6-3}=0$ (22)

(ΟΕΣ 5)

Ντούμας ΟΕΣ 74, 1.34(δ)

$$\int x \arcsin(x^2)\,dx = \int \left(\frac{x^2}{2}\right)' \arcsin(x^2)\,dx = \quad (1)$$

$$= \frac{x^2}{2}\arcsin(x^2) - \int \frac{x^2}{2}\left(\arcsin x^2\right)' dx = \quad (2)$$

$$= \frac{1}{2}x^2\arcsin(x^2) - \frac{1}{2}\int x^2 \frac{1}{\sqrt{1-x^4}}\, 2x\, dx = \quad (3)$$

$$= \frac{1}{2}x\arcsin(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\,dx \qquad \frac{u = 1-x^4}{du = -4x^3\,dx} \quad (4)$$

$$x^3\,dx = -\frac{1}{4}\,du$$

$$= \frac{1}{2}x\arcsin(x^2) - \int \frac{1}{\sqrt{u}}\left(-\frac{1}{4}\right)du = \quad (5)$$

$$= \frac{1}{2}x\arcsin(x^2) + \frac{1}{2}\int \frac{1}{2\sqrt{u}}\,du = \quad (6)$$

$$= \frac{1}{2}x\arcsin(x^2) + \frac{1}{2}\sqrt{u} + c = \quad (7)$$

$$= \frac{1}{2}x\arcsin(x^2) + \frac{1}{2}\sqrt{1-x^4} + c, \quad (8)$$

Ντούγιας σελ 75, 1.36 (ε)

$$\int \frac{dx}{\sqrt{e^{2x}+4e^x+1}} \xrightarrow[\substack{du=e^x dx \\ \Rightarrow \frac{du}{u}=dx}]{u=e^x} \int \frac{1}{\sqrt{u^2+4u+1}} \frac{du}{u}$$

$$= \int \frac{1}{u\sqrt{u^2+2\cdot 2\cdot u+2^2-4+1}} du = \int \frac{1}{u\sqrt{(u+2)^2-3}} du$$

$$= \int \frac{1}{\sqrt{3}\, u \sqrt{\left(\frac{u+2}{\sqrt{3}}\right)^2-1}} du \xrightarrow[\substack{u=\frac{\sqrt{3}}{\cos t}-2 \\ du=\frac{\sqrt{3}\sin t}{\cos^2 t}dt}]{\frac{u+2}{\sqrt{3}}=\frac{1}{\cos t}} \int \frac{1}{\cancel{\sqrt{3}}\left(\frac{\sqrt{3}}{\cos t}-2\right)\sqrt{\frac{1}{\cos^2 t}-1}} \cdot$$

$$\cdot \frac{\cancel{\sqrt{3}}\sin t}{\cos^2 t} dt = \int \frac{1}{\frac{\sqrt{3}-2\cos t}{\cancel{\cos t}} \cdot \frac{\sin t}{\cancel{\cos t}}} \frac{\cancel{\sin t}}{\cancel{\cos^2 t}} dt$$

$$= \int \frac{1}{\sqrt{3}-2\cos t} dt \xrightarrow[\substack{dt=\frac{2dy}{1+y^2} \\ \cos t=\frac{1-y^2}{1+y^2}}]{\tan\frac{t}{2}=y} \int \frac{1}{\sqrt{3}-2\frac{1-y^2}{1+y^2}} \frac{2}{1+y^2} dy$$

$$= \int \frac{2}{\sqrt{3}+\sqrt{3}y^2-2+2y^2} dy = \int \frac{2}{(\sqrt{3}+2)y^2-(2-\sqrt{3})} dy$$

$$= \int \frac{2}{\left(\sqrt{\sqrt{3}+2}\, y\right)^2-\left(\sqrt{2-\sqrt{3}}\right)^2} dy = \int \frac{2\, dy}{\left(\sqrt{\sqrt{3}+2}\, y-\sqrt{2-\sqrt{3}}\right)\left(\sqrt{\sqrt{3}+2}\, y+\sqrt{2-\sqrt{3}}\right)}$$

$$= \frac{2}{2\sqrt{2-\sqrt{3}}} \int \left(\frac{1}{\sqrt{\sqrt{3}+2}\, y-\sqrt{2-\sqrt{3}}} - \frac{1}{\sqrt{\sqrt{3}+2}\, y+\sqrt{2-\sqrt{3}}}\right) dy$$

$$= \frac{1}{\sqrt{2-\sqrt{3}}\sqrt{2+\sqrt{3}}} \ln\left|\sqrt{\sqrt{3}+2}\, y-\sqrt{2-\sqrt{3}}\right| - \frac{1}{\sqrt{2-\sqrt{3}}\sqrt{2+\sqrt{3}}} \ln\left|\sqrt{\sqrt{3}+2}\, y+\sqrt{2-\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{4-3}} \ln\left| \frac{\sqrt{2+\sqrt{3}}\, y - \sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}\, y + \sqrt{2-\sqrt{3}}} \right| + c$$

$$= \ln\left| \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2-\sqrt{3}}} \right| + c$$

$$= \ln\left| \frac{y - (2-\sqrt{3})}{y + (2-\sqrt{3})} \right| + c =$$

$$= \ln\left| \frac{\tan\frac{t}{2} - (2-\sqrt{3})}{\tan\frac{t}{2} + (2-\sqrt{3})} \right| + c =$$

$$\cos t = \frac{\sqrt{3}}{2+u}$$

$$t = \arccos\frac{\sqrt{3}}{2+u}$$

$$\ln\left| \frac{\tan\left(\arccos\frac{\sqrt{3}}{2+u}\right) - 2 + \sqrt{3}}{\tan\left(\arccos\frac{\sqrt{3}}{2+u}\right) + 2 - \sqrt{3}} \right| + c$$

$$= \ln\left| \frac{\tan\left(\arccos\frac{\sqrt{3}}{2+e^x}\right) - 2 + \sqrt{3}}{\tan\left(\arccos\frac{\sqrt{3}}{2+e^x}\right) + 2 - \sqrt{3}} \right| + c$$