Μάθημα 26

Ντούγιας σελ 75   1.35(γ)

$$\int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx \;\underset{\substack{=1\cdot\sin x + x\cos x - \sin x\\ = x\cos x}}{\overset{(x\sin x+\cos x)'=}{=\!=}}\; \int \frac{(x\sin x+\cos x)'}{(x\sin x+\cos x)^2}\,dx$$

$$= -\frac{1}{x\sin x+\cos x} + c$$

1.37 (δ)

$$\int \frac{1}{1+\sin x+\cos x}\,dx \;\overset{u=\tan\frac{x}{2}}{=\!=}\; \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du$$

$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$dx = \frac{2}{1+u^2}\,du$$

$$= \int \frac{2}{1+u^2+2u+1-u^2}\,du = \int \frac{1}{1+u}\,du = \ln|1+u| + c =$$

$$= \ln\left|1+\tan\frac{x}{2}\right| + c$$

1.37(η)

$$\int \frac{dx}{x\sqrt{(x-a)(b-x)}}$$

Παρατηρούμε ότι $(x-a)(b-x) = xb - x^2 - ab + ax =$

$$= -\left(x^2 - (a+b)x + ab\right) = -\left(x^2 - 2\,\frac{a+b}{2}\,x + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab\right)$$

$$= -\left(\left(x - \frac{a+b}{2}\right)^2 - \frac{(a+b)^2 - 4ab}{4}\right) =$$

$$= -\left(\left(x-\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right) = \left(\frac{a-b}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2$$

Άρα $I = \int \frac{dx}{x\sqrt{(x-a)(b-x)}} = \int \frac{dx}{x\sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}}$

Θέτουμε $x - \frac{a+b}{2} = \frac{a-b}{2}\sin t$

$$\Rightarrow x = \frac{a-b}{2}\sin t + \frac{a+b}{2}$$

$$dx = \frac{a-b}{2}\cos t \, dt$$

$$I = \int \frac{\frac{a-b}{2}\cos t \, dt}{\left(\frac{a-b}{2}\sin t + \frac{a+b}{2}\right)\sqrt{\left(\frac{a-b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \sin^2 t}} \, dt$$

$$= \int \frac{1}{\frac{a-b}{2}\sin t + \frac{a+b}{2}} \, dt$$

$u = \tan(t/2)$

$\sin t = \frac{2u}{1+u^2}$

$dt = \frac{2}{1+u^2}\,du$

$$= \int \frac{1}{\frac{a-b}{2}\frac{2u}{1+u^2} + \frac{a+b}{2}} \frac{2}{1+u^2}\,du =$$

$$= \int \frac{1}{(a-b)2u + (a+b)(1+u)^2}\,du = \int \frac{1}{(a+b)u^2 + 4au + (a+b)}\,du$$

εκτός αν $a=-b$

$$= \frac{1}{a+b}\int \frac{1}{u^2 + 4\frac{a}{a+b}u + 1}\,du$$

διακρίνουμε περιπτώσεις αν έχει ή όχι ρίζες. Π.χ.

αν $\Delta < 0$ τότε $I = \frac{1}{a+b}\int \frac{1}{\left(u + \frac{2a}{a+b}\right)^2 + \left(1 - \frac{4a^2}{(a+b)^2}\right)}\,du$

και θέτοντας $y = \dfrac{u + \frac{2a}{a+b}}{\sqrt{1 - \frac{4a^2}{(a+b)^2}}}$ η το I οδηγείται

σε $\displaystyle\int \frac{1}{y^2+1}\,dy = \arctan y + c$ ---- (αφήνεται ως άσκηση)

1.38(J)

$$\int \frac{1+x^2}{1+x^4}\,dx = \int \frac{1+x^2}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}\,dx$$

$$\left(1+x^4 = (x^2+\alpha x+\beta)(x^2+\gamma x+\delta) \Longrightarrow \alpha=\sqrt{2}\quad \gamma=-\sqrt{2}\quad \beta=\delta=1\right)$$

$$= \int \frac{Ax+B}{x^2+\sqrt{2}x+1}\,dx + \int \frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1}\,dx \qquad \underline{\underline{\begin{matrix}A=\Gamma=0\\ B=\Delta=1/2\end{matrix}}}$$

$$= \int \frac{1/2}{x^2+\sqrt{2}x+1}\,dx + \int \frac{1/2}{x^2-\sqrt{2}x+1}\,dx =$$

$$= \frac{1}{2}\int \frac{1}{x^2+2\frac{\sqrt{2}}{2}x+\left(\frac{\sqrt{2}}{2}\right)^2-\left(\frac{\sqrt{2}}{2}\right)^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2-2\frac{\sqrt{2}}{2}x+\left(\frac{\sqrt{2}}{2}\right)^2-\left(\frac{\sqrt{2}}{2}\right)^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx + \frac{1}{2}\int \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\frac{1}{2}\left((\sqrt{2}x+1)^2+1\right)}\,dx + \frac{1}{2}\int \frac{1}{\frac{1}{2}\left((\sqrt{2}x-1)^2+1\right)}\,dx$$

$$= \frac{\arctan(\sqrt{2}x+1)}{\sqrt{2}} + \frac{\arctan(\sqrt{2}x-1)}{\sqrt{2}} + c$$

Εφαρμογή $\displaystyle\int \sqrt{\tan x}\, dx \xrightarrow[\arctan(u^2)=x]{u=\sqrt{\tan x}} \int \frac{2u^2}{1+u^4}\, du$

$$dx = \frac{2u}{1+u^4}\, du$$

Το $(1+u^4)$ παραγοντοποιείται ως εξής: γράφουμε

$1+u^4 = (x^2+\alpha x+\beta)(x^2+\gamma x+\delta)$ και προκύπτει

$\alpha = \sqrt{2} \quad \beta = 1 = \delta \quad \gamma = -\sqrt{2}$

Άρα $\displaystyle\int \frac{2u^2}{1+u^4}\, du = \int \frac{2u^2}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)}\, du$

$\displaystyle= \int \left(\frac{Au+B}{u^2+\sqrt{2}u+1} + \frac{\Gamma u+\Delta}{u^2-\sqrt{2}u+1} \right) du$ και βρίσκουμε

$A = -\frac{\sqrt{2}}{2} \qquad \Gamma = \frac{\sqrt{2}}{2} \qquad B = \Delta = 0$

$$\int \frac{-\frac{\sqrt{2}}{2}u}{u^2+\sqrt{2}u+1}\, du + \int \frac{\frac{\sqrt{2}}{2}u}{u^2-\sqrt{2}u+1}\, du$$

$$= -\frac{\sqrt{2}}{2}\,\frac{1}{2} \int \frac{2u}{u^2+\sqrt{2}u+1}\, du + \frac{\sqrt{2}}{2}\,\frac{1}{2} \int \frac{2u}{u^2-\sqrt{2}u+1}\, du$$

$$= -\frac{\sqrt{2}}{4} \int \frac{2u+\sqrt{2}-\sqrt{2}}{u^2+\sqrt{2}u+1}\, du + \frac{\sqrt{2}}{4} \int \frac{2u-\sqrt{2}+\sqrt{2}}{u^2-\sqrt{2}u+1}\, du$$

$$= -\frac{\sqrt{2}}{4} \ln\left| u^2+\sqrt{2}u+1 \right| + \frac{\sqrt{2}}{4} \ln\left| u^2-\sqrt{2}u+1 \right|$$

$$-\frac{\sqrt{2}}{4} \int \frac{-\sqrt{2}}{u^2+\sqrt{2}u+1}\, du + \frac{\sqrt{2}}{4} \int \frac{\sqrt{2}}{u^2-\sqrt{2}u+1}\, du$$

Νεγρεπόντης σελ 376
24–4 (ii)

$$I = \int \frac{x^4}{1+x^2} \arctan x \, dx$$

Θέλουμε να απαλλαγούμε από την $\arctan x$. Αυτό γίνεται με παραγοντική ολοκλήρωση. Άρα χρειαζόμαστε αντιπαράγωγο του $\frac{x^4}{1+x^2}$. Την υπολογίζουμε ξεχωριστά: $\int \frac{x^4}{1+x^2}\,dx \xlongequal[\text{διαίρεση ή}]{\text{κάνουμε}} \int \frac{x^4-1+1}{1+x^2}\,dx$

$$= \int \frac{(x^2-1)(x^2+1)+1}{1+x^2}\,dx = \int \left((x^2-1) + \frac{1}{x^2+1}\right)dx =$$

$$= \frac{x^3}{3} - x + \arctan x + c. \quad \text{Άρα}$$

$$I = \int \left(\frac{x^3}{3} - x + \arctan x\right)' \arctan x \, dx =$$

$$= \left(\frac{x^3}{3} - x + \arctan x\right)\arctan x - \int \left(\frac{x^3}{3} - x + \arctan x\right)\frac{1}{1+x^2}\,dx$$

Μένει να υπολογιστούν τα $\int \frac{x^3}{1+x^2}\,dx \xlongequal{\text{διαίρεση}} \dots$

$$\int \frac{x}{1+x^2}\,dx \xlongequal{u=1+x^2} \dots$$

και

$$J = \int \frac{\arctan x}{1+x^2}\,dx = \int (\arctan x)' \arctan x \, dx$$

$$= \arctan^2 x - \int \arctan x \frac{1}{1+x^2}\,dx$$

$$= \arctan^2 x - J \Rightarrow J = \frac{1}{2}\arctan^2 x$$

Άλλως(;) $I = \int (x^4 \arctan x)(\arctan x)'\,dx =$

$$= x^4 \arctan^2 x - \int \left(4x^3 \arctan x + \frac{x^4}{1+x^2}\right)\arctan x \, dx =$$

$$= x^4 \arctan^2 x - 4\int x^3 \arctan^2 x - I = ??$$

Πρέπει να υπολογιστεί το $\int 4x^3 \arctan^2 x \, dx$