Μάθημα 32

$$I = \int \frac{1}{\cos^5 x}\, dx = \int \frac{1}{\cos^3 x} \frac{1}{\cos^2 x}\, dx = \int \frac{1}{\cos^3 x} (\tan x)'\, dx \quad (1)$$

$$= \frac{\tan x}{\cos^3 x} - \int \tan x \, (-3)\cos^{-4} x \cdot (-\sin x)\, dx \quad (2)$$

$$= \frac{\sin x}{\cos^4 x} - 3 \int \frac{\sin^2 x}{\cos^5 x}\, dx = \frac{\sin x}{\cos^4 x} - 3 \int \frac{1-\cos^2 x}{\cos^5 x}\, dx \quad (3)$$

$$(4)$$

$$= \frac{\sin x}{\cos^4 x} - 3 \cdot I + 3 \int \frac{1}{\cos^3 x}\, dx$$

$$(5)$$

$$\Rightarrow \boxed{I = \frac{\sin x}{4\cos^4 x} + \frac{3}{4} \int \frac{1}{\cos^3 x}\, dx}$$

$$(6)$$

$$J = \int \frac{1}{\cos^3 x}\, dx = \int \frac{1}{\cos x} (\tan x)'\, dx = \frac{\tan x}{\cos x} - \int (\tan x)(-1)\cos^{-2} x \, (-\sin x)\, dx$$

$$= \frac{\sin x}{\cos^2 x} - \int \tan x \frac{1}{\cos^2 x} \sin x \, dx = \quad (7)$$

$$= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x}\, dx = \frac{\sin x}{\cos^3 x} - J + \int \frac{1}{\cos x}\, dx \quad (8)$$

$$(9)$$

$$\Rightarrow J = \frac{\sin x}{2\cos^3 x} + \frac{1}{2} \underbrace{\int \frac{1}{\cos x}\, dx}_{\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|} \quad (10)$$

Άρα $$I = \frac{\sin x}{4\cos^4 x} + \frac{3}{4}\frac{\sin x}{2\cos^2 x} + \frac{3}{4}\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c \qquad (1)$$

$$\longleftrightarrow$$

$$I = \int \frac{1}{\cos^6 x}\,dx = \int \frac{1}{\cos^4 x}(\tan x)'\,dx = \qquad (2)$$

$$= \frac{\tan x}{\cos^4 x} - \int \tan x\,(-4)\cos^{-5} x\,(-\sin x)\,dx \qquad (3)$$

$$= \frac{\tan x}{\cos^4 x} - 4\int \frac{\sin^2 x}{\cos^6 x}\,dx = \frac{\sin x}{\cos^5 x} - 4\int \frac{1-\cos^2 x}{\cos^6 x}\,dx \qquad (4)$$

$$= \frac{\sin x}{\cos^5 x} - 4I + 4\int \frac{1}{\cos^4 x}\,dx \qquad (5)$$

$$\Rightarrow I = \frac{\sin x}{5\cos^5 x} + \frac{4}{5}\int \frac{1}{\cos^4 x}\,dx \qquad (6)$$

Γενικά $$I_n = \int \frac{1}{\cos^n x}\,dx = \int \frac{1}{\cos^{n-2} x}(\tan x)'\,dx \qquad (7)$$

$$= \frac{\tan x}{\cos^{n-2} x} - \int (\tan x)(-(n-2))\cos^{-n+1} x\,(-\sin x)\,dx \qquad (8)$$

$$= \frac{\sin x}{\cos^{n-1} x} - (n-2)\int \frac{\sin^2 x}{\cos^n x}\,dx = \qquad (9)$$

$$= \frac{\sin x}{\cos^{n-1} x} - (n-2)\int \frac{1-\cos^2 x}{\cos^n x}\,dx \qquad (10)$$

$$= \frac{\sin x}{\cos^{n-1} x} - (n-2)I_n + (n-2)\int \frac{1}{\cos^{n-2} x}\,dx \qquad (11)$$

Άρα $I_n + (n-2) I_n = \dfrac{\sin x}{\cos^{n-1} x} + (n-2) I_{n-2}$ (1)

$(n-1) I_n =$ ——//—— (2)

$$\boxed{I_n = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} I_{n-2}}$$ (3)

Άρα $I_6 = \displaystyle\int \frac{1}{\cos^6 x}\,dx \overset{n=6}{=} \frac{\sin x}{5\cos^5 x} + \frac{4}{5} I_4$ (4)

$\overset{n=4}{=} \dfrac{\sin x}{5\cos^5 x} + \dfrac{4}{5}\left(\dfrac{\sin x}{3\cos^3 x} + \dfrac{2}{3} I_2\right)$ (5)

$$\Rightarrow \boxed{I_6 = \int \frac{1}{\cos^6 x}\,dx = \frac{\sin x}{5\cos^5 x} + \frac{4}{5}\,\frac{\sin x}{3\cos^3 x} + \frac{8}{15}\tan x + c}$$ (6)

Δουλέψτε για να βρείτε ένα αναδρομικό τύπο για το (7)

$I_n = \displaystyle\int \frac{1}{\sin^n x}\,dx$ (8)

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ υπερβολικό συνημίτονο (9)

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ υπερβολικό ημίτονο (10)

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ υπερβολική εφαπτομένη (11)

$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ ——//—— συνεφαπτομένη (12) (σελ 4

$$(\cosh x)' = \sinh x \qquad (\sinh x)' = \cosh x \qquad (\tanh x)' = \frac{1}{\cosh^2 x} \tag{1}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x + \sinh^2 x = \cosh(2x) \tag{2}$$

$$\cos^2 x + \sin^2 x = 1 \qquad \cos^2 x - \sin^2 x = \cos(2x)$$

Αποδείξτε τους αναδρομικούς τύπους για το (3)

$$I_{m,n} = \int \sinh^m x \cosh^n x \, dx \quad \text{για } m+n \neq 0 \quad m, n \in \mathbb{Z} \tag{4}$$

(i) $$I_{m,n} = \frac{1}{m+n} \sinh^{m-1} x \cosh^{n+1} x - \frac{m-1}{m+n} I_{m-2,n} \tag{5}$$

(ii) $$I_{m,n} = \frac{1}{m+n} \sinh^{m+1} \cosh^{n-1} x + \frac{n-1}{m+n} I_{m,n-2} \tag{6}$$

Μετά υπολογίστε το $\int \sinh^3 x \cosh^4 x \, dx$ (7)

<u>Λύση</u> (i) $$I_{m,n} = \int \sinh^m x \cosh^n x \, dx = \int \sinh^{m-1} x \left(\frac{\cosh^{n+1} x}{n+1}\right)' dx \tag{8}$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \int (m-1) \sinh^{m-2} x \cosh x \frac{\cosh^{n+1} x}{n+1} dx \tag{9}$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^{n+2} x \, dx \tag{10}$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^{n} x \left(1 + \sinh^2 x\right) dx \tag{11}$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \underbrace{\int \sinh^{m-2} \cosh^n x \, dx}_{I_{m-2,n}} - \frac{m-1}{n+1} \underbrace{\int \sinh^m x \cosh^n x \, dx}_{I_{m,n}} \tag{12}$$

$$\Rightarrow \underbrace{\left(1 + \frac{m-1}{n+1}\right)}_{= \frac{n+m}{n+1}} I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} I_{m-2,n} \tag{13}$$

$$\boxed{I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+m} - \frac{m-1}{n+m} I_{m-2,n}} \tag{14}$$

(ii) ομοίως (να το κάνετε) (1)

Άρα $I_{3,4} \xlongequal[\substack{m=3\\n=4}]{(ii)} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{7} I_{3,2}$ (2)

$$\xlongequal[\substack{m=3\\n=2}]{(ii)} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{7}\left(\frac{1}{5}\sinh^4 x\cosh x + \frac{1}{5} I_{3,0}\right) \quad (3)$$

(4)

$$\xlongequal[\substack{m=3\\n=0}]{(i)} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{35}\sinh^4 x\cosh x + \frac{3}{35}\left(\frac{1}{3}\sinh^3 x\cosh x - \frac{2}{3} I_{1,0}\right)$$

Αλλά $I_{1,0} = \int \sinh x\, dx = \cosh x$ (5)

Άρα $\boxed{I_{3,4} = \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{35}\sinh^4 x\cosh x + \frac{3}{105}\sinh^3 x\cosh x - \frac{6}{105}\cosh x + c}$

(6)