Μάθημα 32

$$I = \int \frac{1}{\cos^5 x}\, dx = \int \frac{1}{\cos^3 x}\,\frac{1}{\cos^2 x}\, dx = \int \frac{1}{\cos^3 x}\,(\qquad)'\, dx \qquad (1)$$

$$= \frac{\qquad}{\cos^3 x} - \int \underline{\qquad}\,(-3)\cos^{-4} x \cdot (\underline{\qquad})\, dx \qquad (2)$$

$$= \frac{\sin x}{\cos^4 x} - 3\int \frac{\sin^2 x}{\cos^5 x}\, dx = \frac{\sin x}{\cos^4 x} - 3\int \frac{1-\cos^2 x}{\cos^5 x}\, dx \qquad (3)$$

$$= \frac{\sin x}{\cos^4 x} - 3\cdot \underline{\qquad} + 3\int \frac{1}{\cos^3 x}\, dx \qquad (4)$$

$$\Rightarrow \boxed{I = \frac{\sin x}{4\cos^4 x} + \frac{3}{4}\int \frac{1}{\cos^3 x}\, dx} \qquad (5)$$

$$J = \int \frac{1}{\cos^3 x}\, dx = \int \frac{1}{\cos x}\,(\qquad)'\, dx = \frac{\qquad}{\cos x} - \int (\qquad)(-1)\cos^{-2} x\,(-\sin x)\, dx \qquad (6)$$

$$= \frac{\sin x}{\cos^2 x} - \int \tan x\,\frac{1}{\cos^2 x}\,\sin x\, dx = \qquad (7)$$

$$= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x}\, dx = \frac{\sin x}{\cos^2 x} - J + \int \frac{1}{\cos x}\, dx \qquad (8)$$

$$\Rightarrow J = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\underbrace{\int \frac{1}{\cos x}\, dx}_{\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|} \qquad (9)$$

$$(10)$$

Άρα $I = \frac{\sin x}{4\cos^4 x} + \frac{3}{4}\frac{\sin x}{2\cos^2 x} + \frac{3}{4}\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$ (1)

$\longleftrightarrow$

$I = \int \frac{1}{\cos^6 x}\,dx = \int \frac{1}{\cos^4 x}(\quad)'\,dx =$ (2)

$= \frac{\tan x}{\cos^4 x} - \int \tan x \,(-4)\cos^{-5} x\,(-\sin x)\,dx$ (3)

$= \frac{\tan x}{\cos^4 x} - 4\int \frac{\sin^2 x}{\cos^6 x}\,dx = \frac{\sin x}{\cos^5 x} - 4\int \frac{1-\cos^2 x}{\cos^6 x}\,dx$ (4)

$= \frac{\sin x}{\cos^5 x} - 4I + 4\int \frac{1}{\cos^4 x}\,dx$ (5)

$\Rightarrow I = \frac{\sin x}{5\cos^5 x} + \frac{4}{5}\int \frac{1}{\cos^4 x}\,dx$ (6)

Γενικά $I_n = \int \frac{1}{\cos^n x}\,dx = \int \frac{1}{\cos^{n-2} x}(\tan x)'\,dx$ (7)

$= \frac{\tan x}{\cos^{n-2} x} - \int (\tan x)(-(n-2))\cos^{-n+1} x\,(-\sin x)\,dx$ (8)

$= \frac{\sin x}{\cos^{n-1} x} - (n-2)\int \frac{\sin^2 x}{\cos^n x}\,dx =$ (9)

$= \frac{\sin x}{\cos^{n-1} x} - (n-2)\int \frac{1-\cos^2 x}{\cos^n x}\,dx$ (10)

$= \frac{\sin x}{\cos^{n-1} x} - (n-2)I_n + (n-2)\int \frac{1}{\cos^{n-2} x}\,dx$ (11)

Άρα $I_n + (n-2) I_n = \frac{\sin x}{\cos^{n-1} x} + (n-2) I_{n-2}$ (1)

(2)

$(n-1) I_n =$

$$\boxed{I_n = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} I_{n-2}}$$ (3)

Άρα $I_6 = \int \frac{1}{\cos^6 x} dx \overset{n=6}{=} \frac{\sin x}{5\cos^5 x} + \frac{4}{5} I_4$ (4)

$\overset{n=4}{=} \frac{\sin x}{5\cos^5 x} + \frac{4}{5}\left(\frac{\sin x}{3\cos^3 x} + \frac{2}{3} I_2\right)$ (5)

(6)

$$\Rightarrow \boxed{I_6 = \int \frac{1}{\cos^6 x} dx = \frac{\sin x}{5\cos^5 x} + \frac{4}{5}\frac{\sin x}{3\cos^3 x} + \frac{8}{15}\tan x + c}$$

Δουλέψτε για να βρείτε ένα αναδρομικό τύπο για το (7)

$I_n = \int \frac{1}{\sin^n x} dx$ (8)

$\cosh x = \frac{e^x + e^{-x}}{2}$ υπερβολικό συνημίτονο (9)

$\sinh x = \frac{e^x - e^{-x}}{2}$ υπερβολικό ημίτονο (10)

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ υπερβολική εφαπτομένη (11)

$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ — // — συνεφαπτομένη (12)

$(\cosh x)' =$ $\qquad (\sinh x)' =$ $\qquad (\tanh x)' =$

$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x + \sinh^2 x = \cosh(2x)$

Αποδείξτε τους αναδρομικούς τύπους για το

$$I_{m,n} = \int \sinh^m x \cosh^n x \, dx \quad \text{για } m+n \neq 0 \quad m, n \in \mathbb{Z}$$

(i) $I_{m,n} = \dfrac{1}{m+n} \sinh^{m-1} x \cosh^{n+1} x - \dfrac{m-1}{m+n} I_{m-2,n}$

(ii) $I_{m,n} = \dfrac{1}{m+n} \sinh^{m+1} \cosh^{n-1} x + \dfrac{n-1}{m+n} I_{m,n-2}$

Μετά υπολογίστε το $\displaystyle\int \sinh^3 x \cosh^4 x \, dx$

<u>Λύση</u> (i) $I_{m,n} = \displaystyle\int \sinh^m x \cosh^n x \, dx = \int \sinh^{m-1} x \left(\frac{\cosh^{n+1} x}{n+1}\right)' dx$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \int (m-1) \sinh^{m-2} x \cosh x \frac{\cosh^{n+1} x}{n+1} dx$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^{n+2} x \, dx$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^{n} x \left(1 + \sinh^2 x\right) dx$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \underbrace{\int \sinh^{m-2} \cosh^n x \, dx}_{I_{m-2,n}} - \frac{m-1}{n+1} \underbrace{\int \sinh^m x \cosh^n x \, dx}_{I_{m,n}}$$

$$\Rightarrow \underbrace{\left(1 + \frac{m-1}{n+1}\right)}_{\frac{n+m}{n+1}} I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} I_{m-2,n}$$

$$\boxed{I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+m} - \frac{m-1}{n+m} I_{m-2,n}}$$

(ii) όμοιως (να το κάνετε) (1)

Άρα $I_{3,4} \underset{\substack{m=3\\n=4}}{\overset{(ii)}{=}} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{7} I_{3,2}$ (2)

$$\underset{\substack{m=3\\n=2}}{\overset{(ii)}{=}} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{7}\left(\frac{1}{5}\sinh^4 x\cosh x + \frac{1}{5} I_{3,0}\right) \quad (3)$$

$$\underset{\substack{m=3\\n=0}}{\overset{(i)}{=}} \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{35}\sinh^4 x\cosh x + \frac{3}{35}\left(\frac{1}{3}\sinh^3 x\cosh x - \frac{2}{3} I_{1,0}\right) \quad (4)$$

Αλλά $I_{1,0} = \int \sinh x\, dx = \cosh x$ (5)

Άρα
$$\boxed{I_{3,4} = \frac{1}{7}\sinh^4 x\cosh^3 x + \frac{3}{35}\sinh^4 x\cosh x + \frac{3}{105}\sinh^3 x\cosh x - \frac{6}{105}\cosh x + c}$$
(6)