MARGULIS LEMMA, OLD AND NEW

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The Bieberbach theorem says that any discrete group of isometries of the Euclidean space contains a finite index subgroup whose elements are translations. In particular, discrete groups of euclidean isometries are virtually abelian.

In the late 70's, G. Margulis gave a non-Euclidean version of this theorem, now called the "Margulis lemma" : there exists a constant $\mu(n)$ such that for any n-dimensional symmetric Riemannian space of non-positive curvature X, any $x \in X$, any discrete group of isometries of X generated by elements g such that $d(x, gx) \leq \mu(n)$ is virtually nilpotent. This statement as well as the Bieberbach theorem rely on the algebraic nature of the isometry group under consideration but M. Gromov extended the Margulis lemma in a purely Riemanniann setting : let X be a simply connected n-dimensional manifold of sectional curvature $-1 \leq K \leq 0$, then any discrete group of isometries generated by elements g such that $d(x, gx) \leq \mu(n)$ is virtually nilpotent.

In the 80's, M. Gromov generalized the Margulis lemma stating that the fundamental group of an "almost flat manifold" is virtually nilpotent : there exists a constant $\epsilon(n)$ such that the fundamental group of any n-dimensional compact riemannian manifold such that the sectional curvature K and the diameter diam satisfy $|K|.\operatorname{diam}^2 \leq \varepsilon(n)$ is virtually nilpotent. The sectional curvature assumption here replaces the homogeneity of the space in the classical Margulis lemma and seems to be crucial. However M. Gromov conjectured that the Margulis lemma should hold under a much weaker curvature assumption namely a lower bound on Ricci curvature. This conjecture has been recently settled by V. Kapovitch and B. Wilking. Their proof relies on the theory of Cheeger and Colding which describes the structure of Gromov-Hausdorff limits of sequences of riemannian manifolds with Ricci curvature bounded below.

The goal of this lecture is to survey some of these results and the involved tools.