# Lecture course: Metric projective geometry

Plan:

- Definition and basic properties of projective structure. Application: Isometries of Hilbert metrics
- Projective invariant equations. Application: Topology in the 2-dimensional case
- Local normal forms in dimension 2. Application: Solution of the problems stated by Sophus Lie in 1882.
- Projectively invariant tenrors. Application: Proof of projective Lichnerowicz conjecture
- Conifications. Application: solution of Weyl-Ehlers problem
- Open problems, generalizations, and possible analogies in the Finsler geometry

# What is a projective structure?

**Informal and inefficient definition:** *Projective structure* is a $\underbrace{\text{sufficiently big}}_{\text{to be explained}}$ family of curves that after a reparameterisation can be geodesics of some affine connection.

**Sufficiently big:** In any point in any direction there exists a curve from the family passing through this point in this direction.

**Simplest example:** the set of all straight lines on $\mathbb{R}^2$: it is sufficiently big, and they are geodesics of the flat (and not only of the flat) connection.

# Efficient definition of projective structure requires theory

**Let us first study the following question:** Suppose we have two symmetric affine connections, $\nabla = (\Gamma^i_{jk})$ and $\bar\nabla = (\Gamma^i_{jk})$. When each geodesic of $\nabla$, possibly after a reparameterization, is a geodesic of $\bar\nabla$?

**Def.** Two connections are $\nabla$ and $\bar\nabla$ said to be projectively equivalent, if any geodesic of $\nabla$, possibly after a reparameterization, is a geodesic of $\bar\nabla$.

**The above question using the new terminology:** Reformulate projective equivalence of $\nabla$, $\bar\nabla$ as an easy-to-check condition

**Theorem 1 (deep classics: Levi-Civita 1896, Weyl 1924).** $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k\delta^i_j + \phi_j\delta^i_k. \qquad (*)$$

**The condition $(*)$ in the index-free form:** for any vector $X$ and any vectorfield $V$,

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**)$$

# Easy control question to the audience:

**Theorem 1 (deep classics: Levi-Civita 1896, Weyl 1924).** $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k\delta^i_j + \phi_j\delta^i_k. \qquad (*)$$

**Question: please answer now:** Does there exist two projectively equivalent symmetric affine connections on $\mathbb{R}^2$ (or $\mathbb{R}^n$) such that they coincide in some neighborhood and are different in another neighborhood?

Of course YES!!! Take a 1-form which is not zero in a neighborhood but is zero outside the neighborhood, take any connection $\nabla = (\Gamma^i_{jk})$ and "deform" it by $(*)$.

# I will prove Theorem 1 in one direction; the second direction is your homework, I will give some hints

**Let us first study the following natural question:** What is a differential equation of a reparameterized geodesic?

We all know that the differential equation of the geodesic parameterized by the natural ("affine") parameter is

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

(Or, in the index notation, $\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0$.)

**Claim.** Any reparameterised geodesic satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$ for a certain function $\alpha(t)$. Any regular curve satisfying this equation is a reparameterized geodesic.

**Physical interpretation:** $\nabla_{\dot\gamma}\dot\gamma$ is an acceleration, so geodesics are the trajectories of particles with no acceleration (= *freely falling particles*). The condition $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$ means that at every point the acceleration is proportional to the velocity, which implies that the particles go along the same trajectory as in no acceleration case but the speed is not constant.

**Claim.** Any reparameterised geodesic satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$ for a certain function $\alpha(t)$. Any regular curve satisfying this equation is a reparameterized geodesic.

**Proof in direction $\Longrightarrow$** . We assume that the curve $\gamma(t)$ is a geodesic, and that the curve $\bar\gamma(\tau)$ is the geometrically the same curve but other parameterized: $\exists \tau(t)$ such that $\bar\gamma(\tau(t)) = \gamma(t)$.

We denote the $t$-derivative by *dot*, the $\tau$ derivative by *prime*, and we clearly have $\dot\gamma = \dot\tau\ \bar\gamma'$.

Then, the geodesic equation $0 = \nabla_{\dot\gamma}\dot\gamma$ reads

$$0 = \nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\tau\gamma'}(\dot\tau\gamma') = \ \ddot\tau\bar\gamma' + (\dot\tau)^2\,\nabla_{\bar\gamma'}\bar\gamma'.$$

We see that the curve $\bar\gamma$ satisfies the equation

$$\nabla_{\bar\gamma'}\bar\gamma' = \alpha(t)\bar\gamma' \qquad\qquad (***)$$

with the function $\alpha(\tau) = -\frac{\ddot\tau}{(\dot\tau)^2}$.

**Proof in direction $\Longleftarrow$.** Just observe that all steps in the proof in the „$\Longrightarrow$“-direction are invertible: if we have a regular $\bar\gamma(\tau)$ such that curve such that $(***)$ is satisfied, find a function $\tau(t)$ such that $\alpha = -\frac{\ddot\tau}{(\dot\tau)^2}$, its existence follows from the theory of ODE, and then go upwards along the formulas in the proof in $\Longrightarrow$-direction. $\square$

# Proof of Theorem 1 in direction $\Longleftarrow$

**Theorem 1 (deep classics: Levi-Civita 1896, Weyl 1924).** $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\bar\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that for all vector fields

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k\delta^i_j + \phi_j\delta^i_k. \qquad (*)$$

We assume that $\nabla = (\Gamma^i_{jk})$ and $\bar\nabla = (\Gamma^i_{jk})$ are related by $(*)$, our goal is to show that they they are projectively equivalent. That is, we need to show that any $\nabla$-geodesic $\gamma$, after some reparameterization, is a geodesic of $\bar\nabla$. Because of the Claim on the previous slide we need to show that $\bar\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$. We obtain this formula by direct calculation: we do it in the index form

$$0 = \bar\nabla_{\dot\gamma}\dot\gamma = \ddot\gamma + \bar\Gamma^i_{jk}\dot\gamma^k\dot\gamma^j \ = \ddot\gamma + (\Gamma^i_{jk}\dot\gamma^k\dot\gamma^j) + (\phi_k\delta^i_j + \phi_j\delta^i_k)\dot\gamma^j\dot\gamma^k = \bar\nabla_{\dot\gamma}\dot\gamma + 2\phi(\dot\gamma)\dot\gamma$$

as we want.

**Remark** The proof may look more easier in the index-free notation: recall that the analog of the formula $(*)$ is $(**)$ and is

$$\boxed{\bar{\nabla}_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**)}$$

and using this we have

$$0 = \bar{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \phi(\dot\gamma)\dot\gamma + \phi(\dot\gamma)\dot\gamma$$

as we want. For the proof in the $\Longrightarrow$-direction, which I leave you as a homework, I recommend you though to use the index notation.

**Homework.** Prove theorem in the $\Longrightarrow$-direction.

**Hint.** Involves some not-completely-trivial linear algebra.

# Definition of projective structure

**Informal and inefficient definition:** *Projective structure* is a sufficiently big family of curves that after a reparameterisation can be geodesics of some affine connection.

**Efficient Def.** By *projective structure* we understand the equivalence class of symmetric affine connections with respect to the equivalence relation "projective equivalence".

**Remark.** From Theorem 1 it follows that two symmetric affine connection correspond to one projective structure, iff their difference has the form $\phi_k\delta^i_j + \phi_j\delta^i_k$ for an 1-form $\phi$.

# 2-dim projective structures and second order ODEs

In dimension $n=2$, because of the symmetries $\Gamma^i_{jk}=\Gamma^i_{kj}$, the components of $\Gamma^i_{kj}$ in coordinates are $\frac{n^2(n+1)}{2}=6$ function. The freedom in choosing the connection in the projective class is $n=2$ "functions" $(\phi_1,\phi_2)$. Thus, locally, projective structure is given by 4 functions of the coordinates. Let us give, following Beltrami 1859, a geometric sense to these 4 functions.

**Theorem 2.** Let $\left[\Gamma^i_{jk}\right]$ be a projective structure on $U\subset\mathbb{R}^2(x,y)$. Consider the following second order ODE

$$y''=\underbrace{-\Gamma^2_{11}}_{K_0}+\underbrace{\left(\Gamma^1_{11}-2\Gamma^2_{12}\right)}_{K_1}y'+\underbrace{\left(2\Gamma^1_{12}-\Gamma^2_{22}\right)}_{K_2}y'^2+\underbrace{\Gamma^1_{22}}_{K_3}y'^3. \qquad (1)$$

Then, for every solution $y(x)$ of (1) the curve $(x,y(x))$ is a (reparametrized) geodesic.

**Corollary.** The coefficients $K_0,...,K_3$ of ODE (1) contain all the information of the projective structure: two connections are projectively related iff the corresponding functions $K_0,...,K_3$ coincide.

**Homework.** Prove the Corollary: show that the kernel of the (linear) mapping $\Gamma^i_{jk}\mapsto(K_0,K_1,K_2,K_3)$ consists of tensors $T^i_{jk}=\phi_k\delta^i_j+\phi_j\delta^i_k$.

**Theorem 2.** Let $\left[\Gamma^i_{jk}\right]$ be a projective structure on $U \subset \mathbb{R}^2(x, y)$. Consider the following second order ODE

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3. \tag{1}$$

Then, for every solution $y(x)$ of (1) the curve $(x, y(x))$ is a (reparametrized) geodesic.

**Example.** The flat projective structure $[\Gamma^i_{jk} \equiv 0]$ corresponds to the ODE $y'' = 0$. The solutions of this ODE are $y(x) = ax + b$, and the curves $x \mapsto (x, y(x)) = (x, ax + b)$ are indeed straight lines.

**Remark 1.** Note that the set of curves of the form $(x, y(x))$ is quite big: at any point in any direction there exists such a curve passing through this point in this direction.

**Remark 2.** We see a special feature of geodesics of affine connections: they are essentially the same as solutions of 2nd order ODE $y'' = F(x, y, y')$ SUCH THAT THE RIGHT HAND SIDE IS POLYNOMIAL IN $y'$ OF DEGREE $\leq 3$. In particular, taking an ODE $y'' = F(x, y, y')$ such that $F$ is NOT A POLYNOMIAL in $y'$ of degree $\leq 3$, the set of the curves of the form $(x, y(x))$ are geodesics of no affine connection. We will return to this problem in the last lecture.

# How many geodesics determine the projective structure?

We will answer this question in DIMENSION $2$, and give an application in the next slides.
We consider a projective structure $\left[\Gamma^i_{jk}\right]$ and the corresponding ODE

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3. \tag{1}$$

**Claim.** For any point $(\hat{x}, \hat{y})$, 4 different geodesics passing through this point determine the coefficients $K_0(\hat{x}, \hat{y}), K_1(\hat{x}, \hat{y}), K_2(\hat{x}, \hat{y}), K_3(\hat{x}, \hat{y})$ at this point.

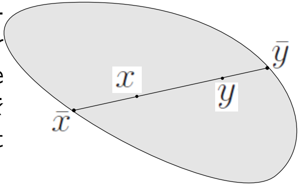
**Proof.** 4 different geodesics passing through $(\hat{x}, \hat{y})$ correspond to 4 different solutions $y_1, y_2, y_3, y_4$ of (1) such that $y_i(\hat{x}) = \hat{y}$. Knowing geodesics implies that we know $y_i'(\hat{x})$ and $y_i''(\hat{x})$ which implies that we known the values of the polynomial

$$P(y') = K_0(\hat{x}, \hat{y}) + K_1(\hat{x}, \hat{y})y' + K_2(\hat{x}, \hat{y})y'^2 + K_3(\hat{x}, \hat{y})y'^3$$

at four points $y_1'(\hat{x}), ..., y_4'(\hat{x})$ and we know that values at 4 points determines a polynomial of degree $\leq 3$. $\square$

# Application in Finsler geometry: proof of the 2-dim de la Harpe conjecture

Let $K \subset \mathbb{R}^n$ be a compact convex body. Hilbert metric is the following distance function $d : int(K) \times int(K) \to \mathbb{R}$ on the interior of $K$: for $x \neq y \in int(K)$ we consider the straight line containing $x, y$ and denote by $\bar{x}$ and $\bar{y}$ the intersection points of this straight line with the boundary of $K$.




Then, we put

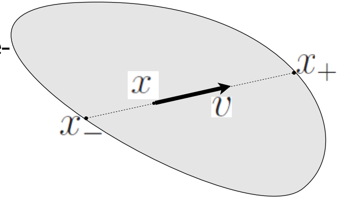
$$d(x,y) := \ln\big((\bar{x}, x; y, \bar{y})\big) := \ln\left(\frac{|\bar{y}-x|}{|\bar{y}-y|} : \frac{|\bar{x}-x|}{|\bar{x}-y|}\right),$$

where $|\cdot|$ denotes the usual Euclidean length.

# Known properties of the Hilbert metrics

Hilbert metric is a Finsler metric, the corresponding Finsler function is given by

$$F(x,v) = \frac{|v|}{|x-x_+|} + \frac{|v|}{|x-x_-|}.$$




- Straight line segments are geodesics
- Projective transformations preserving the convex body preserve the Hilbert metric

**Remark.** If the boundary is not strictly convex, the geodesics are not necessary unique.

# Question of de la Harpe (1991)

**For what $K$ all isometries of $K$ are projective transformations?**

- Example (de la Harpe): If $K$ is simplex, there exist isometries that do not come from projective transformations.
- If $K$ is strictly convex, any isometry is a projective transformation (deep classics; possibly Hilbert)
- 2011: Answer by Walsh and Lemmens for polyhedral $K$.
- 2013: Answer by Walsh for all convex bodies

**Theorem (M$\sim$ – Troyanov 2014, in arXive today).** In dimension two, each isometry $\phi : K \to K$ is a projective transformation unless $K$ is a triangle.

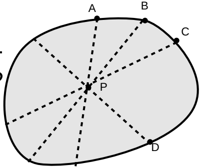
We see that our theorem is not new; but the proof of Walsh is relatively complicated and you will see that our proof is trivial for those who heard the first part on today's lecture.

# Proof .

**Fact (possibly Hilbert 1895, de la Harpe 1991).** A straight line containing an extremal point is UNIQUE MINIMIZING geodesic.

We consider four extremal points $A, B, C, D$ of $K$.

For every point $P \in int(K)$, we consider the intersections of the straight lines containing $A$ and $P$ (resp., $B$ and $P$, $C$ and $P$, $D$ and $P$).




As we learned today, these four straight lines define uniquely a projective structure; this projective structure is projectively flat (all geodesics are straight lines)

The push-forward of this projective structure is a projective structure, since the 4 geodesics are unique, isometry maps sends them to straight lines and therefore the push-forward of the projective structure is projectively flat and $\phi$ is a projective transformation $\square$

# Lecture two: Projectively invariant differential operators

## Plan

1. Definition
2. Two main examples: (projective) Killing and metrization equations
3. Philosophy of metric projective geometry
4. Application: what 2 dim manifolds admit projectively related metrics.

# Projectively invariant operators: def and trivial examples

Projectively invariant = does not depend on the choice of a connection in the projective class and on the coordinate system.

**Not an Example.** Covariant differentiation of vectors or tensors is NOT projectively invariant: If we replace $\nabla$ by a projectively equivalent $\bar{\nabla}$, then the covariant derivative will be CHANGED:

$$\bar{\nabla}_X V - \nabla_X V = \phi(X)V + \phi(V)X.$$

**Trivial Example.** The outer derivative $\omega \mapsto d\omega$ on the space of $k$-forms is projectively invariant. Indeed, it does not depend on a connection at all. (Say, for 1-forms, $d(adx + bdy) = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$)

**Our next goal is to construct two 'nontrivial' projectively invariant differential operations; they will play an important role later today and in other lectures but the price we need to pay now is that we need to introduces weighted tensor fields**

# What are weighted tensors? What is weight?

We assume that our manifold $M$ is orientable and fix an orientation. We consider the bundle $\Lambda_n M$ of positive volume forms on $M$

**Recall.** Volume form is a scew-symmetic form of maximal order, $Vol = f(x)dx^1 \wedge ... \wedge dx^n$ with $f \neq 0$. "Positive" means that if the basis $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ is positively oriented then $f(x) > 0$.

Positive volume bundle is a locally trivial 1-dimensional bundle over our manifold $M$ with the structure group $(\mathbb{R}_{>0}, \cdot)$. That means in particular that for small neighborhood $U \subset M$ we have an isomorphisms between $\Lambda_n U$ and $\mathbb{R}_{>0} \times U$: there are two natural ways to choose the isomorphism, let us discuss them.

1. Choose a section in this bundle, i.e., a volume form, the other sections of this bundle can be thought to be positive functions on the manifold (AND IF WE CHANGE COORDINATES THEY TRANSFORM LIKE FUNCTIONS, I.E., DO NOT TRANSFORM AT ALL). This situation will be actively used later, when the volume form is parallel with respect to an affine connection in the projective class.
2. In local coordinates $x = (x^1, ..., x^n)$, we can choose the volume form $dx^1 \wedge ... \wedge dx^n$, the volume form $\Omega = f(x) dx^1 \wedge ... \wedge dx^n$ corresponds to the function $f(x)$. Its transformation rule is different from that of functions: a coordinate change, $x = x(y)$ transforms $f(x)$ to $\det\left(\frac{dx}{dy}\right) f(x(y))$.

# $(\Lambda_n)^\alpha M$

Let $\alpha \in \mathbb{R} \setminus \{0\}$. Since $t \to t^\alpha$ is an isomorphism of $(\mathbb{R}_{>0}, \cdot)$, for any 1-dimensional $(\mathbb{R}_{>0}, \cdot)$-bundle its power $\alpha$ is well-defined and is also an one-dimensional bundle. We consider $(\Lambda_n)^\alpha M$. It is an 1-dimensional bundle, so its sections locally can be viewed as functions. Again we have two ways to view the sections as functions:

1. Choose a volume form $\Omega$, and the corresponding section $\omega = (\Omega)^\alpha$ of $(\Lambda_n)^\alpha M$. Then, the other sections of this bundle can be thought to be positive functions on the manifold.
2. In local coordinates $x = (x^1, ..., x^n)$, we can choose the section $(dx^1 \wedge ... \wedge dx^n)^\alpha$, then the section $\omega = (f(x)dx^1 \wedge ... \wedge dx^n)^\alpha$ corresponds to the function $(f(x))^\alpha$. Its transformation rule is different from that of functions: a coordinate change, $x = x(y)$ transforms $(f(x))^\alpha$ to $\left(\det\left(\frac{dx}{dy}\right)\right)^\alpha f(x(y))^\alpha$.

# Weighted tensors

**Def.** By a (p,q)-tensor field of projective weight $k$ we understand a section of the following bundle:

$$T^{(p,q)} \otimes (\Lambda_n)^{\frac{k}{n+1}} M \quad (\text{notation } := T^{(p,q)}M(k))$$

If we have a preferred volume form on the manifold, the sections of $T^{(p,q)}M(k)$ can be identified with (p,q)-tensors fields. The identification depends of course on the choice of the volume form.

If we do not have a preferred volume form on the manifold, in a local coordinate system one can choose $(dx^1 \wedge ... \wedge dx^n)$ as the preferred volume, and still think that sections are "almost" (p,q)-tensors: they are also given by $n^{p+q}$ functions but their transformation rule is slightly different from that for tensors: in addition to the usual transformation rule for tensors one needs to multiply by $\left(\det\left(\frac{dx}{dy}\right)\right)^{\alpha}$ with $\alpha = \frac{k}{n+1}$.

# Covariant differentiation of weighted tensor bundles

**Fact (e.g. brute force calculations).** Suppose (projectively equivalent) connections $\nabla = (\Gamma^i_{jk})$ and $\bar{\nabla} = (\Gamma^i_{jk})$ are related by the formula

$$\bar{\nabla}_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**).$$

Then, the covariant derivatives of a volume form $\Omega \in \Gamma(\Lambda_n M)$ in the connections $\nabla$ and $\bar{\nabla}$ are related by

$$\bar{\nabla}_X \Omega = \nabla_X \Omega - (n+1)\phi(X)\Omega.$$

In particular, the covariant derivatives of the section $\omega := \left(\Omega^{\frac{k}{n+1}}\right) \in \Gamma((\Lambda_n)^{\frac{k}{n+1}} M)$ are related by

$$\bar{\nabla}_X \omega = \nabla_X \omega - k\phi(X)\omega. \qquad (2)$$

# First example of projectively invariant differential operation

$$\boxed{\bar\nabla_X \omega = \nabla_X \omega - k\phi(X)\omega. \qquad (2)}$$

Let $K \in \Gamma\left(T^{(0,1)}M(-2)\right)$ be an 1-form of projective weight $(-2)$. We calculate the difference their $\nabla$- and $\bar\nabla$- derivatives assuming

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**):$$

$$\bar\nabla_X K = \nabla_X K \underbrace{-\phi(X)K - K(X)\phi}_{\text{because of } (**)} + \underbrace{2\phi(X)K}_{\text{because of } (2)} = \nabla_X K + \phi(X)K - K(X)\phi. \quad (3)$$

**Theorem.** For $(0,1)$-tensors of projective weight (-2) the operation

$$K \mapsto \mathsf{Symmetrization\_Of}(\nabla K) \quad (\mathrm{K1})$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

**Proof.** Observe in (3) that the difference between $(\bar\nabla_X K)(Y)$ and $(\nabla_X K)(Y)$ is scewsymmetric in $X, Y$ and vanishes after symmetrization.

**Remark.** In the index notation, the mapping $(K1)$ reads $K_i \mapsto K_{i,j} + K_{j,i}$. The equation $(K1) = 0$ is called projective Killing equation for weighted 1-forms.

**Theorem.** For $(0,1)$-tensors of projective weight (-2) the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

**Corollary 1.** For $(0,2)$-tensors of projective weight $(-4)$ the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

**Proof.** Decompose $(0,2)$ tensors of weight $-4$ into the sum of symmetric tensor product of $(0,1)$ tensors of weight $-2$ and apply Corollary 1.

**Notation.** The equation in Corollary 1 is called projective Killing equation; it will play important role at the end of the lecture.

# One more important projectively invariant operation

**Theorem.** For $(1,0)$-tensors of projective weight 1 the operation

$$\sigma \mapsto \mathit{Trace\_Free\_Part\_Of}\,\nabla(\sigma) = \sigma^i{}_{,j} - \tfrac{1}{n}\sigma^s{}_{,s}\delta^i_j.$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

**Proof.** By calculations which are essentially the same as in Corollary 1.

**Corollary 3.** For symmetric $(2,0)$-tensors of projective weight 2 the operation

$$\sigma^{ij} \mapsto \sigma^{ij}{}_{,k} - \tfrac{1}{n+1}(\sigma^{is}{}_{,s}\delta^j_k + \sigma^{js}{}_{,s}\delta^i_k) \qquad (4)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

**Proof.** Decompose $(2,0)$ tensors of weight 2 into the sum of symmetric tensor products of $(0,1)$ tensors of weight $-2$ and apply Corollary 2

**Remark.** In the index-free notation the operation (4) reads

$$\sigma \mapsto \mathrm{Trace\_Free\_Part\_Of}\,(\nabla\sigma)\,,$$

though $\nabla\sigma$ is a (2,1)-(weighted)-tensor and trace is a (weighted) vector.

# Geometric importance of the operator $\sigma \mapsto \text{Trace\_Free\_Part\_Of}\left(\nabla\sigma\right)$.

**(Metrization) Theorem 3 (Eastwood-M$\sim$ 2006).** Suppose the Levi-Civita connection of a metric $g$ lies in a projective class $[\nabla]$. Then, $\sigma^{ij} := g^{ij} \otimes \left(Vol_g\right)^{\frac{2}{n+1}}$ is a solution of

$$\text{Trace\_Free\_Part\_Of}\left(\nabla\sigma\right) = 0. \tag{5}$$

Moreover, for every solution of the equation (5) such that $\det(\sigma) \neq 0$ there exists a metric whose Levi-Civita connection lies in the projective class.

**Proof in the direction $\Rightarrow$.** We assume that $\nabla^g \in [\nabla]$. Since our equation is projectively invariant, we may assume that we work in the connection $\nabla^g$. In this connection the metric and therefore all objects constructed by the metric are parallel so $\nabla^g(\sigma) = 0$ which of course implies (5)

**(Metrization) Theorem 3.** Suppose the Levi-Civita connection of a metric $g$ lies in a projective class $[\nabla]$. Then, $\sigma^{ij} := g^{ij} \otimes \left(Vol_g\right)^{\frac{2}{n+1}}$ is a solution of

$$\text{Trace\_Free\_Part\_Of}\left(\nabla\sigma\right) = 0. \qquad (5)$$

Moreover, for every solution of the equation (5) such that $\det(\sigma) \neq 0$ there exists a metric whose Levi-Civita connection lies in the projective class.

**Proof in the direction $\Leftarrow$ is your homework:** Observe that though equation

$$\sigma^{ij} \mapsto \sigma^{ij}{}_{,k} - \frac{1}{n+1}\left(\sigma^{is}{}_{,s}\delta^j_k + \sigma^{js}{}_{,s}\delta^i_k\right)$$

does not depend on the choice of a connection in the projective class, the part of it marked by *blue color* does depend. Find out how it depends and prove that there exists a connection in the projective class such that the *blue* part is zero, show then that this connections preserves a metric such that $\sigma$ is obtained by the metric by the formula in Theorem 3.

# Relation between (nondegenerate) solutions $\sigma$ of metrization equations and metrics in coordinates

Let us work in a coordinate system and choose $dx^1 \wedge ... \wedge dx^n$. as a volume form.

- If we have a metric $g_{ij}$, then the corresponding solution of the metrization equation is given by

$$\sigma^{ij} := \left( g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}} \right) = g^{ij} |\det g|^{\frac{1}{n+1}}.$$

- For a solution $\sigma = \sigma^{ij}$ of the metrisation equation such that its determinant in not zero, the corresponding metric is given by

$$g^{ij} := |\det(\sigma)| \sigma^{ij}.$$

# Metrization equations in dimension 2 in coordinates:

**Theorem 3 (Metrization equations in all dimensions:)** Trace_Free_Part_Of $(\nabla\sigma) = 0.$ (5)

As we remember from Lecture 1, in dimension 2 the four functions $K_0, K_1, K_2, K_3$ (which are the coefficients of the equation) (essentially R. Liouville 1889)

$$y'' = \underbrace{-\Gamma_{11}^2}_{K_0} + \underbrace{(\Gamma_{11}^1 - 2\Gamma_{12}^2)}_{K_1} y' + \underbrace{(2\Gamma_{12}^1 - \Gamma_{22}^2)}_{K_2} y'^2 + \underbrace{\Gamma_{22}^1}_{K_3} y'^3.$$

encode the projective class of the connection $\Gamma_{jk}^i$.

In this setting, the metrization equations in the following system of 4 PDE on three unknown functions:

$$\left\{\begin{array}{rcl} {\sigma^{22}}_x - \frac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} & = & 0 \\ {\sigma^{22}}_y - 2 {\sigma^{12}}_x - \frac{4}{3} K_2 \sigma^{22} - \frac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} & = & 0 \\ -2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \frac{2}{3} K_2 \sigma^{12} + \frac{4}{3} K_1 \sigma^{11} & = & 0 \\ {\sigma^{11}}_y + 2 K_3 \sigma^{12} + \frac{2}{3} K_2 \sigma^{11} & = & 0 \end{array}\right.$$

**Corollary.** Generic projective structure is not metrizable.
**Explanation (formal proof in Bryant et al 2009).** The system is overdertermined: 4 equations on three unknown functions, and generic overdetermined systems have no solution.

# Metric Projective Geometry: philosophy and goals

One can of course study projective structures without thinking about whether there is a (Levi-Civita connection of a) metric in the projective class.

- Luck of "easy to formulate, hard to prove" results.
- Virtually no applications in physics

Let us study metrizable projective structures, i.e., such that there exists a metric in the projective class.

Generic metrizable projective structure has only one, up to a scaling, metric in the projective class. In this case, all geometric questions can be reformulated as questions on this metric.

We will study metrisable projective structures such that there exists at least two nonproportional metrics in the projective class.

- Many "easy to formulate, hard to prove" results. Many named problems. Applications in physics
- Many questions and methods can be generalized for Finsler manifolds

# Next goal: easy to formulate result for today

**Theorem (M$\sim$-Topalov 1998).** Let $(M^2, g)$ be a two-dimensional closed (compact, no boundary) Riemannian manifold. Assume a metric $\bar{g}$ is projectively related to $g$ and is nonproportional to $g$. Then, $M^2$ has nonnegative Euler characteristic.

I will give an easy proof of this theorem using what we learned today.

# I use: projective Killing equation is projectively invariant.

**Corollary 2.** For $(0,2)$-tensors of projective weight $-4$ the operation

$$K \mapsto \mathsf{Symmetrization\_Of}(\nabla K)$$

is projective invariant.

**Example.** The (Levi-Civita connection of the) metric $g$ does have one nontrivial solution of this equation, namely

$$K = g \otimes (Vol_g)^{\frac{-4}{n+1}}.$$

But the projective Killing equation does not depend on the choice of connection in the projective class.

Thus, any metric in the projective class allows us to construct a solution of the projective Killing equation. Say, if we have another metric $\bar g$ in the same projective class, then

$$\bar K = \bar g \otimes (Vol_{\bar g})^{\frac{-4}{n+1}}.$$

is (still) a solution.

## Geometric sense of Killing equations: conservative quantities.

**Theorem 4.** Suppose $K$ is a solutions of the projective Killing equation. Then, for any metric $g$ in the projective class the tensor field

$$\hat{K} := K \otimes (Vol_g)^{\frac{4}{n+1}}.$$

is a Killing tensor, this means that for any parameterized $g$-geodesic $\gamma$ the function

$$t \mapsto I(\gamma(t), \dot\gamma(t)) = K(\dot\gamma(t), \dot\gamma(t))$$

is constant.

**Proof.** We need to show that

$$\nabla_{\dot\gamma}\left(K(\dot\gamma, \dot\gamma)\right) = 0. \qquad (\star)$$

Because of the definition of geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$, and $(\star)$ reduces to

$$\nabla K(\dot\gamma, \dot\gamma, \dot\gamma) = 0,$$

which follows from $\text{Symmetrization\_Of}(\nabla K) = 0$. $\square$

# Trivial conservative quantity: energy

**Example above.** The following section of $T^{(0,2)}M(-4)$

$$K = \bar{g} \otimes (Vol_{\bar{g}})^{\frac{-4}{n+1}}$$

is a solution of the projective Killing equation for $g$.

**Theorem 4.** $\hat{K} := K \otimes (Vol_g)^{\frac{4}{n+1}}$. is a Killing tensor.

If we take the $K$ from the first frame, and use it to constract $\hat{K}$ from the second frame, then we obtain $\hat{K} = g$, which is of course a Killing tensor; the corresponding conservative quantity is the kinetic energy.

# Nontrivial conservative quantity, if we have two nonproportional metric in the projective class

**Example above.** For a metric $\bar{g}$ in the projective class the following section of $T^{(0,2)}M(-4)$

$$\bar{K} = \bar{g} \otimes (Vol_{\bar{g}})^{\frac{-4}{n+1}}$$

is a solution of the projective Killing equation.

**Theorem 4.** $\hat{K} := K \otimes (Vol_g)^{\frac{4}{n+1}}$. is a Killing tensor.

If we take the $\bar{K}$ from the first frame, and use it to constract $\hat{K}$ from the second frame, then we obtain $\hat{K} = \left|\frac{\det g}{\det \bar{g}}\right|^{\frac{2}{n+1}} \bar{g}$, which is now nonproportional to the trivial (=always existing) Killing tensor $g_{ij}$. The corresponding is given by

$$I(x,\xi) = \left|\frac{\det g}{\det \bar{g}}\right|^{\frac{2}{n+1}} \bar{g}(\xi,\xi)$$

**Historical remark.** There are of course direct proofs that $I$ is a conservative quantity, the most classical is possibly due to Painleve 18... I will possibly show you other proofs in lecture VI, when I speak about Finlser metrics.

# Proof of announced theorem

**Theorem (M$\sim$-Topalov 1998).** Let $(M^2, g)$ be a two-dimensional closed (compact, no boundary) Riemannian manifold. Assume a metric $\bar g$ is projectively related to $g$ and is nonproportional to $g$. Then, $M^2$ has nonnegative Euler characteristic.

In dimension 2, the conservative quantity constructed

$$I_0(\xi) := \left|\frac{\det(g)}{\det(\bar g)}\right|^{\frac{2}{3}} \bar g(\xi, \xi).$$

Assume the surface is neither torus nor the sphere. The goal is to show that $g$ and $\bar g$ are proportional.

Because of topology, there exists $x_0$ such that $g_{|x_0} = \mathrm{const} \cdot \bar g_{|x_0}$. W.l.o.g. we assume const $= 1$. We assume $g_{|x_1} \neq \bar g_{|x_1}$ and find a contradiction.

# Homework

Use Killing tensors to show that the space of solutions of the metrization equation is finite-dimensional (for $n = 2$ first if it makes your life easier and then for any dimension since there are no two-dimensional phenomenas in the proof)

# Lecture 3

## Plan

- Local normal forms of projectively related Riemannian metrics
- Problems of Lie and their solution
- How we solved the problems of Lie

# Our first goal is to prove the Dini's Theorem 1869

**Local normal form question (Beltrami 1865):** Given two projectively related metric, how do they look in "the best" coordinate system (near a generic point)? How unique is such best coordinate system?

**Theorem (Dini 1869).** Let $g$ and $\bar g$ are projectively related 2 dim Riemannian metrics. Then, in a neighborhood of almost every point there exists a coordinate system such that in this coordinate system the metrics are

$$
\begin{aligned}
g &= \begin{pmatrix} X(x) - Y(y) & \\ & X(x) - Y(y) \end{pmatrix} \\
\bar g &= \begin{pmatrix} \frac{X(x)-Y(y)}{X(x)Y(y)^2} & \\ & \frac{X(x)-Y(y)}{X(x)^2Y(y)} \end{pmatrix}
\end{aligned}
$$

The coordinates are unique modulo $(x, y) \mapsto (\pm x + b, \pm y + d)$.

**Remark.** The answer in higher dimensions is also known (Levi-Civita) In other signatures the answer to the Beltrami questions is also known (Darboux/Lie for dim 2, Bolsinov-Matveev 2013 for all dimensions).

**Rem.** In the 2 dim case of splitted signature, there are two more cases: when $g^{-1}\bar g$ has complex eiganvalues, and when $g^{-1}\bar g$ has Jordan block.

# Proof: coordinates such that $g$ and $\bar{g}$ are diagonal

Such coordinates exist near every generic points:

Indeed, at the points where $g$ is not proportional to $\bar{g}$ the (1,1)-tensor $g^{-1}\bar{g} = g^{is}\bar{g}_{js}$ has two different eigenvalues. We consider the coordinate system $(x, y)$ such that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are eigenvectors.

Since the eigenvectors are orthogonal w.r.t. $g$ and w.r.t. $\bar{g}$, in this coordinates the metrics are diagonal.

# Plugging diagonal $\sigma, \bar\sigma$ in the metrization theorem

Thus, we may assume that in the coordinate system $(x, y)$ the metrics are diagonal and theirfore the corresponding solutions of the metrization equation $\sigma = \left(g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}}\right) = g^{ij}(\det g)^{\frac{1}{n+1}}$,
$\bar\sigma = \left(\bar g^{ij} \otimes (Vol_{\bar g})^{\frac{2}{n+1}}\right) = \bar g^{ij}(\det \bar g)^{\frac{1}{n+1}}$ are diagonal.

Consider the (1,1)-tensor field $A = \bar\sigma(\sigma)^{-1} = \bar\sigma^{is}\sigma_{js}$, it is also diagonal:

$$\sigma = \begin{pmatrix} \sigma^{11} & \\ & \sigma^{22} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}, \quad \bar\sigma = \begin{pmatrix} A_1\sigma^{11} & \\ & A_2\sigma^{22} \end{pmatrix}.$$

Let us now plug these $\sigma$ and $\bar\sigma$ in the metrization theorem whose two-dimensional version is in Lecture 2:

$$\begin{array}{rcl}
{\sigma^{22}}_x - \frac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} & = & 0 \\
{\sigma^{22}}_y - 2 {\sigma^{12}}_x - \frac{4}{3} K_2 \sigma^{22} - \frac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} & = & 0 \\
-2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \frac{2}{3} K_2 \sigma^{12} + \frac{4}{3} K_1 \sigma^{11} & = & 0 \\
{\sigma^{11}}_y + 2 K_3 \sigma^{12} + \frac{2}{3} K_2 \sigma^{11} & = & 0
\end{array}$$

# 8 equations on 6 unknown is too much – elementary tricks solve the system

$$\begin{array}{rcl|rcl}
{\sigma^{22}}_x - \frac{2}{3}\,K_1\,\sigma^{22} & = & 0 & A_2{\sigma^{22}}_x + (A_2)_x\sigma^{22} - \frac{2}{3}\,K_1\,A_2\sigma^{22} & = & 0\\
{\sigma^{22}}_y - \frac{4}{3}\,K_2\,\sigma^{22} + 2\,K_0\,\sigma^{11} & = & 0 & A_2{\sigma^{22}}_y + (A_2)_y\sigma^{22} - \frac{4}{3}\,K_2\,A_2\sigma^{22} + 2\,K_0\,A_1\sigma^{11} & = & 0\\
{\sigma^{11}}_x - 2\,K_3\,\sigma^{22} + \frac{4}{3}\,K_1\,\sigma^{11} & = & 0 & A_1{\sigma^{11}}_x + (A_1)_x\sigma^{11} - 2\,K_3\,A_2\sigma^{22} + \frac{4}{3}\,K_1\,A_1\sigma^{11} & = & 0\\
{\sigma^{11}}_y + \frac{2}{3}\,K_2\,\sigma^{11} & = & 0 & A_1{\sigma^{11}}_y + (A_1)_y\sigma^{22} + \frac{2}{3}\,K_2\,A_1\sigma^{11} & = & 0
\end{array}$$

**Message: systems of PDE with more equations are as a rule easier to solve than that with less equations**

**How we proceed:** Solve the first 4 questions with respect to $K_0, ..., K_3$ and substitute the result in the last 4 equations. One obtains the equations

$$\left(\begin{array}{rcl}
(A_1)_y & = & 0\\
(A_2)_x & = & 0\\
((A_1-A_2)\sigma^{11}(\sigma^{22})^2)_x & = & 0\\
((A_1-A_2)\sigma^{22}(\sigma^{11})^2)_y & = & 0.
\end{array}\right) \text{ implying } \left(\begin{array}{rcl}
A_1 & = & X(x)\\
A_2 & = & Y(y)\\
(X(x)-Y(y))\sigma^{11}(\sigma^{22})^2 & = & \frac{1}{Y_1(y)}\\
(X(x)-Y(y))\sigma^{22}(\sigma^{11})^2 & = & \frac{1}{X_1(x)}.
\end{array}\right)$$

$$\boxed{\left(\begin{array}{ccc} A_1 & = & X(x) \\ A_2 & = & Y(y) \\ (X(x)-Y(y))\sigma^{11}(\sigma^{22})^2 & = & \frac{1}{Y_1(y)} \\ (X(x)-Y(y))\sigma^{22}(\sigma^{11})^2 & = & \frac{1}{X_1(x)}. \end{array}\right)}$$

Observe now, because of the relation $g^{ij} = |\det(\sigma)|\sigma$ and because of the matrices $\sigma, \bar{\sigma}$ are diagonal, we have $\sigma^{11}(\sigma^{22})^2 = g^{22}$ and $\sigma^{22}(\sigma^{11})^2 = g^{11}$. Thus, we obtain that

$$g = (X-Y)(X_1 dx^2 + Y_1 dy^2) \text{ and } A = diag(X, Y).$$

By a coordinate change $x = x(x_{new}), y = y(y_{new})$, one can "hide" $X_1$ and $Y_1$ in $dx^2$ and $dy^2$ and obtain the formulas of Dini $\square$

# Projective transformations

**Def.** Projective transformation of a projective structure $[\Gamma]$ is a (local) diffeomorphism that preserves $[\Gamma]$.

**Geometric (equivalent) definition**. Projective transformations are diffeomorphisms that send geodesics to geodesics.

**Example.** Affine transformations from 1st year linear algebra course (i.e., $x \mapsto Ax + b$ with nondegenerate matrix $A$) are projective transformations of the flat (i.e., when $\Gamma^i_{jk} \equiv 0$) projective structure.

**Example.** Projective transformations from linear algebra are projective transformations of the flat projective structure.

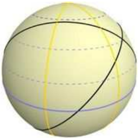
**Def.** A vector field is projective w.r.t. $[\Gamma]$, if its (local) flow acts by projective transformations.

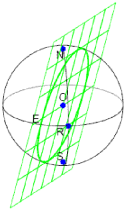
**Example.** A Killing vector field of a metric is projective w.r.t. the projective structure of the metric.

# Beltrami example: projective algebra of the round sphere is $sl(n+1)$.

We consider the standard $S^n \subset R^{n+1}$ with the induced metric.

**Fact.** Geodesics of the sphere are the great circles, that are the intersections of the 2-planes containing the center of the sphere with the sphere.





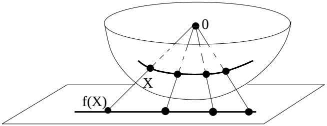

Beltrami (1865) observed:

For every $A \in SL(n+1) \xrightarrow{\text{we construct}} a : S^n \to S^n, a(x) := \frac{A(x)}{|A(x)|}$

- $a$ is a diffeomorphism
- $a$ takes great circles (geodesics) to great circles (geodesics)
- $a$ is an isometry iff $A \in O(n+1)$.

Thus, $Sl(n+1)$ acts by projective transformations on $S^n$. Its stabilizator is discrete and therefore the algebra of projective vector fields is $sl(n+1)$; in dimension $n = 2$ it has dimension $(n+1)^2 - 1 = 8$.

# Example of Lagrange 1789




Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, the projective structure of the plane is the same as that of the sphere and also has 8-dimensional projective algebra.
Everything survives to all dimensions and all signatures and for negative curvature

# “Nice” result of todays lecture: Problems of Lie

Lie 1882:


**Problem I:** *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

**English translation:**

**Describe all 2 dim metrics admitting**

- **Problem I: one projective vector field**
- **Problem II: many projective vector fields**

Lie 1882:


**Problem II:** *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

*Both problems are local, in a neighborhood of a generic point*

# Solution of the 2nd Lie Problem

**Theorem (Bryant, Manno, M∼ 2007)** *If a two-dimensional metric $g$ of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point $p$, then there exist coordinates $x, y$ in a neighborhood of $p$ such that the metrics are as follows.*

1. $\varepsilon_1 e^{(b+2)\,x} dx^2 + \varepsilon_2 b e^{b\,x} dy^2$, *where* $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
2. $a\left(\varepsilon_1 \frac{e^{(b+2)\,x} dx^2}{(e^{b\,x}+\varepsilon_2)^2} + \frac{e^{b\,x} dy^2}{e^{b\,x}+\varepsilon_2}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
3. $a\left(\frac{e^{2\,x} dx^2}{x^2} + \varepsilon \frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, *and* $\varepsilon \in \{-1, 1\}$
4. $\varepsilon_1 e^{3x} dx^2 + \varepsilon_2 e^{x} dy^2$, *where* $\varepsilon_i \in \{-1, 1\}$,
5. $a\left(\frac{e^{3x} dx^2}{(e^{x}+\varepsilon_2)^2} + \frac{\varepsilon_1 e^{x} dy^2}{(e^{x}+\varepsilon_2)}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $\varepsilon_i \in \{-1, 1\}$,
6. $a\left(\frac{dx^2}{(cx+2x^2+\varepsilon_2)^2 x} + \varepsilon_1 \frac{x dy^2}{cx+2x^2+\varepsilon_2}\right)$, *where* $a > 0$, $\varepsilon_i \in \{-1, 1\}$, $c \in \mathbb{R}$.

**Theorem (M$\sim$ 2008):** *Let $v$ be a projective vector field on $(M^2, \bar{g})$. Assume the restriction of $\bar{g}$ to no neighborhood has an infinitesimal homothety. Then, there exists a coordinate system in a neighborhood of almost every point such that certain metric $g$ geodesically equivalent to $\bar{g}$ is given by*

1. $ds_g^2 = (X(x) - Y(y))(X_1(x)dx^2 + Y_1(y)dy^2), v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, where
   - 1.1 $X(x) = \frac{1}{x}, \ Y(y) = \frac{1}{y}, X_1(x) = C_1 \cdot \frac{e^{-3x}}{x}, \ Y_1(y) = \frac{e^{-3y}}{y}$.
   - 1.2 $X(x) = \tan(x), \ Y(y) = \tan(y), X_1(x) = C_1 \cdot \frac{e^{-3\lambda x}}{\cos(x)}$, $Y_1(y) = \frac{e^{-3\lambda y}}{\cos(y)}$.
   - 1.3 $X(x) = C_1 \cdot e^{\nu x}, \ Y(y) = e^{\nu y}, X_1(x) = e^{2x}, \ Y_1(y) = \pm e^{2y}$.
2. $ds_g^2 = (Y(y) + x)dxdy, \ v = v_1(x, y)\frac{\partial}{\partial x} + v_2(y)\frac{\partial}{\partial y}$, where
   - 2.1 $Y = e^{\frac{3}{2y}} \cdot \frac{\sqrt{y}}{y-3} + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi$, $v_1 = \frac{y-3}{2}\left(x + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi\right), \ v_2 = y^2$.
   - 2.2 $Y = e^{-\frac{3}{2}\lambda \arctan(y)} \cdot \frac{\sqrt[4]{y^2+1}}{y-3\lambda} + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi$, $v_1 = \frac{y-3\lambda}{2}\left(x + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi\right), \ v_2 = y^2 + 1$.
   - 2.3 $Y(y) = y^{\nu}, \ v_1(x, y) = \nu x, \ v_2 = y$.

# Why the problems of Lie were not solved before? What know-how allowed us to solve it?

- Many people tried (including Lie and his students)
- One immediately reformulates the problem as a (quasilinear) 2ND ORDER system of PDE on the components of metric and of the vector field; the system is to hard to solve by hands.
- Our new viewpoint on the problem which allowed to solve it was to use projectively-invariant objects.
- This allowed to reduced the PDE-reformulation to MORE equations of the 1ST order which can be solved by hands.

# How we solve the problems of Lie: first observation:

We had two projectively invariant equations: Killing equations and metrization equations: let us compare them in dimension 2:

Metrization equation in dimension 2:

$$\begin{aligned} {\sigma^{22}}_x - \tfrac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} &= 0 \\ {\sigma^{22}}_y - 2 {\sigma^{12}}_x - \tfrac{4}{3} K_2 \sigma^{22} - \tfrac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} &= 0 \\ -2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \tfrac{2}{3} K_2 \sigma^{12} + \tfrac{4}{3} K_1 \sigma^{11} &= 0 \\ {\sigma^{11}}_y + 2 K_3 \sigma^{12} + \tfrac{2}{3} K_2 \sigma^{11} &= 0 \end{aligned}$$

Killing equation in dimension 2:

$$\begin{aligned} a_{11x} - \tfrac{2}{3} K_1 a_{11} + 2 K_0 a_{12} &= 0 \\ a_{11y} + 2 a_{12x} - \tfrac{4}{3} K_2 a_{11} + \tfrac{2}{3} K_1 a_{12} + 2 K_0 a_{22} &= 0 \\ 2 a_{12y} + a_{22x} - 2 K_3 a_{11} - \tfrac{2}{3} K_2 a_{12} + \tfrac{4}{3} K_1 a_{22} &= 0 \\ a_{22y} - 2 K_3 a_{12} + \tfrac{2}{3} K_2 a_{22} &= 0 \end{aligned}$$

We see that the equations coincide after renaming the variables:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = Comatrix\left(\begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{12} & \sigma^{22} \end{pmatrix}\right) = \begin{pmatrix} \sigma^{22} & -\sigma^{12} \\ -\sigma^{12} & \sigma^{11} \end{pmatrix}. \qquad (*)$$

**Remark.** The operation of taking the comatrix is a "geometric" operation: it does not depend on the coordinate system, it is invertible, and in dimension 2 it gives a linear bijection between $(2,0)$-tensors of projective weight 2 and $(0,2)$-sections of projective weight $-4$.

We just have proved the following theorem:

**Theorem.** In dimension 2, solutions of metrization equations are in one-to-one correspondence to the solutions of projective Killing equations (Killing tensors are assumed symmetric), the correspondence is given in coordinates by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = Comatrix \left( \begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{12} & \sigma^{22} \end{pmatrix} \right). \qquad (*)$$

**Corollary.** Suppose there exists a solution $\sigma^{ij}$ of the metrisation equation such that $a_{ij}$ is degenerate but nonzero (in every point). Then, there exists a (projective) Killing 1-form, and in the case we have a metric in the projective class, a Killing vector field.

**Proof.** In dimension 2, comatrix of a nonzero degenerate matrix is a nozero degenerate matrix. It has therefore rank 1 and degenerate nonzero projective Killing tensor has the form $a = \pm K \otimes K$ for some 1-form $K$. This 1-form is Killing: one can see it from equations but let us see it geometrically in metric situation which is sufficient for our goals.

The corresponding conservative quantity is $K(\xi) = \sqrt{\pm a(\xi, \xi)}$. Since the function $a(\xi, \xi)$ is preserved along geodesics, the function $K(\xi)$ is also preserved along geodesics, so that $K$ is a Killing one form (and after raising the index we obtain a Killing vector field).

# We may assume that our metrics do not have Killing vector fields

**Corollary.** Suppose there exists a solution $\sigma^{ij}$ of the metrisation equation such that $a_{ij}$ is degenerate but nonzero (in every point). Then, there exists a (projective) Killing 1-form, and in the case we have a metric in the projective class, a Killing vector field.

**Killing vector field is automatically projective, so without loss of generality in the solution of the first Lie problem we assume nonexistence of a Killing vector field and therefore we assume that all nonzero solutions of the metrization equation we meet are nondegenerate**.

# Lie derivative w.r.t. projective vector field sends solutions of the metrization equations to solutions

**Claim.** Let $v$ be projective for $[\Gamma]$ and suppose $\sigma$ is a solution of the metrization equation. Then, $\mathcal{L}_v\sigma$ is also a solution of the metrization equation.

**Proof in the 2 dim case.** Everything is coordinate-invariant, so w.l.o.g. we can work in coordinates $(x, y)$ such that $v = \frac{\partial}{\partial x}$. In this coordinates, the coefficients $K_0, ..., K_3$ do not depend on $x$, so the coefficients of the metrization equation

$$\boxed{\begin{array}{rcl} {\sigma^{22}}_x - \frac{2}{3}\,K_1\,\sigma^{22} - 2\,K_0\,\sigma^{12} & = & 0 \\ {\sigma^{22}}_y - 2\,{\sigma^{12}}_x - \frac{4}{3}\,K_2\,\sigma^{22} - \frac{2}{3}\,K_1\,\sigma^{12} + 2\,K_0\,\sigma^{11} & = & 0 \\ -2\,{\sigma^{12}}_y + {\sigma^{11}}_x - 2\,K_3\,\sigma^{22} + \frac{2}{3}\,K_2\,\sigma^{12} + \frac{4}{3}\,K_1\,\sigma^{11} & = & 0 \\ {\sigma^{11}}_y + 2\,K_3\,\sigma^{12} + \frac{2}{3}\,K_2\,\sigma^{11} & = & 0 \end{array}}$$

do not depend on $x$ as well. THEN, FOR ANY SOLUTION $\sigma^{ij}$ THE $\frac{\partial}{\partial x}$-LIE DERIVATIVE (WILL BE EXPLAINED ON TRIVIAL LANGUAGE ON THE NEXT SLIDE), which is simply $\frac{\partial}{\partial x}$ is also a solution $\square$

**Remark.** The proof actually works in other dimensions as well – we simply need to observe that in coordinates $(x^1, ..., x^n)$ such that $v = \frac{\partial}{\partial x^1}$ the coefficients of the metrization equation does not depend on $x^1$.

We consider the following linear system of PDE:

$$\sum_{k,i} c_j^{i,k} \frac{\partial u_i}{\partial x_k} + \sum_i c_j^i u_i = 0 \ , \quad j = 1, \ldots, m. \tag{1}$$

Here $(u_1, \ldots, u_\ell)$ are the unknown functions to find, the coefficients $c_j^{i,k}$ and $c_j^i$ are functions thought to be known, everything lives in a small neighborhood $W \subset \mathbb{R}^n$ and is at least as smooth as I need in the proofs.

**Fact (1st year calculus).** Assume the coefficients $c_j^{i,k}$ and $c_j^i$ are independent of $x_1$. Then, for any solution $(u_1, \ldots, u_\ell)$ of (1), the tuple $\left(\frac{\partial}{\partial x_1} u_1, \ldots, \frac{\partial}{\partial x_1} u_\ell\right)$ is also a solution.

**Proof.** We differentiate the equations (1) and interchange the partial derivatives to obtain

$$\frac{\partial}{\partial x_1}\left(\sum_{k,i} c_j^{i,k} \frac{\partial u_i}{\partial x_k} + \sum_i c_j^i u_i\right) = \sum_{k,i} c_j^{i,k} \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_1} u_i\right) + \sum_i c_j^i \left(\frac{\partial}{\partial x_1} u_i\right) = 0 \, .$$

# The action of the Lie derivative of a projective vector field on the space of solutions of the metrization equation

**Notation.** We denote the space of solutions of metrization equation by $Sol([\Gamma])$.

**Fact (Liouville 1889 in dim 2, Sinjukov 1959 in dim n), will be possibly explained later and was your homework:** $\dim\left(Sol([\Gamma])\right) \leq \frac{(n+1)(n+2)}{2} < \infty$.

Consider the linear mapping $\mathcal{L}_v : Sol([\Gamma]) \to Sol([\Gamma])$. Well-defined because by Claim above Lie derivative of a solution is a solution.

**Fact from linear algebra.** If $\dim\left(Sol([\Gamma])\right) \geq 2$, there exists a two-dimensional invariant subspace of $L_v$, we will work with this subspace and forget the rest.

By linear algebra, there exists a basis $\sigma, \bar\sigma \in Sol([\Gamma])$ such that in this basis the matrix of $\mathcal{L}_v$ is given by the following (real) Jordan normal form.

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

As we explained above, w.l.o.g. we may assume that $\sigma$ and $\bar\sigma$ are nondegenerate; then they correspond to some metrics.

Assume that the matrix of $\mathcal{L}_v$ is $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, and assume in addition that $\sigma, \bar{\sigma} \in (Sol([\Gamma]))$ corresponds to Riemannian metrics. Then, by the Dini Theorem the metrics $g$ and $\bar{g}$ corresponding to $\sigma$ and $\bar{\sigma}$ are given by

$$\boxed{\begin{aligned} g &= \begin{pmatrix} X(x) - Y(y) & \\ & X(x) - Y(y) \end{pmatrix} \\ \bar{g} &= \begin{pmatrix} \frac{X(x)-Y(y)}{X(x)Y(y)^2} & \\ & \frac{X(x)-Y(y)}{X(x)^2Y(y)} \end{pmatrix} \end{aligned}}$$

Observe that the projective vector field $v$ preserves the pencil of the solutions $\alpha\sigma + \beta\bar{\sigma}$, and therefore any object constructed by these solutions, in particular the lines of the coordinates $(x, y)$. Then, the vector field $v = (v^1(x), v^2(y))$. Now, from $\mathcal{L}_v\sigma = \lambda\sigma$ it follows that $v$ preserves the conformal structure of the metric $g$, so it is a holomorphic vector field. Then, it is constant or linear, i.e., up to a coordinate change and factor it is either

$$v = (x, y) \ \text{ or } \ v = (\mathrm{const}_1, \mathrm{const}_2).$$

In both cases the flow is given by precise formulas, which after some work give formulas for $X(x)$ and $Y(y)$ from case 1.1. of Theorem.

# Few words about the solution of the second problem of Lie

**Recall:** Difference between the first and the second problem: in the 1st problem we look for metrics with one projective vector field, and in the 2nd problem with many.

- One should of course check whether the metrics we obtained do not have another projective vector field.
  - This is an algorithmically doable problem assuming we can differentiate and arithmetic operations.
  - The algorithm is essentially due to Lie and is build in Maple and since the metrics in Theorem are explicit Maple can work with them and gives an answer.
- Then, the only additional problem to solve is to omit the assumption that their exists no Killing vector field. But if there exists a Killing vector field, one can use again projective invariance of the Killing equation (I do not go into details at this point)

**Historical Remark.** In this lecture I first solved the 1st problem, and then used it in the 2nd problems of Lie, historically first the 2nd problem was solved (Bryant, Manno, M$\sim$ 2006) and then the 1st (M$\sim$ 2008); but the solution of the 2nd without having the 1st is computationally quite hard.

# What Lie did not know? Why he did not solved his problems himself?

**Ingredients of the proof.**

- Local normal forms of projectively equivalent metrics? **Lie knew it**
- Linear algebra? **Lie understood it much better than we. In that time some people said Lienear algebra because of him**
- Quite big calculations (9 cases etc)? **Read any paper of Lie and see how good he was in calculations before asking such questions again.**

• He did not know the projective invariance of the metrization equation!!!

• Message of this lecture: projective invariance is important!!!

• In the next lecture we will construct another type of projectively invariant objects and prove a classical conjecture with their help

## Plan

- Tensor invariants of the projective structure: Weyl and Liouville tensors
- Proof of Lichnerowicz conjecture

# What are tensor invariants?

Tensor invariants of a projective structure are tensor fields canonically constructed by an affine connection in the projective structure such that they do not depend on the choice of affine connection within this projective structure.

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \phi_k \delta^i_j + \phi_j \delta^i_k. \qquad (*)$$

**Example.** Curvature and Ricci tensors are NOT tensor invariants. Indeed, if we replace a connection $\Gamma$ by the connection $\bar{\Gamma}$, then the direct calculations using the straightforward formula

$$R^m{}_{ikp} = \partial_k \Gamma^m{}_{ip} - \partial_p \Gamma^m{}_{ik} + \Gamma^a{}_{ip} \Gamma^m{}_{ak} - \Gamma^a{}_{ik} \Gamma^m{}_{ap}$$

give us the following relation between the curvature tensors of $\Gamma$ and $\bar{\Gamma}$:

$$\bar{R}^h{}_{ijk} = R^h{}_{ijk} + (\phi_{j,k} - \phi_{k,j}) \delta^h{}_i + \delta^h{}_k (\phi_{i,j} - \phi_i \phi_j) - \delta^h{}_j (\phi_{i,k} - \phi_i \phi_k).$$

Contracting this formula with respect to $h, k$, we obtain the following relation of the Ricci curvatures of $\Gamma$ and $\bar{\Gamma}$:

$$\bar{R}_{ij} = R_{ij} + (n-1)(\phi_{i,j} - \phi_i \phi_j) + \phi_{i,j} - \phi_{j,i}.$$

# Projective Weyl tensor.

It is the following tensor field:

$$W^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-1}\left(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}\right) + \frac{1}{n+1}\left(\delta^h{}_i R_{[jk]} - \frac{1}{n-1}\left(\delta^h{}_k R_{[ji]} - \delta^h{}_j R_{[ki]}\right)\right).$$

**Theorem (Weyl, Schouten).** Projective Weyl tensor is a tensor invariant of a projective structure, i.e. it does not depend on the choice of connection within the projective structure.

**Proof.** Substituting the formulas

$$\bar{R}^h{}_{ijk} = R^h{}_{ijk} + \left(\phi_{j,k} - \phi_{k,j}\right)\delta^h{}_i + \delta^h{}_k\left(\phi_{i,j} - \phi_i\phi_j\right) - \delta^h{}_j\left(\phi_{i,k} - \phi_i\phi_k\right).$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\left(\phi_{i,j} - \phi_i\phi_j\right) + \phi_{i,j} - \phi_{j,i}.$$

in the definition of the projective Weyl tensor changing the covariant derivative in $\Gamma$ by the covariant derivative in $\bar{\Gamma}$, we see after an hour of calculations that all $\phi$'s disappear.

# Liouville invariant

In dimension 2, Weyl tensor is necessary identically zero, since each $(1, 3)$ tensor with its symmetries is zero. Fortunately and exceptionally, there is one more tensor invariant in dimension 2:

**Theorem (Liouville 1889).** The tensor field $L_{ijk} := R_{ij,k} - R_{ik,j}$ is a tensor invariant in dimension 2.

**Proof.** Substituting

$$\boxed{\bar{R}_{ij} = R_{ij} + (n-1)\left(\phi_{i,j} - \phi_i\phi_j\right) + \phi_{i,j} - \phi_{j,i}.}$$

in the definition of $L$ and changing the covariant derivative in $\Gamma$ by the covariant derivative in $\bar{\Gamma}$ we again see that all terms containing $\phi$ disappear (assuming $n = 2$). □

**Remark.** There is a similar story in conformal geometry: conformal Weyl tensor vansihes for $n \leq 3$ but in dimension 3 there exists an additional conformal invariant and in dimension 2 conformal geometry is not interesting all. There is a deep explanation of this similarity and there are many results in $n + 1$ dimensional conformal geometry that are visually similar to results in n-dimensional projective geometry; we will not discuss in in this lecture course but just remember that many ideas from my course can be effectively used in the conformal geometry as well.

# How many essential components does $L_{ijk}$ have and when it vanishes?

**Theorem (Liouville 1889).** The tensor field $L_{ijk} := R_{ij,k} - R_{ik,j}$ is a tensor invariant in dim 2.

The tensor $L_{ijk}$ is skew-symmetric in $j, k$, assuming $n = \dim M = 2$ it implies that it has two essential components and can be written in the form $L = (L_1 dx^1 + L_2 dx^2) \otimes (dx^1 \wedge dx^2)$.

**Theorem.** Let $\nabla^g = (\Gamma^i_{jk})$ be the Levi-Civita connection of $g$ on 2-dim $M$. Then, $L_{ijk} \equiv 0$ if and only if $g$ has constant curvature.

**Proof.** It is well-known (and follow from the symmetries of the curvature tensor) that the 2-dim manifold are automatic Einstein in the sense that

$$R_{ij} = \frac{1}{2} R g_{ij}.$$

Calculating $L_{ijk}$ gives

$$L_{ijk} = R_{ij,k} - R_{ij,k} = \frac{1}{2}\left(R_{,k} g_{ij} - R_{,j} g_{ik}\right).$$

Since $g$ is nondegenerate, vanishing of $L$ implies vanishing of $dR$ and hence the constancy of the curvature. $\square$

**Remark.** We also see (or can easily check) that $L_{ijk} = dR \otimes (dx \wedge dy)$.

# $W \equiv 0$ implies constant curvature

**Theorem.** Let $\Gamma$ be the Levi-Civita connection of $g$ on $M$ with $n > 2$. Then, $W^h{}_{ijk} \equiv 0$ if and only if $g$ has constant sectional curvature.

**Proof.** For Levi-Civita connections the Ricci tensor is symmetric so the formula for $W$ reads

$$W^h{}_{ijk} = R^h{}_{ijk} - \tfrac{1}{n-1}\left(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}\right).$$

If $W \equiv 0$, we obtain

$$R^h{}_{ijk} = \tfrac{1}{n-1}\left(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}\right).$$

After lowing the index we have therefore

$$R_{hijk} = \tfrac{1}{n-1}\left(g_{hk} R_{ij} - g_{hj} R_{ik}\right).$$

We see that the left-hand-side is symmetric with respect to $(h, i, j, k) \longleftrightarrow (j, k, h, i)$, so should be the right-hand-side, which implies that $R_{ij}$ is proportional to $g_{ij}$, $R_{ij} = \frac{R}{n} g_{ij}$ so we have

$$R_{hijk} = \tfrac{R}{n(n-1)}\left(g_{hk} g_{ij} - g_{hj} g_{ik}\right)$$

which is equivalent to "sectional curvature is constant". $\square$

# By-product: Beltrami Theorem

**Theorem.** Let $\nabla^g = (\Gamma^i_{jk})$ be the Levi-Civita connection of $g$ on 2-dim $M$. Then, $L_{ijk} \equiv 0$ if and only if $g$ has constant curvature.

**Theorem.** Let $\Gamma$ be the Levi-Civita connection of $g$ on $M$ with $n > 2$. Then, $W^h_{ijk} \equiv 0$ if and only if $g$ has constant sectional curvature.

**Corollary (Beltrami Theorem; Beltrami 1865 for dim 2; Schur 1886 for dim¿2)**. A metric projectively equivalent to a metric of constant curvature has constant curvature.

# Nice result for today: projective Lichnerowicz conjecture

**Theorem.** Let $(M, g)$ be a compact Riemannian manifold such that the sectional curvature is not constant positive. Then, any projective vector field is a Killing vector field.

**Remark.** We have seen in Lecture 3 that the algebra of projective vector fields of the round sphere is $sl(n+1)$ and is bigger than the algebra of isometries which is $so(n+1)$.

**Remark.** We have also seen that in dimension 2 there are (local) metrics of nonconstant curvature admitting projective vector fields. One can construct similar examples in all dimensions. Theorem above says that these examples can not be extended to a closed manifold.

# Was a very popular conjecture

Special cases were proved before by French, Japanese and Soviet geometry schools.

| France (Lichnerowicz) | Japan (Yano, Obata, Tanno) | Soviet Union (Raschewskii) |
|---|---|---|
| Couty (1961) proved the conjecture assuming that $g$ is Einstein or Kähler | Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant | Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic and that $n \geq 3$. |

**Remark.** Stronger statements are also true:

- The statement remains true if one replaces "closed" by "complete", assumes in addition that the projective vector field is complete, and also allows flat metrics:

  **Theorem.** *On a compete Riemannian manifold such that its curvature is not nonnegative constant* $Proj_0 = Iso_0$.

- One can show that on closed manifolds $|Proj/Iso| \leq 2n$ (Zeghib 2014). Actually, one can even slightly improve the result:

  **Theorem (obtained in plane from Munich to Athen).** On closed manifolds such that the curvature is not positive constant $|Proj/Iso| \leq 2$

**A difficulty of dimensions $n \geq 3$ which I avoid by additional assumption.**
In dimension 2, in the solution of the 1st Lie problem, we assumed w.l.o.g. that $\dim(Sol([\Gamma]) = 2$.

The argument was: there exists a 2-dimensional invariant subspace of $Sol([\Gamma]$ and if the solutions from this subspace are degenerate there exists a Killing vector field.

The latter arguments does not work in dimensions $\geq 3$, but still we may assume that $\dim(Sol([\Gamma]) \leq 2$ because of the following nontrivial theorem which will not be proved in this lecture. I will posssibly touch it in the 5th lecture.

**Theorem (M$\sim$ 2003).** On a closed Riemannnian manifold such that its sectional curvature is not constant positive, $\dim\left(Sol([\Gamma])\right) \leq 2$.

# Plan of the proof of the Lichnerowicz conjecture.

**Setup.**

- Our manifold is closed and Riemannian.
- The projective structure of the metric admits a projective vector field.
- We assume that $\dim\left(Sol([\Gamma])\right) \leq 2$.
- Our goal is to show that this vector field is a Killing vector field unless $g$ has constant sectional curvature

## The case $2 > \dim(Sol([\Gamma]) \quad = 1$

**If** $\dim\left(Sol([\Gamma])\right) = 1$, every two projective related metrics are proportional. Then, a projective vector field $v$ is a homothety vector field. Since our manifold is closed, every homothety is isometry so our vector field is a Killing vector field as we want.

# The case $\dim(Sol([\Gamma]) = 2$

**Important observation already used in the solution of Lie problems.** $L_v : Sol([\Gamma]) \to Sol([\Gamma])$, where $L_v$ is the Lie derivative.

After appropriate choice of a basis in $Sol([\Gamma])$, we obtained that the $v$-Lie derivative $\sigma, \bar{\sigma}$ are given by

$$\left[\begin{array}{lll} L_v\sigma & = \lambda\sigma & \\ L_v\bar{\sigma} & = & \mu\bar{\sigma} \end{array}\right] \quad \left[\begin{array}{lll} L_v\sigma & = \lambda\sigma & +\mu\bar{\sigma} \\ L_v\bar{\sigma} & = -\mu\sigma & +\lambda\bar{\sigma} \end{array}\right] \quad \left[\begin{array}{lll} L_v\sigma & = \lambda\sigma & +\bar{\sigma} \\ L_v\bar{\sigma} & = & \lambda\bar{\sigma} \end{array}\right].$$

Thus, the evolution of the solutions along the flow $\Phi_t$ of $v$ is

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar{\sigma} & = & e^{\mu t}\bar{\sigma} \end{array}\right]$$

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & +e^{\lambda t}\sin(\mu t)\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = -e^{\lambda t}\sin(\mu t)\sigma & +e^{\lambda t}\cos(\mu t)\bar{\sigma} \end{array}\right]$$

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\sigma & +te^{\lambda t}\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = & e^{\lambda t}\bar{\sigma} \end{array}\right].$$

We will consider all these three cases separately.

The simplest case is when the evolution is given by

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & + e^{\lambda t}\sin(\mu t)\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = -e^{\lambda t}\sin(\mu t)\sigma & + e^{\lambda t}\cos(\mu t)\bar{\sigma} \end{array}\right].$$

Suppose our metric corresponds to the element $a\sigma + b\bar{\sigma}$.
Its evolution is given by

$$\begin{aligned} \Phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t}\cos(\mu t)\sigma + e^{\lambda t}\sin(\mu t)\bar{\sigma}) \\ &\quad + b(-e^{\lambda t}\sin(\mu t)\sigma + e^{\lambda t}\cos(\mu t)\bar{\sigma}) \\ &= e^{\lambda t}\sqrt{a^2 + b^2}(\cos(\mu t + \alpha)\sigma + \sin(\mu t + \alpha)\bar{\sigma}), \end{aligned}$$

where $\alpha = \arccos(a/(\sqrt{a^2 + b^2}))$.

Now, we use that the metric is Riemannian. Then, for any point $x$ there exists a basis in $T_xM$ such that $\sigma$ and $\bar{\sigma}$ are given by diagonal matrices: $\sigma = diag(s_1, s_2, ...)$ and $\bar{\sigma} = diag(\bar{s}_1, \bar{s}_2, ...)$.

Then, $\Phi_t^*(a\sigma + b\bar{\sigma})$ at this point is also diagonal with the $i$th element $e^{\lambda t}\sqrt{a^2 + b^2}(\cos(\mu t + \alpha)s_i + \sin(\mu t + \alpha)\bar{s}_i)$.

Clearly, for a certain $t$ we have that $\Phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption, $\square$

The proof is is similar when the evolution is given by

$$\left[\begin{array}{lcc} \Phi_t^*\sigma & = e^{\lambda t}\sigma & + te^{\lambda t}\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = & e^{\lambda t}\bar{\sigma} \end{array}\right].$$

We again suppose that our metric corresponds to the element $a\sigma + b\bar{\sigma}$.

Its evolution is given by

$$\begin{aligned} \Phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t}\sigma + e^{\lambda t}t\bar{\sigma}) + b(e^{\lambda t}\bar{\sigma}) \\ &= e^{\lambda t}(a\sigma + (b + at)\bar{\sigma}). \end{aligned}$$

We again see that unless $a \neq 0$ there exists $t$ such that $\Phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption.

Now, if $a = 0$, then $g$ corresponds to $\bar{\sigma}$ and $v$ is its Killing vector field, $\square$

The most complicated case is when the evolution is given by the matrix

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar{\sigma} & = & e^{\mu t}\bar{\sigma} \end{array}\right]. \tag{2}$$

The case $\lambda = \mu$ is trivial, in this case the projective vector field is homothety vector field. We assume $\lambda > \mu$.

We may assume that $g$ corresponds to the solution $\sigma + \bar{\sigma}$.
Consider, for each $t \in \mathbb{R}$, the $(1,1)$-tensor

$$A_t = (\sigma + \bar{\sigma})^{-1} \Phi_t^* (\sigma + \bar{\sigma}) = (\sigma + \bar{\sigma})^{-1} (e^{\lambda t} \sigma + e^{\mu t} \bar{\sigma}).$$

Take a point $p$ and consider a basis such that

$$g = \mathrm{diag}(1, ..., 1), \quad \sigma = \mathrm{diag}(s_1, ..., s_n), \bar{\sigma} = \mathrm{diag}(\bar{s}_1, ..., \bar{s}_n)$$

(Since $\sigma + \bar{\sigma}$ corresponds to $g$, we have $\bar{s}_i = 1 - s_i$).

In this basis, we have

$$A_t = \mathrm{diag}(s_1 e^{\lambda t} + \bar{s}_1 e^{\mu t}, ...).$$

Next, for each $t \in \mathbb{R}$, consider the tensor

$$G_t = g^{-1} \Phi_t^* g.$$

Because of the relation $g^{-1} = \sigma \, |\det(\sigma)|$ (see Lecture 2), we have

$$G_t = \mathrm{diag}\left( \frac{1}{(s_1 e^{\lambda t} + \bar{s}_1 e^{\mu t}) \prod_i (s_i e^{\lambda t} + \bar{s}_i e^{\mu t})}, ... \right).$$

$$\boxed{G_t = \mathrm{diag}\left(\frac{1}{(s_1 e^{\lambda t}+\bar s_1 e^{\mu t})\prod_i (s_i e^{\lambda t}+\bar s_i e^{\mu t})}, \dots\right).}$$

Let us assume for simplicity that all $s_i, \bar s_i \neq 0$. Since $\lambda > \mu$,

$$G_t \overset{t\to+\infty}{\longrightarrow} \mathrm{diag}(e^{-(n+1)\lambda t}, \dots) \text{ and } G_t \overset{t\to-\infty}{\longrightarrow} \mathrm{diag}(e^{(n+1)\mu t}, \dots). \qquad (\star)$$

Consider now the function $f = (|W|_g)^2 = W^i{}_{jk\ell} g_{ii'} g^{jj'} g^{kk'} g^{\ell\ell'} W^{i'}{}_{j'k'\ell'}$. It is a smooth function on the manifold. At points such that $W \neq 0$ we have $f(p) \neq 0$.

Since $\Phi_t^*(g) = gG_t$ and because of $(\star)$ we have that $\Phi_t^*(g)$ has asymptotic $e^{-2(n+1)\lambda t}$ for $t \to +\infty$ and $e^{-2(n+1)\mu t}$ for $t \to -\infty$.

Now, BECAUSE $W$ IS PROJECTIVELY INVARIANT, $\Phi_t^* W = W$. Thus, for $t \to \infty$,

$$f(\Phi_t(p)) = \Phi_t^* f(p) = |\Phi_t^* W|^2_{\Phi_t^* g} \sim \mathrm{const}\, e^{2(n+1)\lambda t}$$

(where const $= 0$ iff $W = 0$)

and for $t \to -\infty$ we have $f(\Phi_t(p)) \sim \mathrm{const}\, e^{-2(n+1)\mu t}$.

$$f(\Phi_t(p)) = \Phi_t^* f(p) = |\Phi_t^* W|^2_{\Phi_t^* g} \sim \text{const}\, e^{2(n+1)\lambda t}$$

(where const $= 0$ iff $W = 0$) and for $t \to -\infty$ we have

$$f(\Phi_t(p)) \sim \text{const}\, e^{-2(n+1)\mu t}$$

We see that if $W(p) \neq 0$ then the smooth function $f$ on a compact manifold is unbounded, which gives a contradiction. $\square$

**Remark.** We had an additional assumption: all $s_i \neq 0$. It is not essential, one simply should be slightly more careful.

**Remark.** In the 2 dim case one should replace $W$ by the Liouville invariant $L_{ijk}$.

# Summary of the proof of the projective Lichnerowicz conjecture

**Theorem (Lichnerowicz conjecture).** Let $(M,g)$ be a compact Riemannian manifold such that the sectional curvature is not constant positive. Then, any projective vector field is a Killing vector field.

- We assumed in addition that $\dim(Sol([\Gamma])) = 2$ and justified this assumption by certain fact we did not prove.
- Then, we used the invariance of the metrization equation and obtained that the evolution of the solutions along the flow of the projective vector field is given by one of the three cases:

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar\sigma & = & e^{\mu t}\bar\sigma \end{bmatrix}, \begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & +e^{\lambda t}\sin(\mu t)\bar\sigma \\ \Phi_t^*\bar\sigma & = -e^{\lambda t}\sin(\mu t)\sigma & +e^{\lambda t}\cos(\mu t)\bar\sigma \end{bmatrix}, \begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & +te^{\lambda t}\bar\sigma \\ \Phi_t^*\bar\sigma & = & e^{\lambda t}\bar\sigma \end{bmatrix}.$$

- In all three cases some geometrically constructed (and therefore continuous ) function is unbounded which can not happen on a closed manifold: in the blue and **black** cases it $\frac{\det(g)}{\det(\Phi_1^* g)}$. In the red case the function is $f = (|W|_g)^2 = W^i{}_{jk\ell} g_{ii'} g^{jj'} g^{kk'} g^{\ell\ell'} W^{i'}{}_{j'k'\ell'}$. It is unbounded unless $W \equiv 0$. In the proof we have used that $W$ is projectively invariant, and that $W = 0$ implies constant curvature.

# A bit of philosophy

Felix Klein, 1873, Vergleichende Betrachtungen über neuere geometrische Forschungen

**Problem I:** *Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickle die auf die Gruppe bezugliche Invariantentheorie.*

**English Translation.** Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group.

In our case we had a manifold, a group of projective transformations, a projective invariants for them, and used them to prove Lichnerowicz conjecture.

In the previous lecture we also had projectively invariants objects, and they were extremely effective.

Wait for new projectively-invariant objects in the 5th lecture!

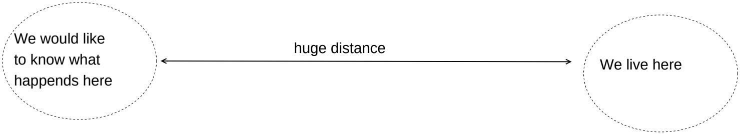
# Lecture 5

## Plan

- Nice result for today (Weyl-Ehlers problem and $\dim(Sol[\Gamma]) \leq 2$ for compact manifolds)
- Petrov's solution of the simplest (Riemannian) version of the Weyl-Ehlers problem
- Conification and its application:
  - first metric projectively equivalent objects
  - proofs of the results announced above.

Sorry, this time one important statement comes as "black boxe" (=no proof), I will still try to explain the effects and give the precise references.

Suppose we would like to understand the structure of the space-time (i.e., a 4-dimensional metric of Lorenz signature) in a certain part of the universe.



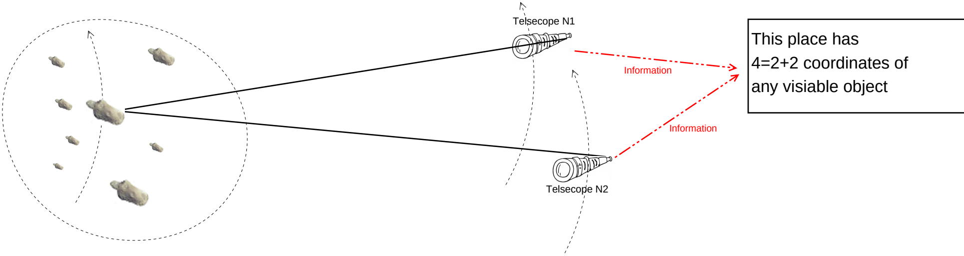

We assume that this part is far enough so the we can use only telescopes (in particular we can not send a space ship there).

We still assume that the telescopes can see sufficiently many objects in this part of universe.

Then, if the relativistic effects are not negligible (that happens for example is the objects in this part of space time are sufficiently fast or if this region of the universe is big enough),
**we obtain as a rule the world lines of the objects as unparameterized curves.**

# In many cases, we do can get unparameterized geodesics with the help of astronomic observations

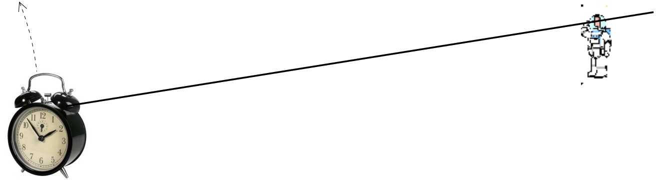
One can obtain unparameterized geodesics by observation:




We take 2 freely falling observers that
measure two angular coordinates of the visible objects
and send this information to one place. This place will have
4 functions angle(t) for every visible object
which are in the generic case 4 coordinates of the object.

# In many cases, the only thing one can get by observations are unparametrised geodesics

If one can not register a periodic process on the observed body, one can not get the own time of the body



This situation is extremely rare

# Problem 1. How to reconstruct a metric by its unparameterized geodesics?

**The mathematical setting:** *We are given a projective structure given as in inefficient definition from Lecture 1, as a family* $\gamma(t;\alpha)$ *in* $U \subseteq \mathbb{R}^4$*; we assume that the family is sufficiently big in the sense that* $\forall x_0 \in U$

$$\Omega_{x_0} := \{\xi \in T_{x_0}U \mid \ \exists \alpha \ \textit{and} \ \exists t_0 \ \textit{with} \ \tfrac{d}{dt}\gamma(t;\alpha)_{|t=t_0} \ \textit{is proportional to} \ \xi\}$$

*contains an open subset of* $T_{x_0}U$*.*

*We need to find a metric* $g$ *such that all* $\gamma(t;\alpha)$ *are reparameterized geodesics.*

The problem was explicitly stated by the famous physists

**Jürgen Ehlers 1972**, who said that "We reject clocks as basic tools for setting up the space-time geometry and propose ... freely falling particles instead. We wish to show how the full space-time geometry can be synthesized ... . Not only the measurement of length but also that of time then appears as a derived operation."

# Problem 1 can be naturally divided in two subproblems

**Subproblem 1.1.** *Given a family of curves $\gamma(t; a)$, how to understand whether these curves are reparameterised geodesics of a certain affine connection? How to reconstruct this connection effectively?*

In Lecture 1, we considered a two-dimensional analog of this problem, and have seen that 4 geodesics at any points allows us to construct by linear algebraic manipulations the coefficients of an affine connection in the projective class.

Now we do the same in any dimension using the same ideas

**Subproblem 1.2.** *Given an affine connection $\Gamma = \Gamma^i_{jk}$, how to understand whether there exists a metric $g$ in the projective class of $\Gamma$? How to reconstruct this metric effectively?*

We know that the existence of the metric is equivalent to the existence of a nondegenerate solution of the metrization equation; the input of this lecture is few tricks that help.

## Problem 2 (implicitely, Weyl). In what situations interesting for physics the reconstruction of a metric by the unparameterised geodesics is unique (up to the multiplication of the metric by a constant)?

In other words, what metrics 'interesting' for relativity allow nontrivial projective equivalence?

We already have seen (Lagrange example) that constant curvature metrics have many projectively equivalent metrics (all having constant curvature). Let us construct one more example interesting for physics.

**Example.** The so-called Friedman-Lemaitre-Robertson-Walker metric

$$g = -dt^2 + R(t)^2 \frac{dx^2 + dy^2 + dz^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)} \ ; \quad \kappa = +1; 0; -1,$$

is not projectively rigid.

Indeed, $\forall c$ the metric

$$\bar{g} = \frac{-1}{(R(t)^2 + c)^2} dt^2 + \frac{R(t)^2}{c(R(t)^2 + c)} \frac{dx^2 + dy^2 + dz^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)}$$

is geodesically equivalent to $g$ (essentially Levi-Civita 1896; repeated by many relativists (Nurowski, Gibbons et al, Hall) later).

One can of course check that the metrics are projectively equivalent by direct calculations.

# Goals: physically interesting projectively rigid examples

**Theorem (Petrov 1961).** Let $g$ and $\bar{g}$ be two projectively equivalent Ricci-flat 4 dim metrics (of arbitrary siganture). Then, they are flat or proportional.

I will give a proof of this results in the Riemannian case, this will be easy and requires no theory. All other results will need some additional results which I will introduce as black boxes (=without proofs).

**Theorem (Kiosak-M$\sim$ 2009).** *Let $g$ and $\bar{g}$ be projectively equivalent 4 dim metrics of arbitrary signature. Assume $g$ is Einstein (i.e., $Ricc = \frac{Scal}{4} g$). Then, Levi-Civita connections of $g$ and $\bar{g}$ coincide unless metrics have constant curvature.*

There exist counterexamples in higher dimensions. By the next theorem, counterexamples are local.

**Theorem (Kiosak-M$\sim$ 2012 + Mounoud-M$\sim$ 2013).** *Let $g$ and $\bar{g}$ be projectively equivalent metrics of arbitrary signature on a compact manifold of dimension $\geq 2$. Assume $g$ is Einstein. Then, Levi-Civita connections of $g$ and $\bar{g}$ coincide unless metrics have constant curvature.*

# I will also explain the statement we have used in the proof of the Lichnerowciz conjecture

**Theorem (Kiosak - M$\sim$ 2012+ M$\sim$-Mounoud 2013) (Riemannian case: M$\sim$ 2003).** On a closed manifold of arbitrary curvature such that its sectional curvature is not constant, $\dim(Sol([\Gamma])) \leq 2$.

# Algorithm how to reconstruct $\Gamma$ by sufficiently many geodesics

Repeat: $\frac{d^2\gamma^a}{dt^2} + \Gamma^a_{bc}\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = f\left(\frac{d\gamma}{dt}\right)\frac{d\gamma^a}{dt}$. $\quad (*)$

Take a point $x_0$; our goal it to reconstruct $[\Gamma(x_0)^i_{jk}]$. Take $\gamma(t_0;\alpha)$ such that $\gamma(t_0;\alpha) = x_0$ and the first component $\left(\frac{d\gamma^1}{dt}\right)_{|t=t_0} \neq 0$. For $\gamma(t_0;\alpha)$, we rewrite the equation $(*)$ at $t = t_0$ in the following form:

$$\begin{array}{rcl} f\left(\frac{d\gamma}{dt}\right) & = & \left(\frac{d^2\gamma^1}{d^2t} + \Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt}\right)/\frac{d\gamma^1}{dt} \\ \frac{d\gamma^2}{dt}\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\Gamma^2_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^2}{d^2t}\frac{d\gamma^1}{dt} - \frac{d\gamma^2}{dt}\frac{d^2\gamma^1}{d^2t} \\ & \vdots & \\ \frac{d\gamma^n}{dt}\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\Gamma^n_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^n}{d^2t}\frac{d\gamma^1}{dt} - \frac{d\gamma^n}{dt}\frac{d^2\gamma^1}{d^2t}. \end{array} \tag{3}$$

The first equation of (3) is equivalent to the equation $(*)$ for $a = 1$ solved with respect to $f\left(\frac{d\gamma}{dt}\right)$. We obtain the second, third, etc, equations of (3) by substituting the first equation of (3) in the equations $(*)$ with $a = 2, 3, \mathrm{etc.}$

Note that the subsystem of (3) containing the the second, third, etc. equations of (3) does not contain the function $f$ and is therefore a linear system on $\Gamma^i_{jk}$.

$$\begin{array}{rcl}
\frac{d\gamma^2}{dt}\,\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\,\Gamma^2_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^2}{d^2t}\,\frac{d\gamma^1}{dt} - \frac{d\gamma^2}{dt}\,\frac{d^2\gamma^1}{d^2t} \\
& \vdots & \\
\frac{d\gamma^n}{dt}\,\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\,\Gamma^n_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^n}{d^2t}\,\frac{d\gamma^1}{dt} - \frac{d\gamma^n}{dt}\,\frac{d^2\gamma^1}{d^2t}.
\end{array} \qquad (3')$$

Then, for every 'geodesic' $\gamma(t_0,\alpha)$ gives us $n-1$ linear (inhomogeneous) equations on the components $\Gamma(x_0)^i_{jk}$. We take a sufficiently big number $N$ (if $n=4$, it is sufficient to take $N=12$) and substitute $N$ generic geodesics $\gamma(t;\alpha)$ passing through $x_0$ in this subsystem.

At every point $x_0$, we obtain an inhomogeneous linear system of equations on $\frac{n^2(n+1)}{2}$ unknowns $\Gamma(x_0)^i_{jk}$.

*In the case the solution of this system does not exist (at least at one point $x_0$), there exists no connection whose (reparameterized) geodesics are $\gamma(t;\alpha)$. In the case it exists, the solution of the system above gives us the projective class of the connection.*

# Proof of Petrov's result for Riemannian metrics

**Theorem (Petrov 1961).** Let $g$ and $\bar{g}$ be two projectively equivalent Riemannian 4 dim Ricci-flat metrics. Then, they are flat or proportional.

**Proof.** Consider the projective Weyl tensor

$$\boxed{W^i{}_{jk\ell} := R^i{}_{jk\ell} - \frac{1}{n-1}\left(\delta^i_\ell\, R_{jk} - \delta^i_k\, R_{j\ell}\right)}$$

We know (Lecture 4) the the projective Weyl tensor does not depend of the choice of metric within the projective class.

Now, from the formula for the Weyl tensor, we know that, if the searched $\bar{g}$ is Ricci-flat, projective Weyl tensor coincides with the Riemann tensor $\bar{R}^i{}_{jk\ell}$ of $\bar{g}$. Thus, if we know the projective class of the Ricci-flat metric $\bar{g}$, we know its Riemann tensor.

Then, the metric $\bar{g}$ must satisfy the following system of equations due to the symmetries of the Riemann tensor:

$$\begin{cases} \bar{g}_{ia} W^a{}_{jkm} + \bar{g}_{ja} W^a{}_{ikm} = 0 \\ \bar{g}_{ia} W^a{}_{jkm} - \bar{g}_{ka} W^a{}_{mij} = 0 \end{cases} \tag{4}$$

The first portion of the equations is due to the symmetry ($\bar{R}_{ijkm} = -\bar{R}_{jikm}$), and the second portion is due to the symmetry ($\bar{R}_{kmij} = \bar{R}_{ijkm}$) of the curvature tensor of $\bar{g}$.

We see that for every point $x_0 \in U$ the system (4) is a system of linear equations on $\bar{g}(x_0)_{ij}$. The number of equations (around 100) is much bigger than the number of unknowns (which is 10). It is expected therefore, that a generic projective Weyl tensor $W^i{}_{jkl}$ admits no more than one-dimensional space of solutions (by assumtions, our $W$ admits at least one-dimensional space of solutions). The expectation is true, as the following classical result shows

**Theorem (Folklore – Petrov, Hall, Rendall, Mcintosh)** Let $W^i{}_{jk\ell}$ be a tensor in $\mathbb{R}^4$ such that it is skew-symmetric with respect to $k, \ell$ and such that its traces $W^a{}_{ak\ell}$ and $W^a{}_{ja\ell}$ vanish. Then, if $W \neq 0$, two positively definite solutions of the equations (4) are proportional.

By this theorem, the metrics $g$ and $\bar{g}$ are conformally equivalent which by result of Weyl implies that they are proportional

# The trick in proof of Petrov's result simplifies the reconstruction of the metric

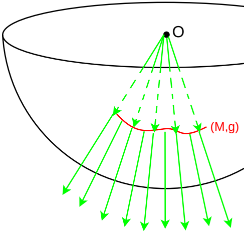
We have seen that (in dim 4) given $[\Gamma]$ which is not flat we can construct $W$ and in the Riemannian case or under additional mild assumptions on $W$ the conformal class of the metric $g$.

Then, the all (there are 20 of them) metrization equations are equations on ONE unknown function and one can be solved by integration of an explicitly given 1-form (M$\sim$-Trautmann 2014).

# Main tool in the proof of all others theorem: conification

Let $M$ be a manifold and $g = \begin{pmatrix} & & \\ & g_{ij} & \\ & & \end{pmatrix}$ a Riemannian metric on it. The cone over this manifold is a manifold $\mathbb{R}_{>0} \times M$ with the metric $dt^2 + t^2 g = \left(\begin{array}{c|c} 1 & \\ \hline & t^2 g_{ij} \end{array}\right)$

**Geometric picture behind the word "conification"**




Think that the manifold $M^n$ (or arbitrary dimension $n$) is imbedded in the sphere (of arbitrary dimension $N \geq n$) and carries the induced metric $g$. Consider the union of all rays connecting the origin of the sphere with the points of $M$. Then, this union is a $n+1$-dimensional manifold, and the restriction of the standard Euclidean metric to it is the cone metric.

# Metrization equations in the presence of metric

Recall that the solutions $\sigma^{ij}$ of the metrization equations are weighted tensors.
In the case we work with the Levi-Civita connection of a metric, we can choose $Vol_g$ as the reference volume form. Then, weighted tensors can be viewed as tensors. Since the volume form is parallel, the covariant derivatives is the usual covariant derviative of tensor fields, and metrization equations can be rewritten as below:

**Theorem (Sinjukov 1962).** Let $g$ be a metric. The metrics $\bar{g}$ that are projectively equivalent to $g$ are in one-to-one correspondence with the solutions of the following system of PDE on the (0,2)-tensorfield $a = a_{ij}$ and (0,1)-tensorfield $\lambda_i$ such that $\det(a) \neq 0$ at all points:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \qquad (*)$$

The one-to-one correspondence is given by

$$\bar{g} \longrightarrow \left( a = \left( \frac{\det(\bar{g})}{\det(g)} \right)^{\frac{1}{n+1}} g\bar{g}^{-1}g, \lambda = \tfrac{1}{2} d\,\mathrm{trace}_g(a) \right).$$

# Black-Box statement

**Thm (Kiosak-M$\sim$ 2011/2013).** Let $g$ be a metric on an $n \geq 3$-dimensional connected manifold such that $\dim Sol \geq 3$ or $g$ is Einstein.
Then, there exists a constant $B$ such that for any solution $(a, \lambda)$ of the equations $a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}$ there exists a function $\mu$ such that in addition the following two equations are fulfilled:

$$\begin{array}{rcl} \lambda_{i,j} & = & \mu g_{ij} - B a_{ij} \\ \mu_{,i} & = & 2B\lambda_i . \end{array}$$

In the case $g$ is Einstein $B = \frac{Scal}{n(n-1)}$

**Remark.** Proof is technically nontrivial, but standard: We went until 5th prolongation in the Cartan-Kähler prolongation procedure to prove it.

**Remark.** The constant $B$ depends on the metric but is the same for all solutions $(a, \lambda)$. We assume $B \neq 0$, in this case one can make it 1 by scaling the metric.

# Projectively equivalent metrics as parallel (0,2)-tensors on the cone.

**What is explained on the previous slide.** We assume that $n = dim(M) \geq 3$ and $\dim Sol \geq 3$ or $g$ is Einstein. Then, metrics $\bar{g}$ that are projectively equivalent to $g$ are in one-to-one correspondence to the solutions $(a, \lambda, \mu)$ of the following system of equations

$$\boxed{\begin{array}{rcl} a_{ij,k} & = & \lambda_i g_{jk} + \lambda_j g_{ik} \\ \lambda_{i,j} & = & \mu g_{ij} - a_{ij} \\ \mu_{,i} & = & 2\lambda_i \end{array}}$$

**Principal Observation.** The solutions of these equations are in one-to-one correspondence with the parallel symmetric $(0,2)$-tensors on the cone $(\hat{M}, \hat{g}) = (\mathbb{R}_{>0} \times M, dt^2 + t^2 g)$.

The one-to-one correspondence is given by

$$(a, \lambda, \mu) \mapsto A := \begin{pmatrix} \mu & -t \cdot \lambda_1 & \dots & -t \cdot \lambda_n \\ -t \cdot \lambda_1 & t^2 \cdot a_{11} & \dots & t^2 \cdot a_{1n} \\ \vdots & \vdots & & \vdots \\ -t \cdot \lambda_n & t^2 \cdot a_{n1} & \dots & t^2 \cdot a_{nn} \end{pmatrix}.$$

**Proof is an easy exercise** – write down the Levi-Civita connection of the cone metric $dt^2 + t^2 g$ and see that the condition that $A$ is parallel is equivalent to the above equations on $a, \lambda, \mu$.

I do not have any geometric explanation of this phenomenon.

**Principal Observation.** Under our assumptions, metrics projectively equivalent to $g$ are essentailly the same as parallel symmetric (0,2)-tensors on $(\hat{M}, \hat{g}) = (\mathbb{R}_{>0} \times M, dt^2 + t^2 g)$.

**Corollary.** Levi-Civita connection of $(\hat{M}, \hat{g})$ does not depend on the choice of the metric $g$ in the projective class.

**This is metrically projecively invariant object!** In order to construct it, we need the metric, but it does not depend on the choice of the metric in the projective class.

**Corollary.** The projection to $M$ of the Riemannian curvature and of the Ricci tensor to the manifold is projectively invariant.

On the manifold $M$, these projections looks as follows:

$$Z^i{}_{jk\ell} := R^i{}_{jk\ell} + \left(\delta^i{}_\ell g_{jk} - \delta^i{}_k g_{j\ell}\right) = R - K,$$

where $K$ is the "algebraic constant curvature tensor"; and the projection of $Ricc$ is its trace. For Einstein metrics, this gives nothing new, since in the Einstein case $Z$ is the projective Weyl tensor, and its trace vanishes. But in the case $\dim Sol \geq 3$ is does give new invariants.

These are again metric projective invariants!

# Cones over compact manifolds do not have parallel (0,2)-tensors

**Theorem ( Tanno 1978, Obata 1978, M$\sim$- Mounoud 2013)**. Non flat cones over closed manifolds do not admit parallel symmetric $(0,2)$ tensors nonproportional to the metrics.

This Theorem proves all announced above Theorems.
**Proof in the Riemannian case.** The existence of nontrivial parallel tensor implies the existence of the local decomposition $\hat{g} = \hat{g}_1 + \hat{g}_2$.

Since cone merics admit homotheties, the metrics $\hat{g}_1$ and $\hat{g}_2$ admit homotheties. One can show the existence of a stable point which by blow up argument of Gromov implies that they are flat $\square$