# Around Model Theory Around Free Groups

## And Around That

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# Formulae

- Langage of groups: $\cdot$, $^{-1}$, 1.
- Group equations with variables $x$
- Finite sentences: And, Or, Not, $\forall$, $\exists$.

No free variables: sentences

Free variables: $->$ definable sets

May allow parameters

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# Definable sets

Fix a group $G$ $(\cdot, \ {}^{-1}, 1)$.

The truth of a sentence is naturally defined.

$$\varphi(x, a)$$

**Definition**

*The set of tuples $g$ of $G$ such that $\varphi(g, a)$ is true is definable.*
*(by the formula $\varphi(x, a)$ with parameters $a$)*

Quantifier elimination?

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- **Exemples of sentences.**

  Axioms of groups.

  Commutativity.

  Bounded simplicity.

- **Exemples of definable sets.**

  Center. Commutators (but not derived subgroups).

  Squares, cubes, etc...

  $C(a)$, $a^G$, translates, etc...

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# Definition

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*For $\Phi$ a set of sentences (or an elementary theory $T$), the $\Phi$-elementary class is the set of groups satisfying all $\varphi$ in $\Phi$.*

- $\Phi$ consistent: The $\Phi$-elementary class is not empty (*cptness*).
- Groups in the $\Phi$-elementary class: Models of $\Phi$.
- $\Phi$ complete: Consistent + Maximal.
- Elementary equivalence: Same complete theory.

**Tarski Problem:** Are $F_n$, $F_m$ elementary equivalent for $n, m \geq 2$?

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**Definition**

*A theory $T$ has quantifier elimination if every formula $\varphi(x)$ is equivalent modulo $T$ to a quantifier-free formula.*

**Fact (Tarski - Chevalley)**

*Algebraically closed fields have quantifier elimination.*

**Fact**

*Abelian groups eliminate up to boolean combination of cosets.*

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# CSA-groups

**Definition (CSA-groups)**

*Maximal abelien subgroups are malnormal.*

**Fact**

$\{CSA\} = \{Centralizers\ are\ abelian\ and\ selfnormalizing\}$

In particular: $CSA$ is an elementary class (universal axioms).

$\{\text{Free gps}\} \subseteq \{\text{torsion-free hyperbolic gps}\} \subseteq \{\text{CSA-gps}\}$

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# More elementary properties

Universal axioms: For $p$ prime, no elementary abelian $p$-group of order $p^{n+1}$.

**Fact**

*The class of CSA-groups with* fixed *rank of maximal abelian p-subgroup and without involutions is closed under:*

- *Free products with* 1-malnormal *amalgamated subgroup.*
- *HNN-extensions on* malnormal separated *subgroups.*

**Corollary (Ould Houcine)**

*Existentially closed CSA-group with* fixed *rank of maximal abelian p-subgroup and without involutions are divisible, wih conjugate maximal abelian subgroups, and boundedly simple.*

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# Definitions

$G_1 \leq G_2$: $G_1$ is a subgroup of $G_2$.

**Example**

- $F_2 \simeq \langle a, b \mid\mid \rangle \leq \langle a, b, r \mid\mid r^n = a \rangle \simeq F_2$
- $F_2 \simeq \langle a, b \mid\mid \rangle \leq \langle a, b, t \mid\mid a^t = b \rangle \simeq F_2$

The most favorable case:

**Definition**

*$G_1 \preceq G_2$ is an elementary extension if $G_1 \leq G_2$ and for every formula $\varphi(x)$ and $g_1$ in $G_1$, $\varphi(g_1)$ true in $G_1$ implies $\varphi(g_1)$ true in $G_2$.*

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# Chains

**Fact (Tarski's test)**

*$G_1 \preceq G_2$ iff for every formula $\varphi(x, y)$ ($y$ 1-uple) and $g_1$ in $G_1$, if $G_2$ satisfies $\exists y \varphi(g_1, y)$ then $\varphi(g_1, \gamma)$ for some $\gamma$ in $G_1$.*

**Fact (Union of chains)**

*Let $(G_i)_{i<\gamma}$ s.t. $G_i \preceq G_j$ whenever $i \leq j$. Then $G_i \preceq \bigcup_{i<\gamma} G_i$.*

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# Generators and relations

A group $G$ is always given by generators $A$ and relations $R$.

$$G = \langle A \mid\mid R \rangle$$

**Fact (Gromov)**

*An arbitrarily chosen finitely presented group is hyperbolic with probability almost one.*

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# Possible relations

$\Gamma$ a (possibly oriented) irreflexive graph on $n$ vertices

**Theorem (Muranov Neman)**

*For most group words $w(x, y)$, the group*

$$\langle a_1, \cdots, a_n \mid\mid w(a_i, a_j);\ \Gamma(a_i, a_j)\rangle$$

*is torsion-free hyperbolic and $w(a_i, a_j) = 1$ iff $\Gamma(a_i, a_j)$.*

**Proof:** $C'(1/6)$, hyperbolicity, asphericity. □

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# Complexity

## Corollary

*Fix an arbitrary group word $w(x, y)$. Then for every infinite set of finite graphs $\Gamma_k$, there exists a torsion-free hyperbolic group $G_k$ with elements $a_1$, ..., $a_k$ s.t.*

$$w(a_i, a_j) = 1 \textit{ iff } \Gamma(a_i, a_j)$$

Maximal local complexity of {t.f. hyperbolic groups}.

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# Shelah

Approach by model-theoretic complexity of definable sets. (usually for an elementary class).

Independence graphs: Finite *bipartite* graphs $\Gamma_n$ on $n + 2^n$ elements coding the powerset of a set of $n$ elements (Vapnik-Chervonenkis).

**Definition**

*A formula $\varphi(x, y)$ has the Independence Property relative to a class $\mathcal{G}$ of groups (not nec. elem.) if for each n there exists $G_n$ in $\mathcal{G}$ such that in $G_n$ the definable set defined by $\varphi(x, y)$ induces an independence graph $\Gamma_n$.*

**Ex:** $[x, y] = 1$ in the group of permutations of finite support of an infinite set (Zilber - Belegradek - Baldwin-Saxl).

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Order graphs: Finite *bipartite* graphs $\Gamma_n$ on $n+n$ elements coding a maximal chain in the powerset of a set of $n$-elements.

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**Ex:** Any infinite linear order, with $x \leq y$.

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*A formula $\varphi(x,y)$ has the Order Property relative to a class $\mathcal{G}$ of groups (not nec. elem.) if for each n there exists $G_n$ in $\mathcal{G}$ such that in $G_n$ the definable set defined by $\varphi(x,y)$ induces an order graph $\Gamma_n$.*

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- Most group words have the IP relative to the class of torsion-free hyperbolic groups (countably many groups).
- Tranfers to existentially closed *CSA*-groups ($2^{\perp}$).
  Phenomenon **antipodal** to algebraically closed fields.

What if *finitely* many t.f. hyperbolic groups? *One*?

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Elementary classes

## Definitions

$\Phi$-elementary class (usually $\Phi$ complete theory).

**Definition**

*$\varphi(x, y)$ defines a stable set if $\varphi(x, y)$ does not have the OP relative to the $\Phi$-elementary class.*

It means: there is a uniform bound $n$ for which $\varphi(x, y)$ encodes order graphs $\Gamma_n$. $->$ stability index of $\varphi$.

$$\forall x_1 \cdots x_{n+1}, y_1 \cdots y_{n+1} \text{ boolean combination of } \varphi(x_i, y_j)$$

Stable sets are closed under boolean combinations and adjunctions of parameters.

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# Stable groups

## Definition

*A group is stable is every definable set $\varphi(x, y)$ is stable.*

## Remark

*The stability index of each definable set $\varphi$ is witnessed by a "$\forall\varphi$" formula in the elementary theory of $G$.*

# Sela

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*Any torsion-free hyperbolic group is stable.*

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# Quotients!

## Example (Folklore)

*Stable groups may have unstable quotients (for ex. plenty of unstable quotients of free groups).*

## Example (Meirembekov)

$$\langle (x_i), z \,||\, x_i^3;\ z^3;\ [x_i, z];\ [x_i, x_j] = z,\ i < j;\ [x_i, x_j] = z^2,\ j > i \rangle$$

$$Z(G) = \langle z \rangle \text{ cyclic of order } 3$$

$$G/Z(G) \text{ elementary abelian of order } 3\ (\aleph_1\text{-categorical})$$

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# Conjectures

**Question (Famous in model theory!)**

*Build new stable groups.*

**Question**

*Is the free product of two stable groups still stable?*

# Corredor's construction

- Start with a free group $F_2$.
- Conjugate maximal abelian subgroups by successive *HNN*-extensions, and take the union.
- Repeat countably many times, and take the union.

*CSA*-group with conjugate maximal abelian subgroups.

Similar if one wants to force divisibility.

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# Modifications

**Lemma**

*For each n, any t.f. CSA-group $G_1$ with cyclic centralizers embeds in such a group $G_2$ in such a way that maximal abelian subgroups of $G_1$ are conjugate and elements of $G_1$ has n-th roots.*

**Proof**

*HNN*-extensions. Then add the $n$-th root. □

Start with $G_1$ a free group (stable).
$G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$ where $G_{n-1} \leq G_n$ as in the lemma.
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$G$ stable group.

**Definition**

*$X \subseteq_{\mathrm{def}} G$ is left generic if finitely many left translates cover $G$.*

**Fact**

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*$G$ is connected iff no partition into two definable generic subsets.*

**Remark (Poizat)**

*If $X$ is a definable generic subset of $F = \langle e_n \mid\mid \rangle$, then all but finitely many $e_n$ are in $X$.*

$F = g_1 X \cup \cdots \cup g_s X$, $e_1$, ..., $e_r$ all generators involved. $e_{r+1} \ldots \in X$.

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