Introduction to Tarski's problem and Model Theory

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Non Positive Curvature and the Elementary Theory of Free Groups,
Anogia, June 10, 2008

A little Model Theory of Groups: Formulas & Sentences

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H is "algebraically closed" in G

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Pb2: Is it true that $Th(F_X)$ decidable ?

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Hence $F_Y \models \phi(\bar{g}; f(b)), f(b) \in F_X$. $\square$

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Tarski's Problem: Universal theory & Equations

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Indeed: $F_2 = \langle a, b | \rangle$, the subgroup $\langle a^{-n} b a^n ; i \in \mathbb{N} \rangle$ is isomorphic to F_ω.
Consequence (combining with Theorem 1):
$Th_\forall(F_X) = Th_\forall(F_Y)$, for $|X|, |Y| \geq 2$.

$\Longrightarrow$ nonabelian free groups have the same universal theory.

Tarski's Problem: Universal theory & Equations

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The answer is yes. $\square$

Consequence: the surface group with presentation

$$\langle x, y, z | x^2 y^2 z^2 = 1 \rangle$$

does not satisfy the universal theory of nonabelian free groups.

Tarski's Problem: Universal theory & Equations

Lemma 3 (Mal'cev, 1962)

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Remark. Lemma 3 holds also in nonabelian models of the univeral theory of nonabelian free groups (Kharlampovich and Myasnikov, 1998).

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Definition

A positive formula $\phi(\bar{z})$, with free variables $\bar{z}$, is a formula

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \bigvee_{1 \leq i \leq n} (\bigwedge_{1 \leq j \leq p_i} w_{ij}(\bar{x}, \bar{y}, \bar{z}) = 1)$$

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$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n (w(\bar{x}, \bar{y}, \bar{z}; a, b) = 1)$$

Tarski's Problem: Universal theory & Equations

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$$F[x_1, \ldots, x_n] \models w\big(x_1, v_1(x_1), \ldots, x_n, v(x_1, \ldots, x_n); \bar{a}\big) = 1.$$

Here $F[x_1, \ldots, x_n] = F * \langle x_1, \ldots, x_n | \rangle$.

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Remark. The decidability of the positive theory follows easilly from (1) and Merzljakov's Theorem.

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Theorem 7 (Lorents 1963, Appel 1968, Chiswell & Remeslennikov 2000)

The set of solutions of a system of equations, with **one** *variable, in a free group, is a finite union of cosets of centralizers.* $\square$

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Remark. Lorents has anounced the theorem without proof and the proof of Appel contains a gap.

Tarski's Problem: Universal theory & Equations

Theorem 8 (Consequence of Sela's work)

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The set of solutions of a system of equations, with **one** *variable, in a torsion-free hyperbolic group, is a finite union of cosets of centralizers.* □

Tarski's Problem: Equations & Sela's approach

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$\implies$ correspondance between solutions of $w(\bar{x}) = 1$ and $\mathrm{Hom}(H_w, G)$.

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$\implies$ a beautiful structure of H(modulo the kernel of the action) and $\mathrm{Hom}(H, F)$.

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- $\ker_\infty(f_n) = \{g \in G | S_2(g) \text{ is finite}\}$.
- H is a *limit group*, if $H = G/ker_\infty(f_n)$ for some group G and a convergent sequence $f_n : G \to F$.

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(3) H is a model of the universal theory of a free group.

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A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

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Furtheremore, for any formula $\phi(\bar{x})$, there exists a boolean combination of $\forall\exists$-formulas, $\varphi(\bar{x})$, such that:
$F \models \forall x(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$, *for any free nonabelian group F.*

Tarski's Problem: the solution

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Theorem 10 (Kharlampovich&Myasnikov, Independently Sela)

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Consequence:
$F_X \prec F_Y$ for $X \subseteq Y, |X| \geq 2$.
$Th(F_X)$ is decidable.

More Model Theory: Stability

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- T is *stable*, if whenever $\phi(\bar{x}, \bar{y})$ is a formula such that $T \cup \{\exists \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})\}$ has a model, $\phi(\bar{x}, \bar{y})$ is without the order property.

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$$\left.\begin{array}{c}\text{Finite Morley rank}\\ \omega\text{-}\textit{stability}\\ \textit{superstability}\end{array}\right\} \implies \text{a good abstract notion of dimension.}$$

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- Free groups of infinite rank in the variety of nilpotent groups of class c of exponent p^n, p is a prime $> c$, are ω-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not ω-stable.
- Nonabelian free groups are not superstable (Gibone, Poizat). More generally a superstable model of the universal theory of nonabelian free groups is abelian (Mustafin-Poizat, independently Ould Houcine).
- Non-cyclic torsion-free hyperbolic groups are not superstable (Ould Houcine).
- A group acting nontrivially and without inversions on a simplicial tree is not ω-stable (Ould Houcine).

More Model Theory: Stability

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- T is stable,
- in any model of T, any order-indiscernible sequence is indiscernible,
- there exists a notion of independence in models of T with good properties.

Theorem 12 (Sela)

A trosion-free hyperbolic group is stable. □

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Consequence: existence of a good notion of independence in groups elementary equivalent to a trosion-free hyperbolic group.

More Model Theory: Stability

Theorem 13

Let G be a group. Suppose that G is stable and $G \prec G * \mathbb{Z}$. Then G is connected and any positive formula is equivalent to an existential positive one. □

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THE END