Splitting Theorems and the JSJ: A survey

E. Swenson

June 9, 2008

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- Analyse how the pieces fit together to solve the original problem.

Graphs of groups

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- Where for a vertex $v \in e$, "$\{v\} \times K_e$" is identified by $\sim$ to $i(K_e) \subset K_v$

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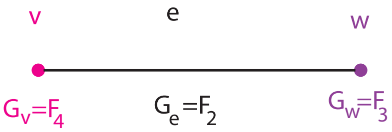
- A graph of groups decomposition Δ of the group G is trivial if K_Δ deformation retracts to a vertex space K_v.
- We say G splits over a subgroup $H < G$ if there is a non-trivial graph of groups decomposition of G with one edge, whose group is H.

Example of the fundamental group of a graph of groups Δ



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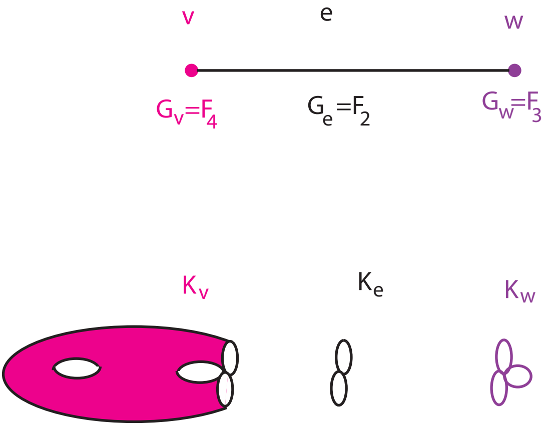

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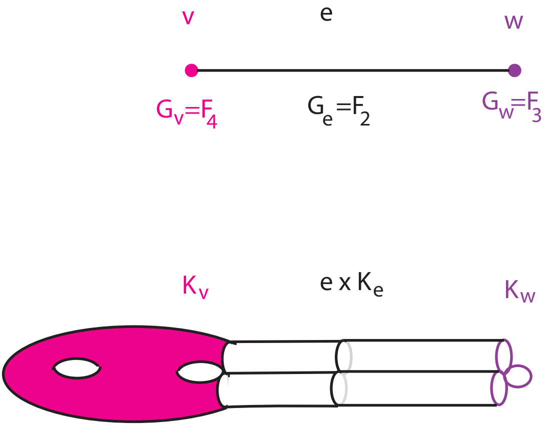

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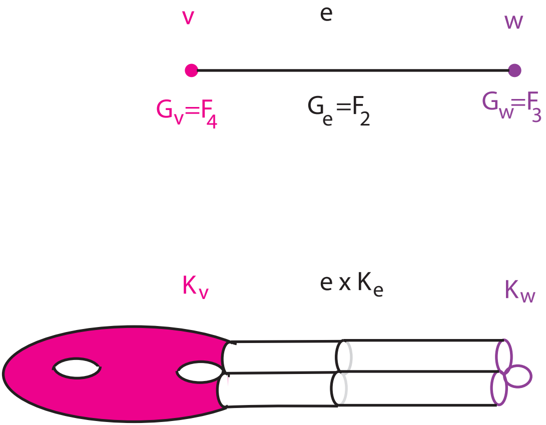

- Thus $\pi_1(\Delta) \cong F_5$.

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- This process is reversible.

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- The Grushko decomposition is "unique" (at least up to the H_i and the rank of F)

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- This process stops since $\operatorname{rank} A \star B = \operatorname{rank} A + \operatorname{rank} B$. (number of vertices is at most the rank).

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- This is false if the p in fp is changed to a g (Dunwoody).

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  - Classifies the splittings of $\pi_1(M)$ over $\mathbb{Z}$ or $\mathbb{Z}^2$.

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- Why are these theorems true?
- Because $\pi_1(M)$ is finitely generated and π_1 of S^2, A and T are virtually polycyclic (Dunwoody, S).

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3. Notice that 2 implies 1.1 but not 1.2

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- However, we think we understand homotopy classes of scc on surfaces.
- We deal with this difficulty by introducing Quadratically Hanging vertex groups

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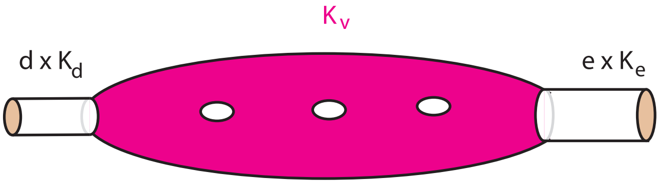

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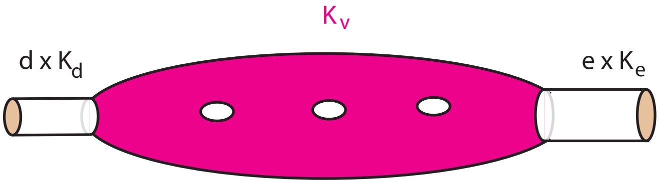

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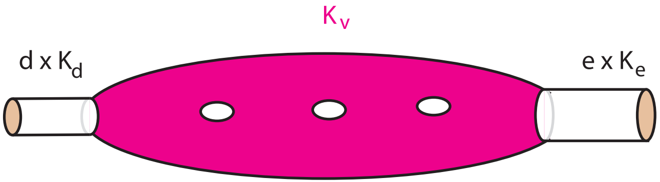
  - Each boundary circle of K_v is identified to exactly one end of an edge cylinder of the form $e \times K_e$



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- Version for when nice = VPC (slender) (Dunwoody-Sageev, Fujiwara-$\Pi\alpha\pi\alpha\zeta o\gamma\lambda o\upsilon$, Scot-Swarup (MOA JSJ))

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  - So G_v splits over a subgroup of C and we can replace the vertex space K_v with a graph of spaces (increasing the number of vertices)

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  - This gives a surface with boundary as a subset of K_Δ which gives (or enlarges) a quadratically hanging vertex space.

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- The action of a group on a tree is k-acylindrical if the diameter of a fixed point set is at most k (non-trivial group element)

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- The idea is to analyse the cut pair structure of ∂G

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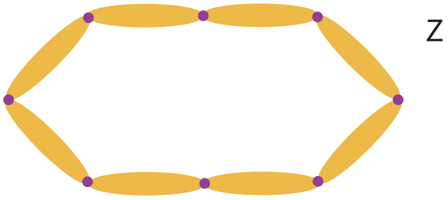

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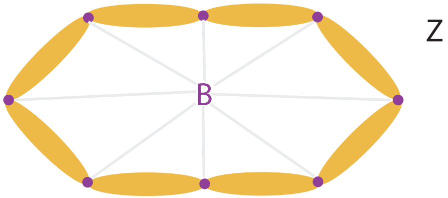

A necklace is a maximal cyclic subset

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The elements of $\mathcal{R}$ are:

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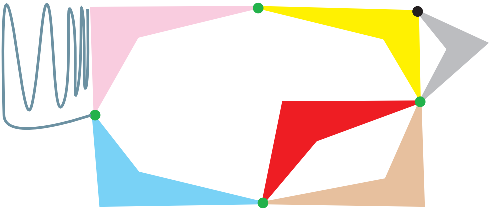
- $x, y \in \mathcal{R}$ are called adjacent if $(x, y) = \emptyset$.

Example



Z

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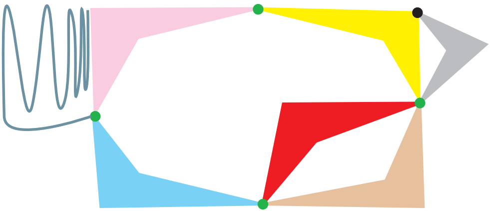

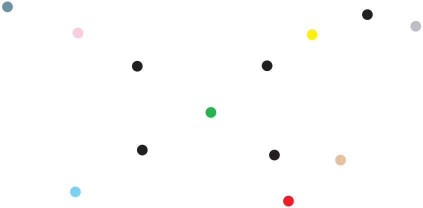

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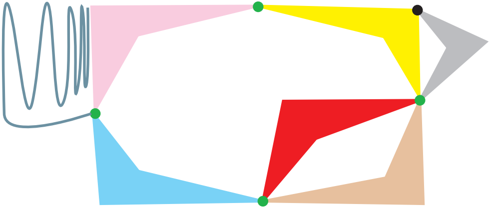

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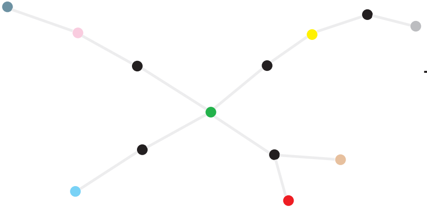

The preetree $\mathcal{R}$ with the interval structure

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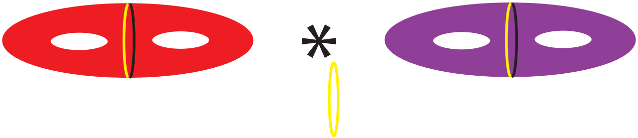
  - The black vertices of T have finite valence

Word hyperbolic example



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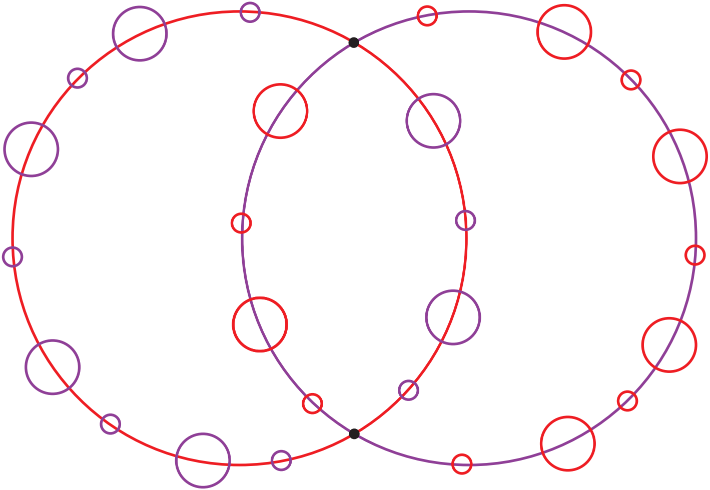

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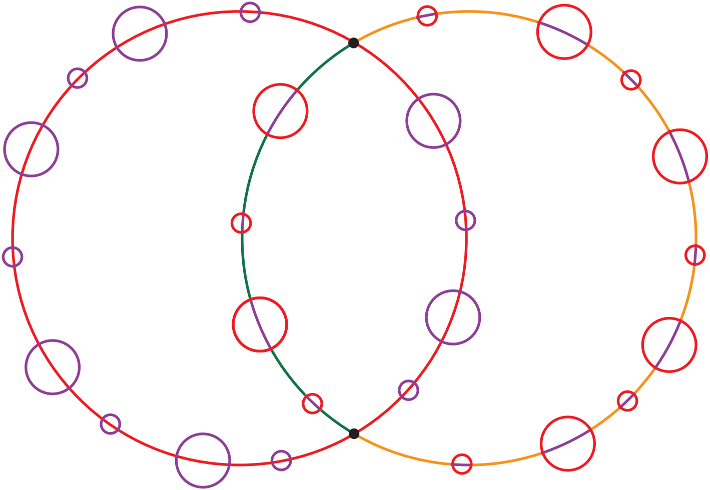

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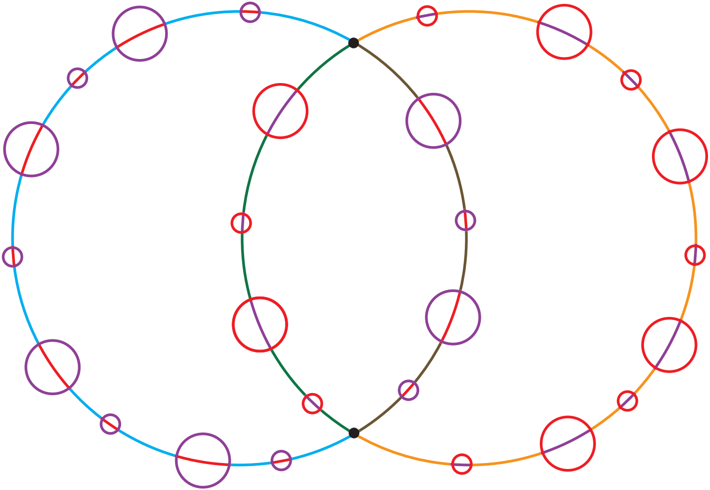
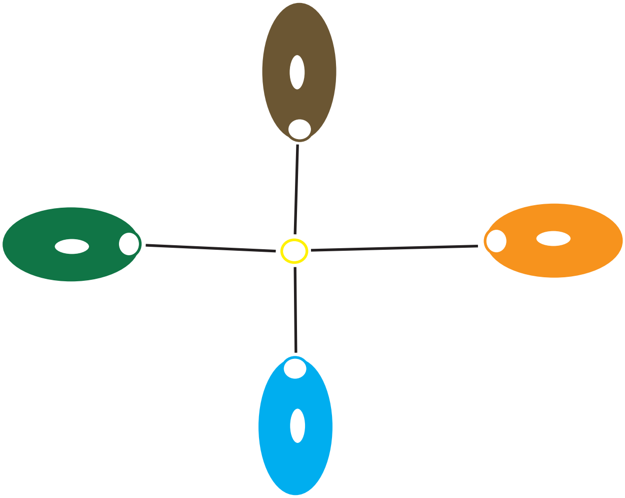
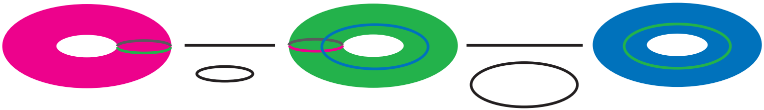


The boundary JSJ



Croke-Kleiner example (not word hyperbolic)

Consider the CAT(0) graph of spaces X consisting of three tori glued along simple closed curves.



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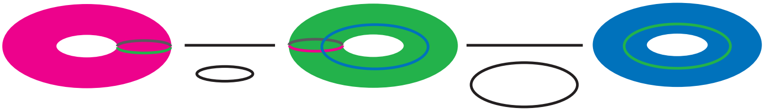

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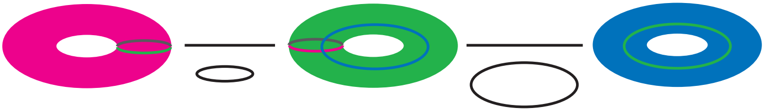

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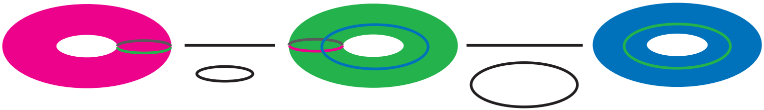
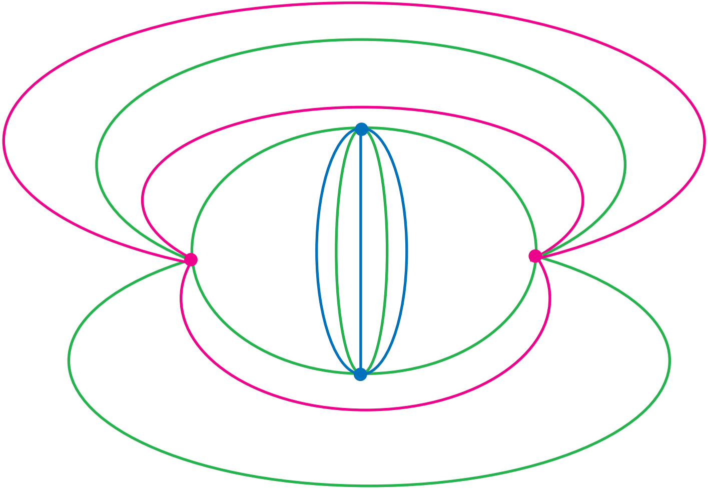
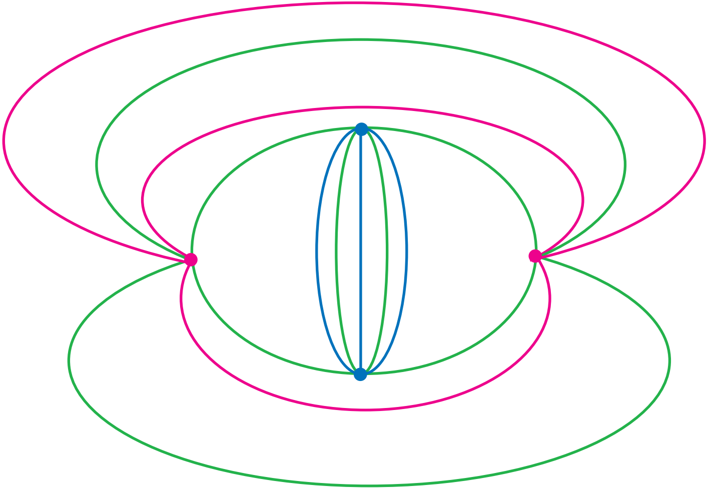


There are three vertex groups (all $\mathbb{Z}^2$) and two $\mathbb{Z}$ edge groups. If we throw out the blue vertex, we are left with $F_2 \times \mathbb{Z}$ whose boundary is the suspension of a Cantor set. Consider a single green circle in ∂X and all circles which intersect it.





There are no necklaces!!

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- Well, yes but we can deform it by pushing a $\mathbb{Z}$ from each F_2 into the central vertex.