Rank Rigidity in CAT(0)

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June 3, 2008

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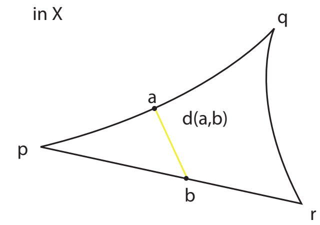
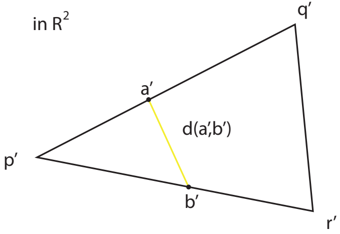
- The triangle Δ satisfies the CAT(0) condition if it is at least as thin as Δ'
- That is for any $a, b \in \Delta$ with comparison points $a', b' \in \Delta'$, we have $d(a, b) \leq d(a', b')$.


in X
q
a
d(a,b)
p
b
r


in R2
q′
a′
d(a′,b′)
p′
b′
r′

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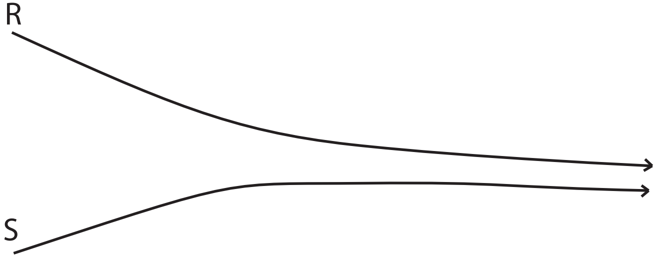
- Notice that $f(s, t) = \bar{\angle}_p(\alpha(s), \beta(t))$ is an increasing function of s and t (so all limits exist)
- When α and β run through q and r resp. then we define

$$\angle_p(q, r) = \lim_{s,t \to 0} \bar{\angle}_p(\alpha(s), \beta(t))$$

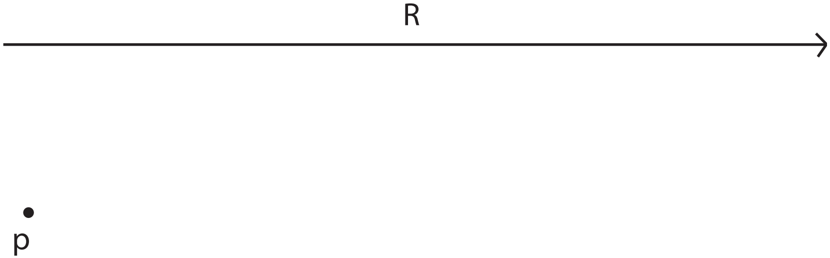
Boundary of X

The ∂X consists of equivalence classes of geodesic rays. Two unit speed geodesic rays $R, S : [0, \infty) \to X$ are equivalent if the function $d(R(t), S(t))$ is bounded.



For any ray R and any point p, there is a ray from p equivalent to R obtained thusly:

Choosing a base point



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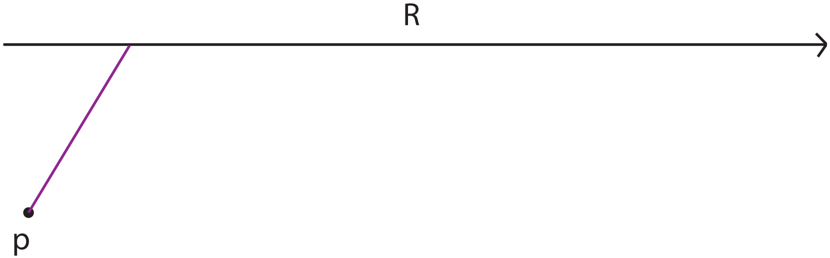

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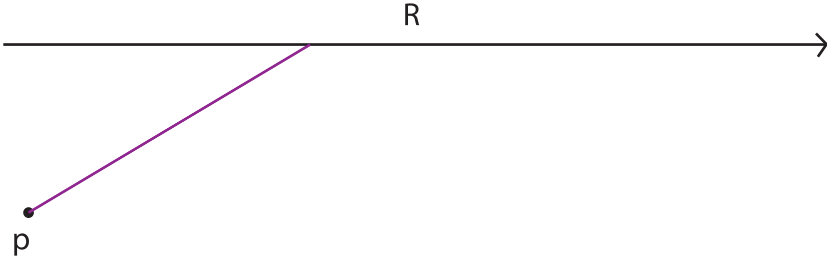

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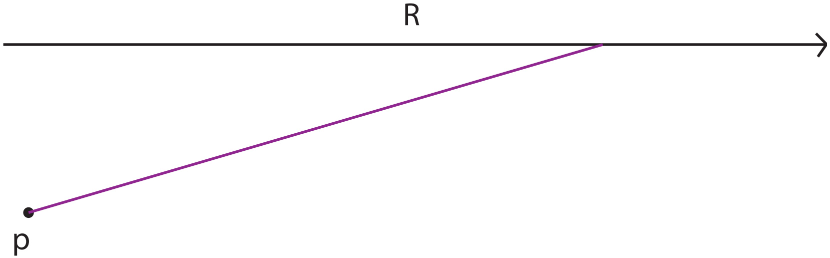

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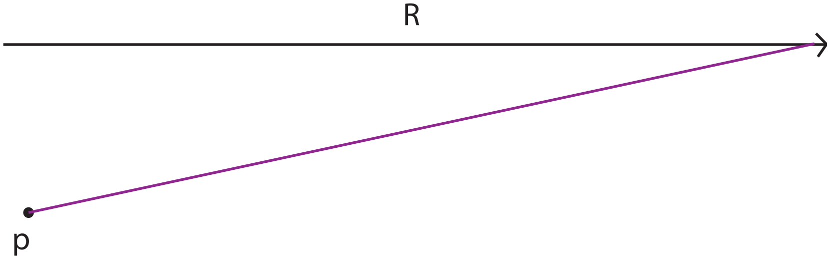

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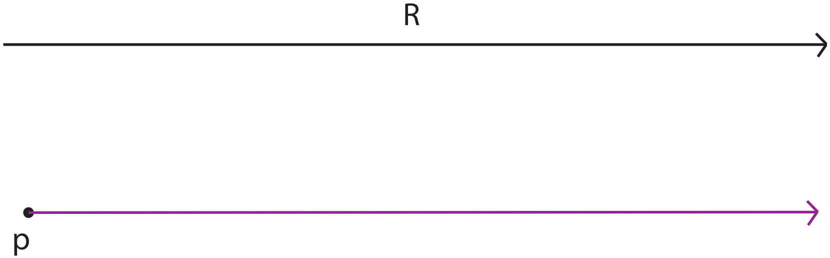

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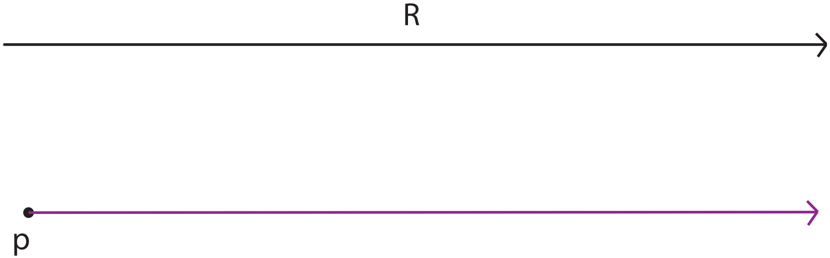

Two rays based at the same point are close if they remain close for a long time. This gives a topology on $\bar{X} = X \cup \partial X$ under which ∂X is compact metrizable and finite dimensional provided X/G is compact. Also G acts on ∂X by homeomorphisms

Comparison angle on $\bar{X}$

- Let $p \in X$ and $q, r \in \bar{X}$, and $\alpha : [0, c] \to \bar{X}$ and $\beta : [0, d] \to \bar{X}$ unit speed geodesics from p to q and r resp., where $c, d \in (0, \infty]$ (Warning: Abuse of notation alert).

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- For any $q, r \in \partial X$, we define $\angle(q, r) = \bar{\angle}_p(q, r)$ (independent of p)

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- If X is δ-hyperbolic, then $d_T(q, r) = \infty$ for $q \neq r$.
- When X is not δ-hyperbolic then "generically" $d_T((\partial X)^2) = [0, \infty]$

π-Convergence

For $(g_i) \subset G$ is a π-convergence sequence if $\exists p, n \in \partial X$ such that for any $\theta \in [0, \pi]$ and for any compact $C \subset X - B(n, \theta)$, $g_i(C) \to B(p, \pi - \theta)$ (Closed Tits balls).

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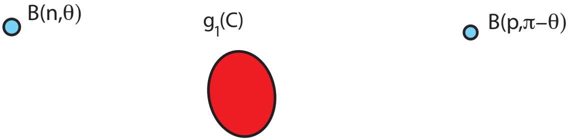

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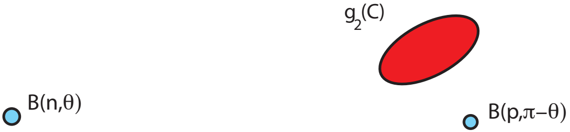

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If G is a discrete group of isometries of X, then every sequence of distinct elements of G has a π-convergence subsequence. (S, Π. $\Pi\alpha\pi\alpha\zeta o\gamma\lambda o\upsilon$)

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Corollary

If the Tits diameter of ∂X is more than 2π then it is ∞.

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- Passing to a subsequence $g_n(i) \to j \in B(p, \frac{\pi}{2} + \delta)$
- Thus $d_T(p, I) \leq \frac{\pi}{2}$, so $\partial X \subset B(p, \frac{3\pi}{2})$

What else does π-convergence make look way too easy?

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- If $H < G$ is not virtually cyclic and $diam_T(\Lambda H) > 2\pi$ then $F_2 < H$ (in particular H is not torsion).

- Let $(g_i) \subset G$, $x \in X$, and $c \in \partial X$ with $g_i^{-1}(x) \to n \in \partial X$ and $g_i(x) \to p \in \partial X$. Assume that $i >> 0$.

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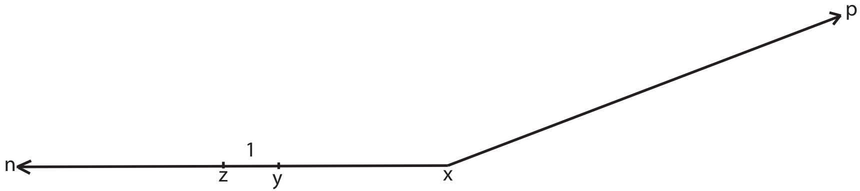

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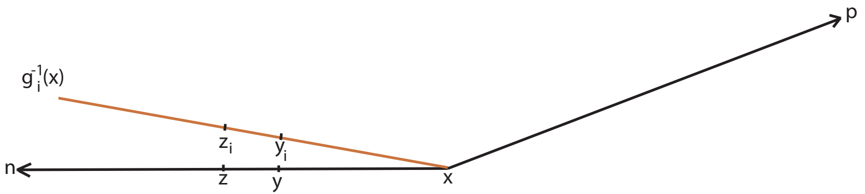

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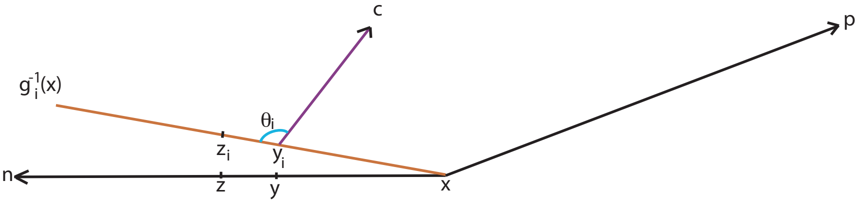

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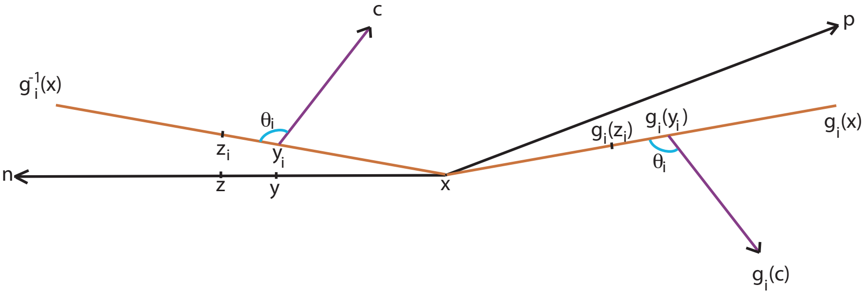

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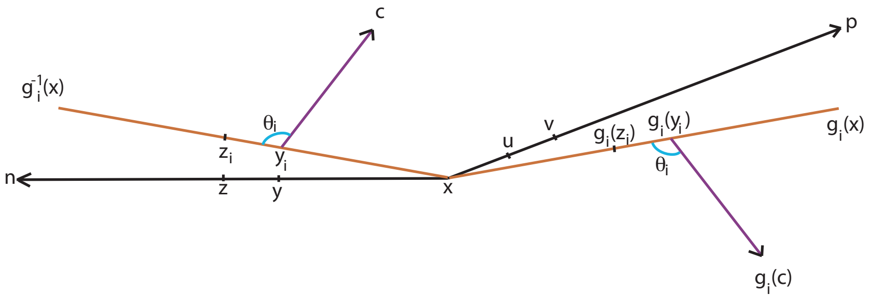

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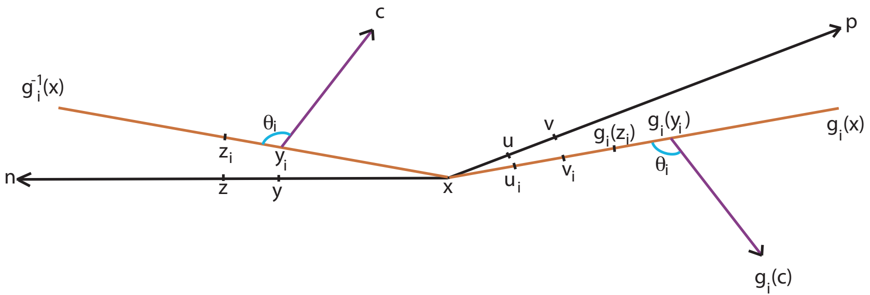

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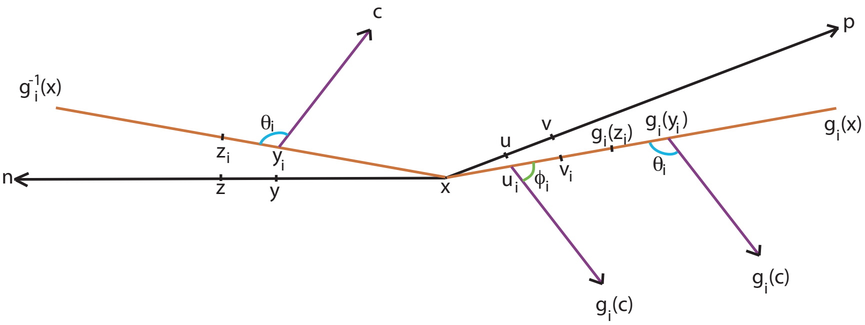
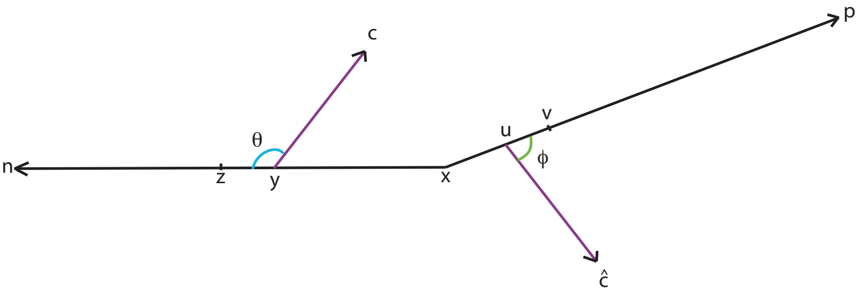
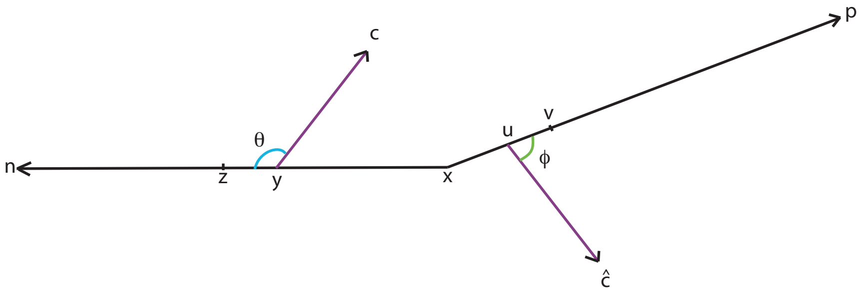
- Let $\theta_i = \bar{\angle}_{y_i}(z_i, c)$ apply g_i so $\theta_i = \bar{\angle}_{g_i(y_i)}(g_i(z_i), g_i(c))$
- Let $u \in [x, p)$, $v \in [u, p)$ with $d(u, v) = 1$,and let u_i, v_i be the projections of u, v resp. onto $[x, g_i(x)]$.
- Let $\phi_i = \bar{\angle}_{u_i}(v_i, g_i(c))$. We will show that $\phi_i + \theta_i \leq \pi$.



- Notice that $\theta_i \to \theta := \bar{\angle}_y(z,c)$ and $\underline{\lim}\, \phi_i \geq \phi := \bar{\angle}_u(v,\hat{c})$ whenever $g_i(c) \to \hat{c}$.



- Notice that $\theta_i \to \theta := \bar{\angle}_y(z, c)$ and $\underline{\lim}\, \phi_i \geq \phi := \bar{\angle}_u(v, \hat{c})$ whenever $g_i(c) \to \hat{c}$.
- If $\phi_i + \theta_i \leq \pi$, then $\phi + \theta \leq \pi$.



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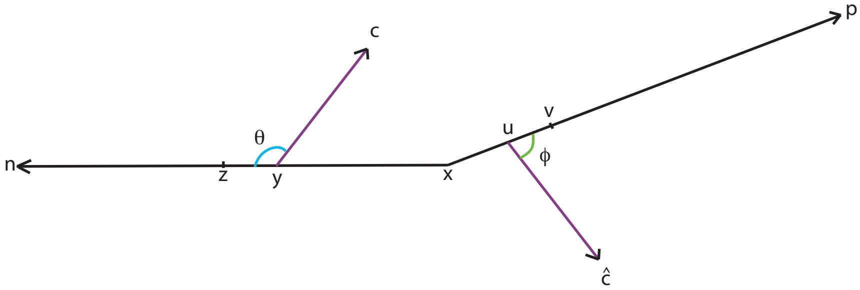

We can choose y, u so that $\angle(n, c) \cong \theta$ and $\angle(p, \hat{c}) \cong \phi$.

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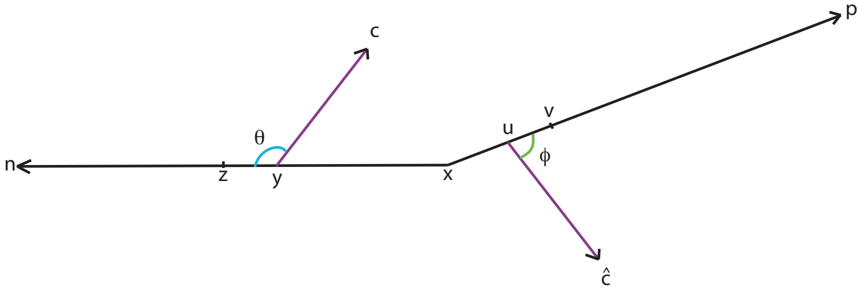
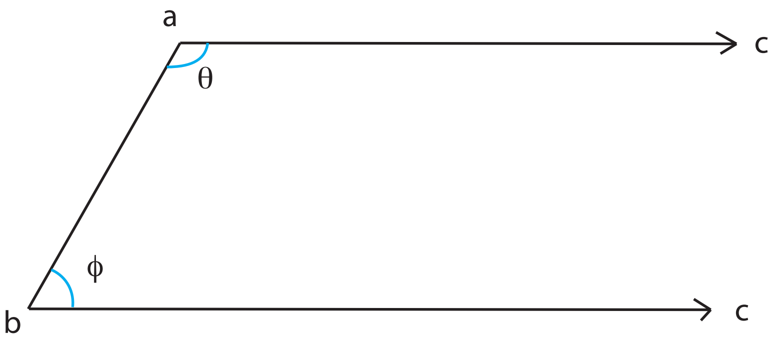
- If $\phi_i + \theta_i \leq \pi$, then $\phi + \theta \leq \pi$.



We can choose y, u so that $\angle(n, c) \cong \theta$ and $\angle(p, \hat{c}) \cong \phi$. Thus $\angle(n, c) + \angle(p, \hat{c}) \leq \pi$ or π-convergence.

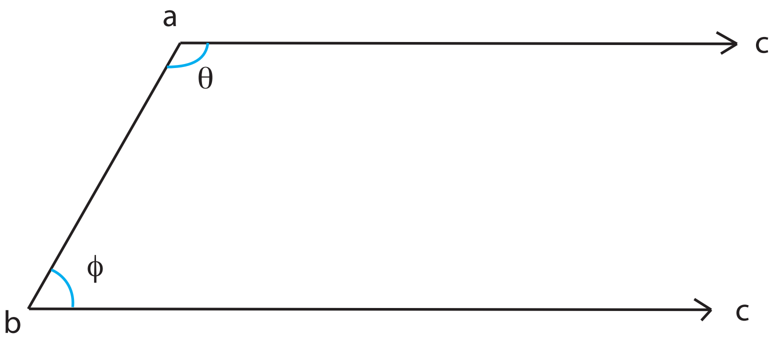
Showing $\phi_i + \theta_i \leq \pi$

- By monotonicity of comparison angles, it suffices to show that the comparison angle sum of an ideal triangle is at most π.



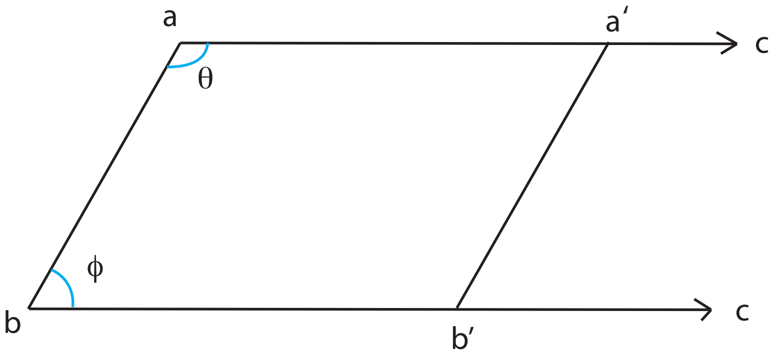
Showing $\phi_i + \theta_i \leq \pi$

- By monotonicity of comparison angles, it suffices to show that the comparison angle sum of an ideal triangle is at most π.
- Let $a, b \in X$ and $c \in \partial X$ and $\theta = \bar{\angle}_a(b, c)$ and $\phi = \bar{\angle}_b(a, c)$.

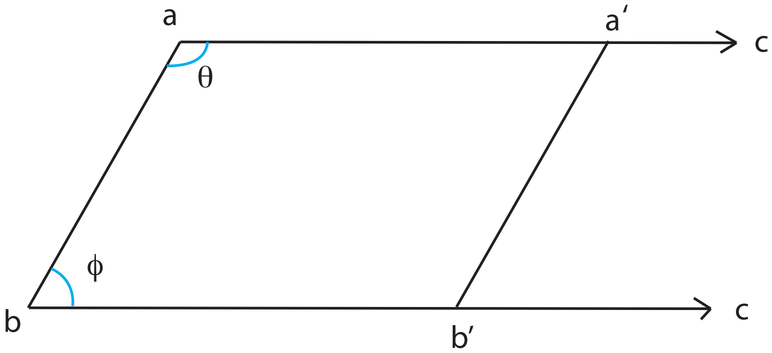


Showing $\phi_i + \theta_i \leq \pi$

- By monotonicity of comparison angles, it suffices to show that the comparison angle sum of an ideal triangle is at most π.
- Let $a, b \in X$ and $c \in \partial X$ and $\theta = \bar{\angle}_a(b, c)$ and $\phi = \bar{\angle}_b(a, c)$.
- If $\theta + \phi > \pi$, then $\exists a' \in [a, c)$, $b' \in [b, c)$ such that $\bar{\angle}_a(b, a') + \bar{\angle}_b(a, b') > \pi$ and $d(a, a') = d(b, b')$.



Comparing the quadrilateral $a'abb'$ to a Euclidian quaduadrilateral we see that $d(a,b) < d(a',b')$ contradicting convexity of the metric on X.



This completes the proof of π-convergence