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Preface

The present volume contains contributions of the participants to the 16th Panhellenic Conference on Analysis. The Panhellenic Conference on Analysis is a major scientific event in Greece on the area of Analysis. The 16th Conference was organized by the Department of Mathematics of the University of Aegean and was held on 25–27 May 2018 at Karlovassi of Samos. The lectures given during the Conference include topics in the following fields: Analytic Number Theory, Combinatorics, Complex Analysis, Convex Analysis, Ergodic Theory, Geometric Analysis, Harmonic Analysis, Numerical Analysis, Operator Theory, Partial Differential Equations, Set Theory.

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THE ERROR TERM IN THE PRIME GEODESIC THEOREM FOR HYPERBOLIC 3-MANIFOLDS

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Abstract. We summarize what is known for the growth of the error term $E_\Gamma(X)$ in the Prime geodesic theorem for Riemann surfaces $\Gamma\backslash\mathbb{H}^2$ and we discuss recent research activity for hyperbolic 3-manifolds $\mathcal{M} = \Gamma\backslash\mathbb{H}^3$. In our recent joint work [4], for $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[i])$ we use the Kuznetsov-Motohashi trace formula to improve the classical pointwise bound of Sarnak $E_\Gamma(X) = O(X^{5/3+\epsilon})$ to $E_\Gamma(X) = O(X^{13/8+\epsilon})$. For a general cofinite Kleinian group, we improve this on average to $E_\Gamma(X) = O(X^{8/5+\epsilon})$ using the 3-dimensional Selberg trace formula.

1. Prime geodesic theorems in dimensions 2 and 3

1.1. **Prime geodesic theorem on Riemann surfaces.** Prime geodesic theorems describe the asymptotic behaviour of primitive closed geodesics on hyperbolic manifolds. They can be viewed as geometric generalizations of the Prime number theorem, which describes the asymptotic behaviour of prime numbers, when we count them up to a fixed height. More precisely, if we define the Chebyshev (or summatory von Mangoldt) function

$$\psi(X) = \sum_{p^k \leq X} \log p,$$

the Prime number theorem states that

$$\psi(X) \sim X. \tag{1.1}$$

The growth of the error term $E(X) = \psi(X) - X$ in (1.1) is still a far open problem and is related to the Riemann Hypothesis (RH). The size of the error is governed by the exponential sum

$$E(X) \sim \sum_{\rho} \frac{X^\rho}{\rho},$$

where the sum runs over all the nontrivial zeros of the Riemann zeta function $\zeta(s)$; the conjectural bound $E(X) = O_\epsilon(X^{1/2+\epsilon})$ is in fact equivalent to RH. It was an amazing discovery of Huber [10, 11] and Selberg (see [13, Thm. 10.5]), later generalized by many authors, that primitive closed geodesics on Riemann surfaces have an asymptotic behaviour similar to that of primes. Effective solutions to such Prime geodesic problems can be viewed as close geometric analogues of the Riemann Hypothesis.

Let $\Gamma\backslash\mathbb{H}^2$ be a Riemann surface, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a cofinite Fuchsian group. We denote by P a typical hyperbolic element of Γ (i.e. an element such that $|\operatorname{tr}(P)| > 2$)

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and by $\{P\}$ the conjugacy class of P. We will state the Prime geodesic theorem on $\Gamma\backslash\mathbb{H}^2$ in terms of the Chebyshev function $\psi_\Gamma(X)$, given by

$$\psi_\Gamma(X) = \sum_{N(P)\leq X} \Lambda_\Gamma(N(P)), \tag{1.2}$$

where the sum in (1.2) is taken over the hyperbolic classes of Γ, $N(P)$ denotes the norm of P and the von Mangoldt function is given by $\Lambda_\Gamma(N(P)) = \log N(P_0)$ if P is a power of a primitive hyperbolic element P_0 (an element which cannot be written as a nontrivial power of another element $\gamma \in \Gamma$) and zero otherwise. If P is conjugate to a matrix of the form

$$P_0^n = \begin{pmatrix} p^{n/2} & 0 \\ 0 & p^{-n/2} \end{pmatrix}, \quad p > 1$$

then the length of the P-invariant primitive closed geodesic on $\Gamma\backslash\mathbb{H}^2$ is equal to $\log N(P_0) = \log p$. Selberg proved that, as $X \to \infty$, we have

$$\psi_\Gamma(X) = M_\Gamma(X) + E_\Gamma(X),$$

where the main term is a finite sum coming from the small eigenvalues of the hyperbolic Laplacian $\lambda_j = s_j(1-s_j) < 1/4$:

$$M_\Gamma(X) = \sum_{1/2<s_j\leq 1} \frac{X^{s_j}}{s_j}$$

and the error term satisfies the bound $E_\Gamma(X) = O(X^{3/4})$.

For arithmetic groups Γ further improvements on the bound for the error term can be deduced as an application of the (Bruggeman-)Kuznetsov formula (see Kuznetsov [15]). Such improvements were first deduced for the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ by Iwaniec [12] and Luo and Sarnak [16]. The crucial step in these works is proving a non-trivial bound on a specific spectral exponential sums over the Laplacian eigenvalues λ_j (equivalently, proving a mean subconvexity estimate for the symmetric square L-function defined by the Laplace eigenfunctions) and applying this to the explicit formula of Iwaniec [12] for $E_\Gamma(X)$. Using an entirely different method, Soundararajan and Young [22] proved the current best known bound

$$E_\Gamma(X) = O_\epsilon(X^{25/36+\epsilon}). \tag{1.3}$$

They did this by using class number formula to relate $\psi_\Gamma(X)$ to Zagier's zeta function, which are further related to quadratic Dirichlet L-functions $L(s,\chi_d)$. They concluded the estimate $E_\Gamma(X) = O_\epsilon(X^{2/3+\theta/6+\epsilon})$, where θ is the subconvexity exponent for these Dirichlet L-functions. Using the estimate $\theta = 1/6$ of Conrey and Iwaniec [8], who improved on Burgess subconvexity bound, they concluded (1.3). This bound was recently recovered by Balkanova and Frolenkov [1] using Iwaniec's method. The natural limit of these methods is the estimate $E_\Gamma(X) = O_\epsilon(X^{2/3+\epsilon})$, which follows from Lindelöf Hypothesis $\theta = 0$. However, it is justified that one can expect

$$E_\Gamma(X) = O_\epsilon(X^{1/2+\epsilon}).$$

This estimate is the natural limit of Iwaniec's explicit formula, it is supported by Ω-results of Hejhal and would follow from a conjecture of Petridis-Risager [20] for spectral exponential sums.

Cherubini and Guerreiro [7] initiated the study the second moment of the error term. They first succeeded to improve on average the error term for general cofinite groups with the use of Selberg trace formula, proving the bound

$$\frac{1}{X}\int_X^{2X}|E_\Gamma(x)|^2dx = O_\epsilon(X^{4/3+\epsilon}).$$

They further applied the Kuznetsov trace formula to prove the refined estimate for the modular surface:

$$\frac{1}{X}\int_X^{2X}|E_\Gamma(x)|^2dx = O_\epsilon(X^{5/4+\epsilon}).$$

Their later result was recently further improved by Balog, Biró, Harcos and Maga [3] to $O(X^{7/6+\epsilon})$. These were the first results breaking the $X^{2/3}$-barrier.

1.2. **Prime geodesic theorem on hyperbolic 3-manifolds.** In our recent joint work with O. Balkanova, G. Cherubini, D. Frolenkov and N. Laaksonen [4] we study the Prime geodesic theorem for 3-dimensional hyperbolic manifolds. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a cofinite Kleinian group acting on the 3-dimensional hyperbolic space $\mathbb{H}^3$ and let $\psi_\Gamma(X)$ be the Chebyshev function attached to Γ, which counts hyperbolic (and loxodromic) conjugacy classes in the group (we remind that γ is called loxodromic if $\mathrm{tr}\,(\gamma) \notin \mathbb{R}$). The small eigenvalues $\lambda_j = s_j(2-s_j) < 1$ provide a finite number of main terms for $\psi_\Gamma(X)$; as $X \to \infty$ we have the asymptotic:

$$\psi_\Gamma(X) \sim M_\Gamma(X) := \sum_{1<s_j\leq 2}\frac{X^{s_j}}{s_j}.$$

The main problem here again is the study of the behaviour of the error term

$$E_\Gamma(X) = \psi_\Gamma(X) - M_\Gamma(X).$$

For a general cofinite group, Sarnak [21] proved the first nontrivial bound

$$E_\Gamma(X) = O_\epsilon(X^{5/3+\epsilon}). \tag{1.4}$$

Koyama [14] proved that this bound can be further improved in the arithmetic case of the Picard group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}[i])$. Adopting Iwaniec's strategy, he proved the conditional bound

$$E_\Gamma(X) = O_\epsilon(X^{11/7+\epsilon})$$

under a mean Lindelöf hypothesis for symmetric square L-functions attached to Maass forms on the Picard manifold $\Gamma\backslash\mathbb{H}^3$. Our main result in [4] is the first unconditional improvement of (1.4) for the Picard group.

Theorem 1.1 ([4])**.** *Let* $\Gamma = \mathrm{PSL}(2,\mathbb{Z}[i])$. *Then*

$$E_\Gamma(X) = O_\epsilon(X^{13/8+\epsilon}). \tag{1.5}$$

We present a sketch of our proof in subsection 3.1. Our proof uses Iwaniec's idea to use the arithmetic trace formula and reduce the estimate of the error term $E_\Gamma(X)$ to the study of the first moment of the Rankin-Selberg L-function $L(u_j \otimes u_j)$ of Hecke-Maass forms u_j, or equivalently the second moment of the symmetric square L-function $L(\mathrm{Sym}^2\, u_j)$. However, in this case we cannot use the Luo-Sarnak method to bound the second moment of $L(\mathrm{Sym}^2\, u_j)$ as the existing large sieve inequality (due to Watt) is not strong enough to provide a power saving. In this part we use Weil's bound for

Kloosterman sums and careful estimates for integral of Bessel functions to deduce the desired power saving.

Recently, in an independent work, Balkanova and Frolenkov [2] used a different method to improve (1.5) to

$$E_\Gamma(X) = O_\epsilon(X^{3/2+\theta/2+\epsilon}), \tag{1.6}$$

where θ is the subconvexity exponent for quadratic Dirichlet L-functions defined over $\mathbb{Z}[i]$. In that case, convexity bound $\theta = 1/4$ recovers (1.5). The Burgess-type subconvexity result of Wu [24] and Nakasuji's bound for Ramanujan conjecture [19] allows them to take $\theta = 103/512$, thus leading to the improved unconditional bound

$$E_\Gamma(X) = O_\epsilon(X^{\frac{1639}{1024}}). \tag{1.7}$$

Notice that the exponent here is ≈ 1.600586.

In [4] we initiate the study of the second moment of $E_\Gamma(X)$ in three dimensions. As an application of the 3-dimensional Selberg trace formula we prove the following second moment estimate.

Theorem 1.2. *Let Γ be a cofinite Kleinian group. Then*

$$\frac{1}{X}\int_X^{2X} |E_\Gamma(x)|^2 dx \ll X^{16/5}(\log X)^{2/5}.$$

Theorem 1.2 says that $E_\Gamma(X) \ll X^{8/5+\epsilon}$ on average. More precisely, we have:

Corollary 1.3. *For every $\eta > 8/5$ there exists a set $A \subseteq [2,\infty)$ of finite logarithmic measure (i.e. for which $\int_A x^{-1}dx$ is finite), such that $E_\Gamma(X) = O(X^\eta)$ for $X \to \infty$, $X \notin A$.*

Notice that for the Picard group Theorem 1.2 impoves the bound of Theorem 1.1 on average (and the improved known bound (1.7)), but not Koyama's conditional exponent. We will return to the study of the second moment for the arithmetic case in a forthcoming work.

One can ask what is the truth order of growth for the error term in this case. However, this problem is still not as well understood as in two dimensions. Existing lower bounds, due to Nakasuji [18], read

$$E_\Gamma(X) = \Omega_\epsilon(X^{1-\epsilon}), \tag{1.8}$$

and one may be tempted to conjecture that

$$E_\Gamma(X) \ll X^{1+\epsilon}.$$

A main difference between the Riemann surfaces and the case of 3-manifolds is the natural limitation of the explicit formula (see (**??**)), which in its current form has a natural barrier $E_\Gamma(X) \ll X^{3/2+\epsilon}$. The work of Balkanova and Frolenkov supports the conjecture that one cannot expect anything better than $X^{3/2+\epsilon}$, but it is an interesting problem to improve the existing lower bounds.

Finally, we refer to the original [4] for the connection between the prime geodesic theorem for $\mathrm{PSL}(2,\mathbb{Z}[i])$ and class numbers for binary quadratic forms over $\mathbb{Z}[i]$.

1.3. **Remark.** We mention that the analogy between the Prime number theorem and Prime geodesic theorems can be further understood by the theory of the Selberg zeta function $Z_\Gamma(s)$. This is a meromorphic function of order 2 (for Fuchsian groups) or 3 (in the case of Kleinian groups), with an Euler product defined over the hyperbolic conjugacy classes of Γ and which satisfies RH. In combination with the explicit formula, this later fact explains the differences in the bounds of the error terms between Prime number theorem and Prime geodesic theorems.

2. Spectral theory on hyperbolic 3-manifolds and trace formulas

The hyperbolic space $\mathbb{H}^3$ is the set of points $p = z + jy = (x_1, x_2, y)$, where $z = x_1 + ix_2 \in \mathbb{C}$ and $y > 0$, endowed with the hyperbolic metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}$$

and the hyperbolic volume

$$dv = \frac{dx_1 dx_2 dy}{y^3}.$$

If we write the points $p \in \mathbb{H}^3$ in the quaternion form, then a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{C})$ acts on p via the orientation-preserving isometric action

$$Mp = \frac{ap + b}{cp + d}$$

(inverse taken in quaternions). A discrete group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ is cofinite if the quotient $\mathcal{M} = \Gamma \backslash \mathbb{H}^3$ has $v(\mathcal{M}) < \infty$. It $\mathcal{M}$ is compact then Γ is called cocompact.

Fixing a cofinite group Γ, the metric ds on $\mathcal{M}$ defines a Laplace–Beltrami operator Δ acting on $L^2(\Gamma\backslash\mathbb{H}^3)$. This operator admits eigenvalues

$$\lambda_j = s_j(2 - s_j), \quad s_j = 1 + ir_j,$$

where $r_j \in \mathbb{R}$ or r_j purely imaginary in the interval $(0, i]$. Let $\Gamma^\sharp$ be the dual lattice of Γ; notice that if $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ is a Bianchi group then $\Gamma^\sharp = \mathcal{O}_K^*/\sim$, where $\mathcal{O}_K^* = \mathcal{O}_K \backslash \{0\}$ are the nonzero elements of $\mathcal{O}_K$ and $n \sim m$ iff they generate the same ideal in $\mathcal{O}_K$. For $n \in K$ define also the norm $N(n) = |n|^2$. If $\Gamma\backslash\mathbb{H}^3$ has cusps, then there is also a continuous spectrum spanning $[1, \infty)$. If λ_j is an eigenvalue, the cusp form u_j attached to λ_j reads [9, §3 Thm. 3.1]

$$u_j(p) = y \sum_{0 \neq n \in \Gamma^\sharp} \rho_j(n) K_{ir_j}(2\pi|n|y) \exp(2\pi i \langle n, z\rangle), \tag{2.1}$$

where $\langle x, y\rangle$ denotes the standard inner product on $\mathbb{R}^2 \cong \mathbb{C}$.

The distribution and the size of the discrete spectrum of the Laplace operator on hyperbolic manifolds of finite volume is a classical problem. For a general cofinite group we do not even know if there are any other eigenvalues at all except the trivial one $\lambda_0 = 0$. The Weyl law describes the asymptotic behaviour of both the discrete and continuous spectrum. In our case Weyl law [9, §6 Thm. 5.4] gives

$$\sharp\{r_j \leq T\} - \frac{1}{4\pi} \int_{-T}^{T} \frac{\varphi'}{\varphi}(1 + ir)\, dr \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2} T^3.$$

In particular, if Γ is cocompact there is no continuous spectrum and we deduce there are infinitely many eigenvalues. In the proof of Theorem 1.2 we will use a result of Bonthonneau [5, Thm. 2], which in our case gives a good error term in the Weyl law:

$$\sharp\{r_j \leq T\} - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(1+ir)\,dr = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O\left(\frac{T^2}{\log T}\right). \tag{2.2}$$

Maass–Selberg relations [9, §3 Thm 3.6] and (2.2) give the following inequality on unit intervals:

$$\sharp\{r_j \in [T, T+1]\} + \int_{T\leq|r|\leq T+1} \left|\frac{\varphi'}{\varphi}(1+ir)\right| dr \ll T^2. \tag{2.3}$$

2.1. **The Selberg trace formula.** For general cofinite groups, the Selberg trace formula is one of the most effective tools available to attack problems in the spectral theory of automorphic forms. This formula relates geometric information attached to a group to spectral data of the hyperbolic Laplacian.

We classify the elements $M \in \mathrm{PSL}(2,\mathbb{C})$, $M \neq I$, by their trace $\mathrm{tr}(M)$. If $\mathrm{tr}(M) \notin \mathbb{R}$ then M is called loxodromic. Otherwise, depending on whether the absolute value of the trace is smaller, equal or larger than 2, M is called elliptic, parabolic or hyperbolic, respectively. Every hyperbolic or loxodromic element M is conjugated in $\mathrm{PSL}(2,\mathbb{C})$ to a unique element

$$\begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}$$

wihere $p = p(M)$ satisfies $|p| > 1$. The quantity $N(M) := |p|$ is called the norm of M and is invariant under conjugation; hence we define the norm of a conjugacy class to be the norm of any of its representatives.

The following Selberg trace formula for $\mathcal{M} = \Gamma\backslash\mathbb{H}^3$ was first derived by Tanigawa [23] (see also [9, §6 Thm. 5.1]). For simplicity, we assume that we only have one cusp at infinity.

Theorem 2.1 (Selberg Trace Formula)**.** *Let h be an even function, holomorphic in $|\Im r| < 1 + \epsilon_0$ for some $\epsilon_0 > 0$, and assume that $h(r) = O((1+|r|)^{-3-\epsilon})$ in the strip. Let g be the Fourier transform of h, defined by $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-irx}dr$. Then*

$$\sum_j h(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}(1+ir)dr \quad = \quad I + E + H + P, \tag{2.4}$$

where the left hand encodes data from the spectrum of $\mathcal{M}$, and the right hand side encodes the information from the conjugacy classes of Γ (length spectrum). In particular, I stands for the contribution of the identity element, E for the elliptic classes not stabilizing the cusp at ∞, H for the the hyperbolic (and loxodromic) classes and P for the parabolic classes. I, E and H are explIcitly given by the formulas

$$I = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{4\pi^2}\int_{-\infty}^{\infty} h(r)r^2dr, \quad E = \sum_{\{R\}\mathrm{nce}} \frac{g(0)\log N(T_0)}{4|\mathcal{E}(R)|\left(\sin\frac{\pi k}{m(R)}\right)^2},$$

$$H = \sum_{\{T\}\mathrm{lox,hyp}} \frac{g(\log N(T))\Lambda_\Gamma(N(T))}{|\mathcal{E}(T)||p^{1/2}(T) - p(T)^{-1/2}|^2}.$$

The formula for P is elementary but more complicated to give it here. We refer to [9] for a formula for P and an explanation of the notation.

2.2. **The Kuznetsov trace formula.** The Kuznetsov trace formula was first independently proved by Bruggeman [6] and Kuznetsov [15] in the case of the modular group, and was later generalized for cofinite Fuchsian groups by Proskurin and for general real rank one groups by Miatello and Wallach. For arithmetic groups, the Kuznetsov formula is a very powerful tool to deduce results that cannot be obtained by means of the Selberg formula, as it relates spectral data of Γ to Kloosterman sums (arithmetical data). For arithmetic Kleinian groups such as $\mathrm{PSL}(2,\mathbb{Z}[i])$ the Kuznetsov formula was worked out by Motohashi, Bruggeman-Motohashi, Lokvenec-Guleska and Qi.

For $m,n,c \in \mathbb{Z}[i]$, with $c \neq 0$ we define the Kloosterman sums

$$S(m,n;c) := \sum_{a \in (\mathbb{Z}[i]/(c))^\times} e(\langle m, a/c\rangle) e(\langle n, a^*/c\rangle),$$

where a^* denotes the inverse of a modulo the ideal (c), that is $aa^* \equiv 1 \bmod c$. The Weil bound for these sums is [14, p. 791] reads $|S(n,n;c)| \ll N(c)^{1/2}|(n,c)|d(c)$ where d is the number of divisors of c.

Theorem 2.2 (Kuznetsov formula for $\mathrm{PSL}(2,\mathbb{Z}[i])\backslash\mathbb{H}^3$ [17])**.** *Let h be an even function, holomorphic in $|\Im r| < 1/2+\epsilon$, for some $\epsilon > 0$, and assume that $h(r) = O((1+|r|)^{-3-\epsilon})$ in the strip. Then, for any non-zero $m,n \in \mathbb{Z}[i]$:*

$$D + C = U + S,$$

with

$$\begin{aligned} D &= \sum_{j=1}^{\infty} \frac{r_j \rho_j(n)\overline{\rho_j(m)}}{\sinh \pi r_j} h(r_j), \qquad C = 2\pi \int_{-\infty}^{\infty} \frac{\sigma_{ir}(n)\sigma_{ir}(m)}{|mn|^{ir}|\zeta_K(1+ir)|^2} dr, \\ U &= \frac{\delta_{m,n}+\delta_{m,-n}}{\pi^2} \int_{-\infty}^{\infty} r^2 h(r) dr, \\ S &= \sum_{c \in \mathbb{Z}[i]^*} \frac{S(m,n;c)}{|c|^2} \int_{-\infty}^{\infty} \frac{ir^2}{\sinh \pi r} h(r) H_{ir}\left(\frac{2\pi\sqrt{mn}}{c}\right) dr, \end{aligned}$$

where $\sigma_s(n) = \sum_{d|n} N(d)^s$ is the divisor function,

$$H_\nu(z) = 2^{-2\nu}|z|^{2\nu} J^*_\nu(z) J^*_\nu(\overline{z}),$$

$\rho_j(n)$ is the Fourier coefficient defined in (2.1), ζ_K is the Dedekind zeta function of $K = \mathbb{Q}[i]$, J_ν is the J-Bessel function of order ν and $J^*_\nu(z) = J_\nu(z)(z/2)^{-\nu}$.

3. Sketch of the Proofs

3.1. **The proof of Theorem 1.1.** In this section we sketch the proof of the pointwise improvement for $E_\Gamma(X)$, which is the main result of [4]. A main ingredient of our proof is the an explicit formula for $E_\Gamma(X)$, which expresses the error as a spectral expansion (a certain spectral exponential sum). This is the analogue of the Riemann-von Mangoldt explicit formula in the theory of prime numbers, and for the modular group it was first derived by Iwaniec [12]. For $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[i])$ it was proved by Nakasuji [18], and it reads

$$E_\Gamma(X) = 2\Re\left(\sum_{0<r_j\leq T} \frac{X^{1+ir_j}}{1+ir_j}\right) + O\left(\frac{X^2}{T}\log X\right), \tag{3.1}$$

for $T \leq X^{1/2}$. This explicit formula provides an effective way to pass from the geometric quantity $E_\Gamma(X)$ to spectral data. From (3.1) we deduce that the study of $E_\Gamma(X)$ is reduced to the study of the spectral exponential sum

$$S(T,X) := \sum_{0<r_j\leq T} X^{ir_j}.$$

By Weyl law, the trivial bound for $S(T,X)$ is $O(T^3)$; if we combine this with (3.1) we recover Sarnak's bound for $E_\Gamma(X)$. The natural limitation of formula (3.1) is $X^{3/2+\epsilon}$ (taking $T = X^{1/2}$). To reach this we need to assume the bound $S(T,X) \ll T^{2+\epsilon}X^\epsilon$ for the finite sum over $r_j \leq T$.

If we ignore the limitiation $T \leq X^{1/2}$, assuming the square root cancellation $S(T,X) \ll T^{3/2+\epsilon}X^\epsilon$ we get $E_\Gamma(X) \ll X^{4/3+\epsilon}$. We conclude the bound $X^{1+\epsilon}$ is probably far from being true, as it can only be reached by assuming the extremely strong estimate $S(T,X) \ll T^{1+\epsilon}X^\epsilon$. This assymetry does not appear in two dimensions, see [12], [20]. Our proof of Theorem 1.1 follows from (3.1) and the following bound for $S(T,X)$.

Theorem 3.1. *Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[i])$, and let $X, T > 2$. Then*

$$S(T,X) \ll T^{2+\epsilon}X^{1/4+\epsilon}.$$

The proof of Theorem 3.1 is an application of the Kuznetsov-Motohashi formula (Theorem 2.2). By Koyama [14] we relate the spectral exponential sum $\sum_{|r_j|\leq T} X^{ir_j}$ with a sum involving the Fourier coefficients $\sum_n |\rho_j(n)|^2$ (see (3.2)). To estimate the last we need an estimate for the Fourier coefficients $\rho_j(n)$. We define the Rankin-Selberg L-function

$$L(s, u_j \otimes u_j) := \sum_{n\in\mathbb{Z}[i]\setminus\{0\}} \frac{|\rho_j(n)|^2}{N(n)^s},$$

attached to the Maass–Hecke cusp form u_j. A core part of the proof is proving a subconvexity estimate for $L(s, u_j \otimes u_j)$ in the spectral aspect. We deduce the following bound:

Theorem 3.2. *Let u_j be a Maass–Hecke cusp form and suppose that $\Re w = \frac{1}{2}$. Then for some fixed A:*

$$\sum_{r_j\leq T} \frac{r_j}{\sinh \pi r_j} |L(w, u_j \otimes u_j)| \ll |w|^{A+\epsilon} T^{7/2+\epsilon}.$$

The convexity bound in the spectral aspect is $T^{4+\epsilon}$, while Lindelöf Hypothesis would give $T^{3+\epsilon}$, therefore our theorem takes us 50% of the way towards the goal. By Koyama, for any N we have

$$\frac{1}{N}\sum_{n\in\mathbb{Z}[i]\setminus\{0\}} f(|n|)\sum_{|r_j|\le T}\frac{r_j|\rho_j(n)|^2}{\sinh \pi r_j}X^{ir_j}\exp(-r_j/T)$$

$$= c\sum_{|r_j|\le T} X^{ir_j}\exp(-r_j/T) + O\left(T^{7/2+\epsilon}N^{-1/2}\right). \tag{3.2}$$

where f is a suitable smooth and compactly supported test function and c is a constant. Picking a test function of the form $h(r) = X^{ir}e^{-r/T} + O(e^{-\pi r})$ and applying Kuznetsov trace formula, we bound $|C| + |U| = O(T^2)$ and $S = O(N^{1/2+\epsilon}T^{1/2+\epsilon}X^{1/2})$. Balancing we finish the proof of Theorems 3.1 and 1.1.

We note that a bound of the form $O(T^{3+\alpha})$ in Theorem 3.2, with $0 < \alpha < 1$, gives the inequalities

$$S(T,X) \ll T^{(7+2\alpha)/4+\epsilon}X^{1/4+\epsilon}, \quad E_\Gamma(X) \ll X^{\frac{11+4\alpha}{7+2\alpha}+\epsilon}. \tag{3.3}$$

Clearly, on taking $\alpha = 0$ one recovers the conditional exponent $11/7$ of Koyama for $E_\Gamma(X)$.

3.2. **The proof of Theorem 1.2.** We now sketch our proof for the second moment estimate using the Selberg trace formula for a suitably chosen family of test functions. For $q(x)$ a smooth, even, non-negative real function with compact support contained in $[-1,1]$ and unit mass, and for $X > 1$, $s = \log X$ and $0 < \delta < 1/4$ consider the functions

$$g_s(x) = 4\left(\sinh^2\frac{x}{2}\right)\mathbf{1}_{[0,s]}(|x|), \quad q_\delta(x) = \delta^{-1}q\left(\frac{x}{\delta}\right), \quad g_\pm(x) = (g_{s\pm\delta}\star q_\delta)(x).$$

We pick $h_s(r)$ the Gourier transform of g_s and smooth approximations $h_\pm$ of h_s. Applying Selberg trace formula to $(g_\pm, h_\pm)$ we bound the contribution of I, E and P by $O(X)$. Using Weyl law we get

$$\frac{1}{X}\int_X^{2X}|E_\Gamma(x)|^2dx \ll \delta^2X^4 + X^3 + \frac{X^2}{\delta^3}(\log\delta^{-1}). \tag{3.4}$$

Balance $\delta = X^{-2/5}\log^{1/5}(X)$ completes the proof.

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DENSITY HALES–JEWETT NUMBERS—WHERE DO WE STAND?

PANDELIS DODOS

Abstract. The *density Hales–Jewett numbers* are central numerical invariants in Ramsey theory. We discuss what is known about these invariants and how they are related with other results in the area. We also make a conjecture about their asymptotic behavior.

Contents

1. Introduction

1.1. In this short note we shall discuss some properties of *discrete hypercubes*, that is, sets of the form

$$A^n := \underbrace{A \times \cdots \times A}_{n-\text{times}} \tag{1.1}$$

where A is a finite set with $|A| \geqslant 2$ and n is a positive integer which is commonly referred to as the *dimension* of the hypercube A^n. We will be mostly interested in the *high-dimensional* case, that is, when the dimension n is large compared with the cardinality of A.

1.2. A classical result concerning the structure of high-dimensional hypercubes was discovered in 1963 by Hales and Jewett [HJ1963]. It asserts that for every partition of A^n into, say, two pieces, one can always find a "subcube" of A^n which is entirely contained in one of the pieces of the partition.

To state the Hales–Jewett theorem we need to introduce some pieces of notation and some terminology. Let A and n be as above, and fix a letter $x \notin A$ which we view as a variable. A *variable word over A of length n* is a finite sequence of length n having values in $A \cup \{x\}$ where the letter x appears at least once. If v is a variable word over A of length n and $\alpha \in A$, then let $v(\alpha)$ denote the unique element of A^n which is obtained by replacing every appearance of the letter x in v with α. (E.g., if $A = \{\alpha, \beta, \gamma\}$ and $v = (\alpha, x, \gamma, \beta, x)$, then $v(\beta) = (\alpha, \beta, \gamma, \beta, \beta)$.) A *combinatorial line* of A^n is

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a set of the form $\{v(\alpha) : \alpha \in A\}$ where v is a variable word over A of length n.

Hales–Jewett theorem. *For every pair k, r of positive integers with $k \geqslant 2$ there exists a positive integer N with the following property. If A is a set with $|A| = k$ and $n \geqslant N$ is an integer, then for every r-coloring of A^n there exists a combinatorial line of A^n which is monochromatic. The least positive integer N with this property is denoted by* $\mathrm{HJ}(k, r)$.

Here, by an *r-coloring of A^n* we mean a function $c \colon A^n \to \{1, \dots, r\}$; moreover, we say that a subset X of A^n is *monochromatic* (*with respect to* c) if $X \subseteq c^{-1}(\{i\})$ for some $i \in \{1, \dots, r\}$.

The Hales–Jewett theorem is often regarded as an abstract version of the van der Waerden theorem [vW1927] and it is considered to be one of the foundational results of Ramsey theory. The exact values of the numbers $\mathrm{HJ}(k, r)$ are still unknown; the best known upper bounds are primitive recursive and are due to Shelah [Sh1998].

1.3. Around 30 years after the discovery of the Hales–Jewett theorem, another fundamental result was proved by Furstenberg and Katznelson [FK1991]. It asserts that every *dense* subset of A^n (that is, every subset of A^n whose cardinality is proportional to that of A^n) must contain a "sub-cube" of A^n.

Density Hales–Jewett theorem. *For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer N with the following property. If A is a set with $|A| = k$ and $n \geqslant N$ is an integer, then every subset D of A^n with $|D| \geqslant \delta |A^n|$ contains a combinatorial line of A^n. The least positive integer N with this property is denoted by* $\mathrm{DHJ}(k, \delta)$.

1.4. We will focus on the following central problem.

Problem. *Which is the asymptotic behavior of the density Hales–Jewett numbers* $\mathrm{DHJ}(k, \delta)$*?*

We shall comment, shortly, on the critical role of the density Hales–Jewett theorem and the importance of the above problem. At this point, let us briefly discuss what is known so far.

The original proof of the density Hales–Jewett theorem was based on the ergodic-theoretic methods pioneered by Furstenberg [F1977]; as such, it provides no quantitative information for the numbers $\mathrm{DHJ}(k, \delta)$. Another (ineffective) ergodic proof was given in [Au2011].

The first effective proof of the density Hales–Jewett theorem was discovered in 2009 by Polymath [P2012]; it yields upper bounds for the numbers $\mathrm{DHJ}(k, \delta)$ which have an Ackermann-type dependence with respect to k. Subsequently, two more combinatorial proofs were given in [DKT2014, T2011]; these proofs give essentially the same upper bounds as in [P2012]. Quite recently, yet another proof was found in [DT2018]; its most important feature is the quantitative improvement of a crucial part which appears (in various forms) in all previous combinatorial proofs. In particular, the results in [DT2018] are a step towards obtaining primitive recursive bounds for the density Hales–Jewett numbers.

2. Related results

We proceed to discuss some consequences of the density Hales–Jewett theorem. We note that all these consequences are deep and significant results on their own, and they can be proved using a variety of tools. We briefly discuss the existing approaches, and we comment on how they are compared (quantitatively) with a proof which is based on the density Hales–Jewett theorem.

2.1. We begin by recalling Szemerédi's theorem [Sz1975].

For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer $\mathrm{Sz}(k,\delta)$ with the following property. If $n \geqslant \mathrm{Sz}(k,\delta)$ is an integer, then every subset D of $\{1,\dots,n\}$ with $|D| \geqslant \delta n$ contains an arithmetic progression of length k.

Szemerédi's theorem is a remarkably influential result. In particular, there are numerous different proofs some of which are discussed in [TV2006, Chapter 11]. The best known upper bounds are due to Gowers [Go2001]:

$$\mathrm{Sz}(k,\delta) \leqslant 2^{2^{\delta^{-2^{2^{k+9}}}}}. \tag{2.1}$$

It is not hard to see that the density Hales–Jewett theorem implies Szemerédi's theorem—see [DK2016, Section 8.4.1] for details. The argument originates from [HJ1963] and it is amenable to generalizations, but it gives very weak upper bounds for the numbers $\mathrm{Sz}(k,\delta)$.

2.2. The next result is known as the *multidimensional Szemerédi theorem* and it is due to Furstenberg and Katznelson [FK1978].

For every pair k, d of positive integers with $k \geqslant 2$ and every $0 < \delta \leqslant 1$ there exists a positive integer $\mathrm{MSz}(k,d,\delta)$ with the following property. If $n \geqslant \mathrm{MSz}(k,d,\delta)$ is an integer, then every $D \subseteq \{1,\dots,n\}^d$ with $|D| \geqslant \delta n^d$ contains a set of the form $\big\{\mathbf{c} + \lambda\mathbf{x} : \mathbf{x} \in \{0,\dots,k-1\}^d\big\}$ for some $\mathbf{c} \in \mathbb{N}^d$ and some positive integer λ.

The first quantitative information for the numbers $\mathrm{MSz}(k,d,\delta)$ became available as a consequence of the *hypergraph removal lemma* [Go2007, NRS2006, RSk2004]. The multidimensional Szemerédi theorem can also be derived using the density Hales–Jewett theorem (see [DK2016, Section 8.4.1]). Despite this progress, the best known upper bounds for the numbers $\mathrm{MSz}(k,d,\delta)$ have an Ackermann-type dependence with respect to k for each fixed $d \geqslant 2$ and $0 < \delta \leqslant 1$.

Remark 1. We note that all known effective proofs of the hypergraph removal lemma are based on an appropriate version of the hypergraph regularity lemma. Recently, in [MS2018], it was shown that there exist Ackermann-type *lower* bounds for all these versions of the hypergraph regularity lemma. It remains an important open problem to decide whether there exist primitive recursive bounds for the hypergraph removal lemma.

2.3. The next result in our list is a version of Szemerédi's theorem for abelian groups. Specifically, let G be an abelian group (written additively) and let r be a positive integer. By $\mathcal{F}_r$ we denote the set of all nonempty subsets of

$\{0, \dots, r-1\}$. An IP_r*-set in* G is a family $(g_\alpha)_{\alpha \in \mathcal{F}_r}$ of elements of G such that $g_{\alpha \cup \beta} = g_\alpha + g_\beta$ whenever $\alpha \cap \beta = \emptyset$. (Observe that $(g_\alpha)_{\alpha \in \mathcal{F}_r}$ is an IP_r-set in G if and only if $g_\alpha = \sum_{m \in \alpha} g_{\{m\}}$ for every $\alpha \in \mathcal{F}_r$.) The following result is due to Furstenberg and Katznelson [FK1985].

For every positive integer k and every $0 < \delta \leqslant 1$ there exist a positive integer $\mathrm{G}(k, \delta)$ and a strictly positive constant $\varepsilon(k, \delta)$ with the following property. Let G be an abelian group, let $r \geqslant \mathrm{G}(k, \delta)$ be an integer, and let $(g^{(0)}_\alpha)_{\alpha \in \mathcal{F}_r}, \dots, (g^{(k-1)}_\alpha)_{\alpha \in \mathcal{F}_r}$ be IP_r-sets in G. Also let J be a nonempty finite subset of G such that

$$\max\big\{ |(g^{(i)}_{\{m\}} + J) \triangle J| : 0 \leqslant i \leqslant k-1 \text{ and } 0 \leqslant m \leqslant r-1 \big\} \leqslant \varepsilon(k, \delta)|J|. \tag{2.2}$$

If $D \subseteq J$ with $|D| \geqslant \delta |J|$, then D contains a set of the form $\{g + g^{(i)}_\alpha : 0 \leqslant i \leqslant k-1\}$ for some $g \in G$ and some $\alpha \in \mathcal{F}_r$.

Of course, if G is a finite abelian group, then we may set "$J = G$" and apply the above result directly to dense subsets of G. This finite version can also be proved using the hypergraph removal lemma; see, e.g., [RSTT2006]. A proof using the density Hales–Jewett theorem can be found in [DK2016, Section 8.4.2]. As expected, both approaches are quantitatively poor.

2.4. The following result is also due to Furstenberg and Katznelson [FK1985]. It is the density version of the affine Ramsey theorem [GLR1972].

For every prime power q, every positive integer d and every $0 < \delta \leqslant 1$ there exists a positive integer $\mathrm{F}(q, d, \delta)$ with the following property. If $\mathbb{F}_q$ is a finite field with q elements and V is a vector space over $\mathbb{F}_q$ of dimension at least $\mathrm{F}(q, d, \delta)$, then every $D \subseteq V$ with $|D| \geqslant \delta |V|$ contains an affine d-dimensional subspace.

The deduction of this result from the density Hales–Jewett theorem is fairly straightforward (see, e.g., [DK2016, Section 8.4.2]).

In 2017, there was a breakthrough in this direction. Specifically, by the results in [CLP2017, EG2017] it follows that for every positive integer d and every $0 < \delta < 1$ we have

$$\mathrm{F}(3, d, \delta) = O_d\Big(\log \frac{1}{\delta} \Big). \tag{2.3}$$

(The implied constant in (2.3) is effective.). The approach in [CLP2017, EG2017] is based on the *polynomial method*, a recent trend in combinatorics which has led to several significant advances.

As we shall see in Section 3, the bound in (2.3) is too strong to be anywhere near the exact value of the numbers $\mathrm{DHJ}(k, \delta)$. Nevertheless, it is conceivable that a further development of the polynomial method might shed light on the behavior of the density Hales–Jewett numbers (though, at present, this possibility seems rather remote).

2.5. The last result which we shall discuss is known as the IP_r*-Szemerédi theorem* and is due to Furstenberg and Katznelson [FK1985]. It is a multiple recurrence result and it is a far-reaching extension of the multidimensional Szemerédi theorem. To state it we need to introduce some terminology.

Let r be a positive integer and recall that by $\mathcal{F}_r$ we denote the set of all nonempty subsets of $\{0,\dots,r-1\}$. An IP_r*-system* is a family $(T_\alpha)_{\alpha\in\mathcal{F}_r}$ of transformations on a nonempty set X (that is, $T_\alpha\colon X\to X$ for every $\alpha\in\mathcal{F}_r$) such that $T_{\{0\}},\dots,T_{\{r-1\}}$ are pairwise commuting, and moreover

$$T_{\{i_0,\dots,i_m\}} = T_{\{i_0\}}\circ\cdots\circ T_{\{i_m\}} \tag{2.4}$$

for every $0\leqslant m\leqslant r-1$ and every $0\leqslant i_0<\cdots<i_m\leqslant r-1$. Two IP_r-systems $(T_\alpha)_{\alpha\in\mathcal{F}_r}$ and $(S_\alpha)_{\alpha\in\mathcal{F}_r}$ of transformations on the same set X are called *commuting* if $S_\beta\circ T_\alpha = T_\alpha\circ S_\beta$ for every $\alpha,\beta\in\mathcal{F}_r$. Also recall that a *measure preserving transformation* on a probability space (X,Σ,μ) is a measurable map $T\colon X\to X$ with the property that $\mu\big(T^{-1}(A)\big)=\mu(A)$ for every $A\in\Sigma$.

For every positive integer k and every $0<\delta\leqslant 1$ there exist a positive integer $\mathrm{IP\text{-}Sz}(k,\delta)$ *and a strictly positive constant $\eta(k,\delta)$ with the following property. Let $r\geqslant \mathrm{IP\text{-}Sz}(k,\delta)$ be an integer and let $(T^{(1)}_\alpha)_{\alpha\in\mathcal{F}_r},\dots,(T^{(k)}_\alpha)_{\alpha\in\mathcal{F}_r}$ be commuting IP_r-systems of measure preserving transformations on a probability space (X,Σ,μ). If $D\in\Sigma$ with $\mu(D)\geqslant\delta$, then*

$$\mu\big(D\cap {T^{(1)}_\alpha}^{-1}(D)\cap\cdots\cap {T^{(k)}_\alpha}^{-1}(D)\big)\geqslant \eta(k,\delta) \tag{2.5}$$

for some $\alpha\in\mathcal{F}_r$.

We note that the only known effective proof of the IP_r-Szemerédi theorem is based on the density Hales–Jewett theorem; see [DK2016, Section 8.4.1].

3. What about lower bounds?

The density Hales–Jewett numbers are understood rather well when $k=2$. Specifically, for every $0<\delta\leqslant 1$ we have

$$\frac{1}{\delta}\leqslant \mathrm{DHJ}(2,\delta)\leqslant 4\Big(\frac{1}{\delta}\Big)^2. \tag{3.1}$$

(The reader should compare this estimate with that in (2.3).)

However, the case $k\geqslant 3$ is quite different. Indeed, by transferring Behrend's classical construction [Be1946] of a set of integers not containing an arithmetic progression of length three, one obtains a quasi-polynomial lower bound. More precisely, for every $0<\delta\leqslant 1$ we have

$$2^{O\left((\log\frac{1}{\delta})^{\ell}\right)}\leqslant \mathrm{DHJ}(k,\delta) \tag{3.2}$$

with $\ell=\Theta(\log k)$; see [P2010] for details.

4. A conjecture

Perhaps the Hales–Jewett numbers are exponential.
—József Beck, *Combinatorial Games: Tic-Tac-Toe Theory.*

It is very difficult—given the current level of understanding—to predict the exact growth of the density Hales–Jewett numbers. That said, the huge gap between the known lower and upper bounds for the numbers $\mathrm{DHJ}(k,\delta)$ seems to reflect the inefficiency of the existing proofs rather than an inherent intricacy of these invariants. In particular, we believe that the lower bound in (3.2) is closer to the numbers $\mathrm{DHJ}(k,\delta)$, and we make the following conjecture.

Conjecture. *For every integer $k \geqslant 2$ and every $0<\delta\leqslant 1$ we have*

$$\mathrm{DHJ}(k,\delta) \leqslant 2^{(\frac{1}{\delta})^{O_k(1)}} \tag{4.1}$$

with a "reasonable" implied constant.

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ESTIMATES OF THE DERIVATIVES OF THE HEAT KERNEL

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Abstract. We obtain pointwise upper bounds on the derivatives of the heat kernel, on spaces that include symmetric spaces and Damek-Ricci spaces.

1. Introduction and statement of the results

In this survey article we discuss our results related to pointwise upper bounds on the derivatives of the heat kernel on a class of Riemannian manifolds. This class include symmetric spaces and Damek-Ricci spaces. Without loss of generality, we shall focus in the case of hyperbolic space, but the general case can be treated similarly.

Let $S = \mathbb{H}^n$ be the n-dimensional hyperbolic space and let us denote by r the geodesic distance in S. Let us denote by h_t the heat kernel on S. Our main result is the following.

Theorem 1. *For all $\epsilon > 0$ and $i \in \mathbb{N}$ there is a constant $c > 0$ such that*

$$\left|\frac{\partial^i h_t}{\partial t^i}(r)\right| \leq c\, t^{-(n/2)-i} e^{-(1-\epsilon)\left(\frac{(n-1)^2}{4}t + \frac{(n-1)}{2}r + \frac{r^2}{4t}\right)}, \tag{1}$$

for all $t > 0$ and $r \geq 0$.

Denote by $H_t = e^{-\Delta_S t}$ the heat semigroup on S. Fix $i \in \mathbb{N}$. Then, for all $\sigma \geq 0$ we consider as in [1], the σ-maximal operator

$$H_{\sigma,max}(f) = \sup_{t>0} e^{\sigma t} t^i \left|\frac{\partial^i}{\partial t^i} H_t f\right|,$$

and the Littlewood-Paley-Stein operator

$$H_\sigma(f)(x) = \left(\int_0^\infty e^{2\sigma t}\left(t^i \left|\frac{\partial^i}{\partial t^i} H_t f(x)\right|^2 + \|\nabla_x H_t f(x)\|^2\right)\frac{dt}{t}\right)^{1/2}.$$

Next, we apply Theorem 1 in order to prove that the operators $H_{\sigma,max}$ and H_σ are bounded on $L^p(S)$, $p \in (1, \infty)$, provided that

$$\sigma < (n-1)^2/pp'. \tag{2}$$

Finally, we discuss without proof some more applications of Theorem 1.

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2. Preliminaries

2.1. **The heat kernel.** If $\kappa = \kappa(r)$ is a locally integrable radial function and $*|\kappa|$ denotes the convolution operator whose kernel is $|\kappa|$, then in [2, Theorem 3.3] it is proved that

$$\| * |\kappa| \|_{L^p(S)\to L^p(S)} = \int_S dx |\kappa(x)| \phi_{i(\frac{1}{p}-\frac{1}{2})(n-1)}(x), \tag{3}$$

where ϕ_λ are the elementary spherical functions.

Using polar coordinates on S, [2, p.656],

$$\phi_{i(\frac{1}{p}-\frac{1}{2})(n-1)}(r) \asymp \begin{cases} e^{-\frac{(n-1)}{p'}r}, & \text{if} \quad 1 \le p < 2, \\ (1+r)e^{-\frac{(n-1)}{2}r}, & \text{if} \quad p = 2. \end{cases} \tag{4}$$

Denote by h_t the heat kernel on S. Then, h_t is a radial right-convolution kernel on S:

$$h_t(x,y) = h_t(d(x,y)).$$

Then, the following estimate holds:

$$h_t(r) \asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(n-1)^2}{4}t-\frac{(n-1)}{2}r-\frac{r^2}{4t}}, \tag{5}$$

for $t > 0$ and $r \ge 0$ (see [2] for details).

Consequently, (5) implies the upper bound

$$h_t(r) \le c\, t^{-\frac{n}{2}}(1+t)^{\frac{n-3}{2}}(1+r)^{\frac{n-1}{2}} e^{-\frac{n^2}{4}t-\frac{n}{2}r-\frac{r^2}{4t}}. \tag{6}$$

Grigory'an in [3] derived Gaussian upper bounds for all time derivatives of the heat kernel, under some assumptions on the on-diagonal upper bound for h_t on an arbitrary complete non-compact Riemannian manifold M. More precisely, he proves that if there exists an increasing continuous function $f(t) > 0$, $t > 0$, such that

$$h_t(x,x) \le \frac{1}{f(t)}, \text{ for all } t > 0 \text{ and } x \in M,$$

then,

$$\left|\frac{\partial^i h_t}{\partial t^i}\right|(x,y) \le \frac{1}{\sqrt{f(t) f_{2i}(t)}}, \text{ for all } i \in \mathbb{N},\ t > 0,\ x, y \in M, \tag{7}$$

where the sequence of functions $f_i = f_i(t)$, is defined by

$$f_0(t) = f(t) \text{ and } f_i(t) = \int_0^t f_{i-1}(s)ds, i \ge 1.$$

3. Proof of Theorem 1

In this section we give the proof of Theorem 1. More precisely we shall prove the following estimate: for all $\epsilon > 0$ and $i \in \mathbb{N}$, there is a $c > 0$ such that

$$\left|\frac{\partial^i h_t}{\partial t^i}(r)\right| \le c\, t^{-(n/2)-i} e^{-(1-\epsilon)\left(\frac{(n-1)^2}{4}t+\frac{(n-1)}{2}r+\frac{r^2}{4t}\right)}, \tag{8}$$

for all $r \ge 0$ and all $t > 0$.

For the proof of (8) we need several lemmata. The following lemma is technical but important for the proof of Theorem 1. It provides a method to obtain estimates for the first derivative of a function, given some upper bounds on the function and its second derivative.

Lemma 2. *Let*

$$\alpha > \beta,\ D \geq D_*,\ B \geq B_*,\ C \geq C_*, \tag{9}$$

and assume that for fixed $r \geq 0$ the function $f_r : (0, +\infty) \to \mathbb{R}$, satisfies

$$|f_r(t)| \leq c\, t^{-\alpha}(1+t)^{\beta}(1+r)^{\gamma} e^{-Dt-Br-Cr^2/(4t)} \tag{10}$$

and

$$\left|\frac{d^2 f_r}{dt^2}(t)\right| \leq c\, t^{-\alpha-2}(1+t)^{\beta}(1+r)^{\gamma} e^{-D_*t-B_*r-C_*r^2/(4t)}. \tag{11}$$

Then, for all $\epsilon \in (0,1)$, there is a constant $c > 0$, that does not depend on r, t, such that for all $r \geq 0$,

$$\begin{aligned}\left|\frac{df_r}{dt}(t)\right| \leq & c\, t^{-\alpha-1}(1+t)^{\beta}(1+r)^{\gamma} \\ & \times e^{-\left((D_*+D)t/2+(B_*+B)r/2+(C_*+C\lambda_\epsilon)r^2/8t\right)},\end{aligned}$$

where $\lambda_\epsilon = \frac{1-\epsilon}{1+\epsilon}$.

The proof follows the following strategy. By applying twice the mean value theorem, one can prove that

$$\left|\frac{df_r}{dt}(t)\right| \leq \frac{1}{\delta}\left(|f_r(t)| + |f_r(t+\delta)|\right) + \delta \sup_{\tau \in (t, t+\delta)} \left|\frac{d^2 f_r}{dt^2}(\tau)\right|, \text{ for all } \delta > 0. \tag{12}$$

We use this formula and we choose δ appropriately in order to balance the exponential terms.

We next apply the estimate (7) of Grigory'an in the case of a hyperbolic space.

Lemma 3. *For all $i \in \mathbb{N}$ there is a constant $c > 0$ such that*

$$\left|\frac{\partial^i h_t}{\partial t^i}(r)\right| \leq c\, t^{-\frac{n}{2}-i}(1+t)^{\frac{n-3}{2}}, \text{ for all } t > 0,\ r \geq 0. \tag{13}$$

Note that the derivative estimates in the above lemma do not involve the distance r. Given the estimates (6), and (13), we can apply Lemma 2 and find improved upper bounds for the $i-th$ derivative, for every $i \geq 1$. In this way, we can refine the upper bound for the $i-th$ derivative given by the above lemma. We can apply an inductive argument in order to obtain $\ell \in \mathbb{N}$ successive upper bounds for the $i-th$ derivative, for every $i \geq 1$.

Lemma 4. *Let us fix $\epsilon \in (0,1)$ and set $\lambda_\epsilon = \frac{1-\epsilon}{1+\epsilon}$. Then, for all $i, \ell \in \mathbb{N}$, there are non-negative constants $c, \beta_\ell^i, \gamma_\ell^i$, such that*

$$\begin{aligned}\left|\frac{\partial^i h_t}{\partial t^i}(r)\right| \leq & c\, t^{-(n/2)-i}(1+t)^{(n-3)/2}(1+r)^{(n-1)/2} \\ & \times e^{-\beta_\ell^i\left(\frac{(n-1)^2}{4}t+\frac{(n-1)}{2}r\right)} e^{-\gamma_\ell^i \frac{r^2}{4t}},\end{aligned} \tag{14}$$

for all $t > 0$ *and* $r \geq 0$, *where* c *is a constant that depends on* ϵ, i, k. *Furthermore the sequences* β_k^i, γ_k^i *satisfy the iteration formulas*

$$\begin{aligned} \beta_\ell^i &= \frac{1}{2}(\beta_{\ell-1}^{i-1} + \beta_{\ell-1}^{i+1}), \\ \gamma_\ell^i &= \frac{1}{2}(\lambda_\epsilon \gamma_{\ell-1}^{i-1} + \gamma_{\ell-1}^{i+1}), \end{aligned} \tag{15}$$

and the initial conditions

$$\beta_0^i = 0,\ \gamma_0^i = 0,\ \text{for all}\ \ i \geq 1,\ \beta_\ell^0 = 1,\ \gamma_\ell^0 = 1,\ \text{for all}\ \ \ell \geq 0. \tag{16}$$

Remark. The constant $c = c(i, \ell, \epsilon)$ in relation (14) of Lemma 4 depends on i, ℓ and ϵ and it increases to infinity (when either $i \to \infty$ or $\ell \to \infty$ or $\epsilon \to 0$), but we only need the fact that it is finite for fixed i, ℓ, ϵ.

In the following Lemma, we prove by induction that the exponential coefficients β_ℓ^i, γ_ℓ^i of Lemma 5 are convergent sequences of ℓ and we shall compute their limits. Using this fact, we shall show that these coefficients, after a sufficiently large number of iterations, can get arbitratily close to 1.

Lemma 5. *For any* $i \in \mathbb{N}$,

$$\lim_{\ell \to \infty} \gamma_\ell^i = \left(1 - \sqrt{1 - \lambda_\epsilon}\right)^i \ \ \text{and} \ \ \lim_{\ell \to \infty} \beta_\ell^i = 1. \tag{17}$$

End of the proof of Theorem 1: To complete the proof of Theorem 1, note that $\lim_{\epsilon \to 0} \left(1 - \sqrt{1 - \lambda_\epsilon}\right)^i = 1$. Thus, taking $\ell \in \mathbb{N}$ sufficiently large and ϵ sufficiently close to zero, one has $\gamma_\ell^i \geq 1 - \epsilon$ and $\beta_\ell^i \geq 1 - \epsilon$. Thus, from (14) and (17) it follows that

$$\left| \frac{\partial^i h_t}{\partial t^i}(r) \right| \leq c\, t^{-(n/2)-i}(1+t)^{(n-3)/2}(1+r)^{(n-1)/2} e^{-(1-\epsilon)\left(\frac{(n-1)^2}{4}t + \frac{(n-1)}{2}r + \frac{r^2}{4t}\right)}.$$

Taking now into account that if $a, b > 0$ then there exists a constant $c = c(a, b)$ such that $x^a \leq c e^{bx}$ for all $x > 0$, we conclude that for every $\epsilon > 0$, there exists a constant $c > 0$ such that

$$\left| \frac{\partial^i h_t}{\partial t^i}(x, y) \right| \leq c\, t^{-(n/2)-i} e^{-(1-\epsilon)\left(\frac{(n-1)^2}{4}t + \frac{(n-1)}{2}r + \frac{r^2}{4t}\right)},$$

and the proof of Theorem 1 is complete.

4. Applications

In this section we apply the estimates of the derivatives of the heat kernel.

We claim that the operators $H_{\sigma,max}$ and H_σ defined in Section 1 are bounded on $L^p(S), p \in (1, \infty)$, provided that

$$\sigma < (n-1)^2/pp'.$$

We shall only sketch the proof for $H_{\sigma,max}$. The proof for H_σ is similar and then omitted.

We consider separately the *small time* maximal operator

$$H^0_{\sigma,max}(f)(x) = \sup_{0<t\leq 1} e^{\sigma t} t^i \left| \frac{\partial^i}{\partial t^i} H_t(f)(x) \right|$$

and the *large time* maximal operator

$$H^{\infty}_{\sigma,max}(f)(x)=\sup_{t\geq 1} e^{\sigma t}t^i|\frac{\partial^i}{\partial t^i}H_t(f)(x)|.$$

As noted in [1], the whole problem comes from the component $H^{\infty}_{\sigma,max}$.

Let

$$k^{\infty}_{\sigma,max}(x)=\sup_{t\geq 1} e^{\sigma t}t^i|\frac{\partial^i}{\partial t^i}h_t(x)|.$$

Then, the component $H^{\infty}_{\sigma,max}$ can be handled by estimating

$$H^{\infty}_{\sigma,max}(f)(x)\leq |f|*k^{\infty}_{\sigma,max} \tag{18}$$

and applying the Kunze-Stein phenomenon. For the estimates of $k^{\infty}_{\sigma,max}$ we apply Theorem 1.

Next, we can show that the component $H^{0}_{\sigma,max}$ is bounded on $L^p(S)$, $p\in(1,\infty)$. Indeed, we split the operator $H^{0}_{\sigma,max}$ into two parts

$$H^{0,0}_{\sigma,max}(f)(x)=\sup_{0<t\leq 1} e^{\sigma t}t^i|\frac{\partial^i}{\partial t^i}f*\psi h_t(x)|$$

and

$$H^{0,\infty}_{\sigma,max}(f)(x)=\sup_{0<t\leq 1} e^{\sigma t}t^i|\frac{\partial^i}{\partial t^i}f*(1-\psi)h_t(x)|$$

using a smooth cutoff fuction $\psi\in C^{\infty}_c(S)$ with $\psi\equiv 1$ near the origin. Then we observe that the second term $H^{0,\infty}_{\sigma,max}$ can be handled like $H^{\infty}_{\sigma,max}$ and the first term $H^{0,0}_{\sigma,max}$ can be handled as in the Euclidean case (see for example [1, 2]).

In a similar way we can study the boundedness of the Littlewood-Paley-Stein operator defined in Section 1, the Riesz transform $R=\nabla(-\Delta_S)^{-1/2}$, and also operators related to the Poisson operator $P_t=e^{-t(-\Delta_S)^{1/2}}$. Also, by applying Theorem 1 we can estimate the L^p norm of the operators $\Delta e^{-t\Delta}$ and $\nabla_x e^{-t\Delta}$. Finally, we can estimate the derivatives of the heat kernel on a Kleinnian group $M=\Gamma\backslash\mathbb{H}^n$ by applying the formula

$$h^M_t(x,y)=\sum_{g\in\Gamma}h_t(x,g(y)), \tag{19}$$

and we can study operators that are functions of the Laplacian on M.

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ON GENERALIZED STIELTJES FUNCTIONS AND THEIR APPROXIMATIONS

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Abstract. We present a survey of some recent results on generalized Stieltjes functions. We give definitions, characterizations, properties and approximations of such functions. Several applications are also provided.

1. Introduction

In this article we survey some recent results on generalized Stieljes functions. These functions are a special case of completely monotonic functions. Completely monotonic functions have a long history, going back to the seminal work of F. Hausdorff [8] who called such functions "total monotone". He also discovered their close relation with moment sequences of finite positive measures on $[0, 1]$. Let us recall the definition.

Definition 1.1. A function $f : (0, \infty) \to \mathbb{R}$ is called *completely monotonic* if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, \quad \text{for all} \quad x > 0 \quad \text{and} \quad n = 0, 1, 2, \dots \tag{1.1}$$

J. Dubourdieu [7] proved that if a non-constant function f is completely monotonic on $(0, \infty)$, then strict inequality holds in (1.1). See also [9] for a simpler proof of this result.

S. N. Bernstein, see [28, pp. 160–161] gave the following characterization of completely monotonic functions.

Theorem 1.2. *The function f is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{1.2}$$

where μ is a non-negative Borel measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

Motivated by some applications on asymptotic expansions of certain special functions, some interesting subclasses of completely monotonic functions have been introduced in [16].

Definition 1.3. Let $\alpha \geq 0$. A function $f : (0, \infty) \to \mathbb{R}$ is called *completely monotonic function of order α* if $x^\alpha f(x)$ is completely monotonic on $(0, \infty)$.

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There is an analogue of Bernstein's theorem mentioned above, for completely monotonic functions of positive order. We recall that the Riemann-Liouville fractional integral $I_\alpha(\mu)(t)$ of order $\alpha > 0$, of a Borel measure μ on $[0,\infty)$ is defined by

$$I_\alpha(\mu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, d\mu(s).$$

The following characterization has been obtained in [16].

Theorem 1.4. *The function $f : (0, \infty) \to \mathbb{R}$ is completely monotonic of order $\alpha > 0$ if and only if f is the Laplace transform of a fractional integral of order α of a non-negative Radon measure μ on $[0, \infty)$, that is,*

$$f(x) = \int_0^\infty e^{-xt}\, I_\alpha(\mu)(t)\, dt,$$

and the integral converges for all $x > 0$.

This characterization takes a simpler form when α is a positive integer. We need first to introduce the following classes of functions.

Definition 1.5. Let A_0 denote the set of non-negative Borel measures μ on $[0,\infty)$ such that $\displaystyle\int_0^\infty e^{-xs}\, d\mu(s) < \infty$ for all $x > 0$. Let A_1 denote the set of functions $t \mapsto \mu([0,t])$, where $\mu \in A_0$. For $n \geq 2$, let A_n denote the set of $n-2$ times differentiable functions $\xi : [0,\infty) \to \mathbb{R}$ satisfying $\xi^{(j)}(0) = 0$ for $j \leq n-2$ and
$\xi^{(n-2)}(t) = \displaystyle\int_0^t \mu([0,s])\, ds$ for some $\mu \in A_0$.

With this definition the characterization can be stated as follows.

Proposition 1.1. *Let r be a positive integer. A function $f : (0,\infty) \to \mathbb{R}$ is completely monotonic of order r if and only if*

$$f(x) = \int_0^\infty e^{-xt}\xi(t)\, dt$$

for some $\xi \in A_r$.

Let us give a non trivial example of a completely monotonic function of positive order. Consider the remainder $r_n(x)$ in the asymptotic expansion of the logarithm of Euler's gamma function.

$$\begin{aligned}
&\log \Gamma(x) \\
&= \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{n} \frac{B_{2k}}{(2k-1)\,2k}\,\frac{1}{x^{2k-1}} \\
&+ (-1)^n\, r_n(x),
\end{aligned}$$

where B_k are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^\infty B_k \frac{t^k}{k!} = 1 - \frac{t}{2} + \sum_{j=1}^\infty B_{2j}\frac{t^{2j}}{(2j)!}, \quad |t| < 2\pi.$$

It can be shown the following result. See [12] and also [15] for a simpler proof.

Proposition 1.2. *(i) The remainder $r_n(x)$ in the above asymptotic expansion is a completely monotonic function of order n on $(0, \infty)$, for all $n \geq 0$. (ii) The following inequality holds true*

$$0 < r_n(x) < (-1)^n \frac{B_{2n+2}}{(2n+1)(2n+2)} \frac{1}{x^{2n+1}},$$

for all $x > 0$ and $n \geq 0$.

Results of this type have been obtained for asymptotic expansions of several other special functions, such as, multiple gamma functions, multiple zeta functions and polygamma functions. See [13], [14], [15], [16], [17], [18], [19], [20].

There are some other subclasses of completely monotonic functions that are of importance in applications.

Definition 1.6. A function $f : (0, \infty) \to (0, \infty)$ is called *logarithmically completely monotonic* if f has derivatives of all orders and $-(\log f)'$ is completely monotonic on $(0, \infty)$.

Applying Leibniz's rule and induction it can be shown that *every logarithmically completely monotonic function is completely monotonic.* The converse need not be true. Consider, for example, the function $f(x) = e^{-x} + e^{-2x}$.

A non trivial example is the following. For $\nu > -1$ the function

$$x^{\nu/2} 2^{-\nu} \{I_\nu(\sqrt{x})\Gamma(\nu+1)\}^{-1}$$

is logarithmically completely monotonic on $[0, \infty)$. Here, $I_\nu(x)$ is the modified Bessel function of the first kind. See [10]. This is a special case of a more general result for entire functions, see [24].

A characterization of logarithmically completely monotonic functions is the following.

Theorem 1.7. *A function $f : (0, \infty) \to (0, \infty)$ is logarithmically completely monotonic if and only if*

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is an infinitely divisible measure on $[0, \infty)$ and the integral converges for all $x > 0$.

We recall that a measure μ on $[0, \infty)$ is called *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a measure μ_n on $[0, \infty)$ such that $\mu = \mu_n * \mu_n * \ldots * \mu_n$ (n times), where $*$ denotes the convolution of measures.

We next consider a subclass of logarithmically completely monotonic functions.

Definition 1.8. A function $f : (0, \infty) \to \mathbb{R}$ is called *a Stieltjes function*, if it is of the form

$$f(x) = c + \int_0^\infty \frac{d\mu(t)}{x+t},$$

where c is a nonnegative constant and μ is a non-negative Borel measure on $[0, \infty)$ making the integral convergent for any $x > 0$.

There is a fundamental relationship between Stieltjes functions and Laplace transforms:

Theorem 1.9.
$$F(x) = \int_0^\infty \frac{d\mu(t)}{x+t}, \quad \textit{for all} \;\; x > 0,$$
where μ *is a non-negative Borel measure on* $[0, \infty)$, *if and only if*
$$F(x) = \int_0^\infty e^{-xt} f(t)\, dt \;\; \textit{with} \;\; f(t) = \int_0^\infty e^{-ts}\, d\mu(s).$$

It is known that every completely monotonic density is infinitely divisible (cf. [10]). In general, it can be shown that every Stieltjes function is logarithmically completely monotonic.

It is easily seen that every Stieltjes function has a holomorphic extension to the cut plane $\mathbb{A} := \mathbb{C} \setminus (-\infty, 0]$. This turns out to be a useful observation. For instance, the function
$$\frac{1}{x(1+x^2)} = \int_0^\infty e^{-xt}\,(1-\cos t)\, dt$$
is obviously completely monotonic, but it *cannot* be a Stieltjes function, since it has poles at $\pm i$.

The following characterization of Stieltjes functions is proved in [1] and there attributed to Krein.

Theorem 1.10. *A function* $f : (0, \infty) \to \mathbb{R}$ *is a Stieltjes function if and only if* $f(x) \geq 0$ *for all* $x > 0$ *and it has a holomorphic extension to the cut plane* $\mathbb{A} = \mathbb{C} \setminus (-\infty, 0]$ *satisfying* $\mathrm{Im}\, f(x+iy) \leq 0$ *for all* $y > 0$.

Before proceeding any further let us give some interesting examples of Stieltjes functions. See [2], [3], [5].

Proposition 1.3. *The following functions are Stieltjes functions.*

(*i*) $\dfrac{x \log x}{\log \Gamma(x+1)}$,

(*ii*) $\Phi(x) := \dfrac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\dfrac{1}{x}\right)^x, \quad \log \Phi(x)$,

(*iii*)
$$\begin{aligned} h(x) : &= (x+1)\left[\mathrm{e} - \left(1+\frac{1}{x}\right)^x\right] \\ &= \frac{\mathrm{e}}{2} + \frac{1}{\pi}\int_0^1 \frac{t^t\,(1-t)^{1-t}\,\sin(\pi t)}{x+t}\, dt, \end{aligned}$$

(*iv*) *For* $a < 1$, $x > 0$,
$$\begin{aligned} F_a(x) : &= e^x\, x^{-a} \int_x^\infty e^{-t}\, t^{a-1}\, dt \\ &= \frac{1}{\Gamma(1-a)} \int_0^\infty \frac{1}{x+s}\, e^{-s} s^{-a}\, ds. \end{aligned}$$

A real-variable characterization of Stieltjes functions has been given by D. V. Widder [27].

Theorem 1.11. *f is a Stieltjes function if and only if*

$$\frac{d^n}{dx^n}\big[x^n\, f(x)\big]$$

is completely monotonic on $(0,\,\infty)$ for all $n = 0, 1, 2 \ldots$.

The aim of this article is to present some other characterizations of Stieltjes functions in terms of their asymptotic expansions. We put this on a more general setting by considering generalized Stieltjes functions.

Definition 1.12. Let λ be a positive real number. A function $f : (0,\,\infty) \to \mathbb{R}$ is called *a generalized Stieltjes function of order* λ, if it is of the form

$$f(x) = c + \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda}\ ,$$

where c is a non-negative constant and μ is a non-negative Borel measure on $[0,\,\infty)$ making the integral convergent for all $x > 0$. The class of these functions is denoted by $\mathcal{S}_\lambda$.

We give some examples that emerge in the study of special functions.

Let

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!}\, x^n$$

be the Gaussian hypergeometric function. If $c > b > 0$, then according to Euler's integral representation [4] we have

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1+xt)^{-a}\, dt.$$

Assume $0 < a \le b$. Then ${}_2F_1(a,b;c;-x) \in \mathcal{S}_a$. Indeed,

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty \frac{\varphi(t)}{(x+t)^a}\, dt,$$

where $\varphi(t) = t^{a-c}(t-1)^{c-b-1}, \ \ t > 1$.

Another interesting example is the following. For $a > 0$, the function

$$F(x) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt}\, t^{a-1}\, (1+t)^{c-a-1}\, dt,$$

is a solution of the differential equation

$$x\, y'' + (c-x)\, y' - a\, y = 0,$$

which is known as the confluent hypergeometric equation, see [4, 188-189]. For $a+1 > c$, we have $F \in \mathcal{S}_a$. This follows easily by applying the following characterization, see [19, Lemma 2.1].

Proposition 1.4. *A function f belongs to $\mathcal{S}_\lambda$ if and only if it is of the form*

$$f(x) = c + \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xs}\, s^{\lambda-1}\, \varphi(s)\, ds,$$

where $\varphi(s) = \int_0^\infty e^{-ts}\, d\mu(t)$ for some non-negative Borel measure μ and c is a non-negative constant. In the affirmative case μ is the measure representing f.

Some properties of generalized Stieltjes functions are given next (cf. [11]).

Proposition 1.5. *(i) If $\alpha < \beta$ then $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$*
(ii) $\displaystyle\bigcap_{\alpha>0} \mathcal{S}_\alpha = \{\textit{non-negative constants}\}$
(iii) $\overline{\displaystyle\bigcup_{\alpha>0} \mathcal{S}_\alpha} = \mathcal{C}$,
where $\mathcal{C}$ is the class of completely monotonic functions and the closure is taken with respect to the pointwise convergence on $(0, \infty)$.

We next give some definitions and notations.

Let $\mathcal{M}^*$ denote the class of non-negative Borel measures on $[0, \infty)$ having finite moments of all orders.

For $\mu \in \mathcal{M}^*$ the moments $\{s_n(\mu)\}$ are defined by

$$s_n(\mu) = \int_0^\infty x^n \, d\mu(x), \quad n \geq 0.$$

The class $\mathcal{S}^*_\lambda$ denotes those functions from $\mathcal{S}_\lambda$ corresponding to $c = 0$ and $\mu \in \mathcal{M}^*$.

We shall be concerned with asymptotic expansions in the complex plane.

Let Ω denote an unbounded domain of the complex plane, not containing 0. We recall that a function g defined in Ω has an asymptotic series

$$g(z) \sim \sum_{k=0}^\infty \frac{b_k}{z^k}$$

if, for any $n \geq 0$

$$z^n \left(g(z) - \sum_{k=0}^{n-1} \frac{b_k}{z^k} \right) \to b_n$$

as $z \to \infty$ within Ω. (In general the series $\sum_{k=0}^\infty b_k/z^k$ may diverge.)

As the sets Ω we shall use sectors of the form

$$S_\theta = \{ z \in \mathbb{C} \setminus \{0\} \,|\, |\arg z| \leq \theta \},$$

where $\arg z$ denotes the principal argument of z. For $\theta < \pi$ these sectors exclude the negative real line.

2. Main results

The following result shows that any function in the class $\mathcal{S}^*_\lambda$ has an asymptotic expansion with a suitable representation for the remainders and this has been obtained in [19]. We shall use the standard notation $(\lambda)_k = (\lambda + k - 1) \cdots (\lambda + 1)\lambda = \Gamma(k + \lambda)/\Gamma(\lambda)$.

Theorem 2.1. *Suppose that*

$$f(z) = \int_0^\infty \frac{d\mu(t)}{(z+t)^\lambda}\,, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\lambda > 0$ and $\mu \in \mathcal{M}^*$. Then the function $z^{\lambda-1} f(z)$ has the asymptotic expansion

$$z^{\lambda-1} f(z) = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!} \frac{(-1)^k s_k(\mu)}{z^{k+1}} + (-1)^n R_n(z),$$

for all $n \geq 0$, where, for $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$R_n(z)z^{1-\lambda} = \frac{(\lambda)_n}{(n-1)!} \int_0^1 (1-s)^{n-1} \int_0^\infty \frac{t^n}{(z+st)^{n+\lambda}} d\mu(t) ds.$$

For $z \in S_{\pi-\delta}$ the remainder R_n satisfies the estimate

$$|R_n(z)| \leq \frac{(\lambda)_n s_n(\mu)}{n! (\sin \delta)^{n+\lambda}} \frac{1}{|z|^{n+1}},$$

and for $z \in S_{\pi/2}$ the estimate

$$|R_n(z)| \leq \frac{(\lambda)_n s_n(\mu)}{n!} \frac{1}{|z|^{n+1}}.$$

For z in the open right half plane the remainder has the representation

$$R_n(z) = \frac{z^{\lambda-1}}{\Gamma(\lambda)} \int_0^\infty e^{-zt}\, t^{\lambda-1} \xi_n(t)\, dt,$$

where ξ_n belongs to $C^\infty([0,\infty))$, and satisfies $\xi_n^{(j)}(0) = 0$ for $j \leq n-1$ and $0 \leq \xi_n^{(n)}(t) \leq s_n(\mu)$ for $t \geq 0$.

We refer to [19] for the details of the proof of Theorem 2.1 and various applications of it. It turns out that a converse of this theorem holds true (cf. [19, Theorem 3.3]).

Theorem 2.2. *Let $\lambda > 0$ and let $\{a_j\}$ be a real sequence. Suppose that $f : (0,\infty) \to \mathbb{R}$ satisfies the following: For any $n \geq 0$ there exists $\xi_n \in A_n$ such that $e^{-xt} t^{\lambda-1} \xi_n(t) \in L^1([0,\infty))$ for all $x > 0$ such that*

$$x^{\lambda-1} f(x) = \sum_{j=0}^{n-1} \frac{(\lambda)_j}{j!} \frac{a_j}{x^{j+1}} + (-1)^n \frac{x^{\lambda-1}}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{\lambda-1}\, \xi_n(t)\, dt.$$

Then f has the representation

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda}, \quad \text{for} \quad x > 0,$$

where $\mu \in \mathcal{M}^*$ and $a_j = (-1)^j\, s_j(\mu)$ for $j \geq 0$.

We note that the condition on ξ_0 is understood as $e^{-xt} t^{\lambda-1} \in \mathcal{L}^1(\xi_0)$ and the integral involving $\xi_0(t)$ is understood as

$$\int_0^\infty e^{-xt}\, t^{\lambda-1}\, d\xi_0(t).$$

Some interesting special cases of the above are given next.

Corollary 2.3. *Let $\lambda \in (0,1]$ and let $\mu \in \mathcal{M}^*([0,\infty))$. Then the asymptotic expansion*

$$x^{\lambda-1} \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!} \frac{(-1)^k\, s_k(\mu)}{x^{k+1}} + (-1)^n\, R_n(x)$$

holds for all $n \geq 0$, where R_n is a completely monotonic function of order n.

Corollary 2.4. *Let $\lambda \in (1,\infty)$ and let $\mu \in \mathcal{M}^*([0,\infty))$. Then, for any $n \geq 0$, the asymptotic expansion*

$$x^{\lambda-1}\int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!}\,\frac{(-1)^k\, s_k(\mu)}{x^{k+1}} + (-1)^n\, R_n(x)$$

holds, where R_n is a completely monotonic function of order $n-\lambda+1$.

We note that in the case where $\lambda > 1$ and $f(x) = \displaystyle\int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda}$ for some $\mu \in \mathcal{M}^*$, if, for some $n \geq 0$ we have

$$x^{\lambda-1} f(x) = \sum_{j=0}^{n-1} \frac{a_j}{x^{j+1}} + (-1)^n R_n(x),$$

where R_n is completely monotonic of order n, then either $\displaystyle\int_0^\infty \frac{d\mu(t)}{t^{n+\lambda}} = \infty$ or $\mu \equiv 0$.

In view of the above, we have the following characterization for ordinary Stieltjes functions.

Corollary 2.5. *The following are equivalent for a function $f : (0,\infty) \to \mathbb{R}$:*

(a) *f has the representation*

$$f(z) = \int_0^\infty \frac{d\mu(t)}{z+t}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\mu \in \mathcal{M}^*$.

(b) *f admits an asymptotic expansion $f(x) \sim \sum_{k=0}^\infty a_k/x^{k+1}$ on $x > 0$ in which the remainder R_n in the expansion*

$$f(x) = \sum_{k=0}^{n-1} \frac{a_k}{x^{k+1}} + (-1)^n R_n(x)$$

is completely monotonic of order n for any $n \geq 0$.

In the affirmative case, $a_k = (-1)^k s_k(\mu)$, and f admits an asymptotic expansion in $S_{\pi-\delta}$ for any $\delta > 0$.

There are some real-variable characterizations of generalized Stieltjes functions. We need first to introduce some differential operators.

(i) For $\lambda > 0$ and n, k non-negative integers

$$[T^\lambda_{n,k}(f)](x) := (-1)^n\, x^{-(n+\lambda-1)}\, \frac{d^k}{dx^k}\left[x^{n+k+\lambda-1}\, f^{(n)}(x)\right]$$

and (ii)

$$[c^\lambda_k(f)](x) := x^{1-\lambda} \frac{d^k}{dx^k}\left[x^{\lambda-1+k} f(x)\right].$$

Lemma 2.6. *The relation*

$$T^\lambda_{n,k}(f)(x) = (-1)^n \left(c^\lambda_k(f)\right)^{(n)}(x)$$

holds for any $n, k \geq 0$ and $x > 0$.

See for details in [22].

Theorem 2.7. *The following are equivalent for a function $f \in C^\infty((0,\infty))$:*

(i) *f is a generalized Stieltjes function of order λ.*

(ii) *$c_k^\lambda(f)$ is completely monotonic for all $k \geq 0$.*

(iii) *$T_{n,k}^\lambda(f) \geq 0$ for all $n \geq 0$ and all $k \geq 0$.*

The proof of this result is given in the recently published paper [22]. See also [26] for related considerations.

We are able to characterize, for any given positive integer N, those functions f for which $c_0^\lambda(f), \ldots, c_N^\lambda(f)$ are completely monotonic. We introduce the classes $\mathcal{C}_N^\lambda$ as

$$\mathcal{C}_N^\lambda = \{f \in C^\infty((0,\infty)) \,|\, c_k^\lambda(f) \in \mathcal{C} \text{ for } k = 0, \ldots, N\}.$$

We need some notation from the theory of distributions. The standard reference is [25]. We recall that the action of a distribution u on a test function φ (an infinitely often differentiable function of compact support in $(0,\infty)$) is denoted by $\langle u, \varphi \rangle$. The distribution ∂u is defined via $\langle \partial u, \varphi \rangle = -\langle u, \varphi' \rangle$.

The following characterization is also obtained in [22].

Theorem 2.8. *Let $\lambda > 0$ be given, and let $N \geq 1$. The following properties of a function $f : (0,\infty) \to \mathbb{R}$ are equivalent.*

(i) *$f \in \mathcal{C}_N^\lambda$;*

(ii) *f can be represented as*

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s),$$

where $c \geq 0$, and μ is a non-negative Borel measure on $(0,\infty)$ for which $\mu_k \equiv (-1)^k s^k \partial^k \mu$, (in distributional sense) is a non-negative Borel measure such that

$$\int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s) < \infty, \quad k = 0, \ldots, N.$$

In the affirmative case,

$$c_k^\lambda(f)(x) = x^{1-\lambda} \left(x^{\lambda-1+k} f(x)\right)^{(k)} = \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s) + (\lambda)_k\, c$$

for $k = 0, \ldots, N$.

The representing measures μ_k are related as follows.

Theorem 2.9. *Suppose that $f \in \mathcal{C}_N^\lambda$, and let for $k = 0, \ldots, N$*

$$c_k^\lambda(f)(x) = \int_0^\infty e^{-xs}\, d\mu_k(s) + b_k,$$

where μ_k is a non-negative Borel measure on $(0,\infty)$ and $b_k \geq 0$. Then, in the distributional sense,

$$(-1)^k s^k \partial^k (s^{1-\lambda} \mu_0) = s^{1-\lambda} \mu_k.$$

(cf. [22, Proposition 2.2]

There is a simple way to construct examples of functions in the class $\mathcal{C}_N^\lambda$.

Proposition 2.1. *Let $\lambda > 0$ and $k \geq 0$. Assume that $p \in C^k((0,\infty))$. Then for the function f given by*

$$f(x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{\lambda-1}\, p(t)\, dt, \quad x > 0,$$

we have

$$[T^\lambda_{n,k}(f)](x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{n+k+\lambda-1}\, (-1)^k p^{(k)}(t)\, dt.$$

We refer to [19] for the proof of the above proposition and related results.

Corollary 2.10. *Assume that $p \in C^N((0,\infty))$ and satisfies*

$$(-1)^k\, p^{(k)}(t) \geq 0, \quad \text{for } k = 0, 1, \ldots N.$$

Then for the function f given by

$$f(x) = \int_0^\infty e^{-xt}\, t^{\lambda-1}\, p(t)\, dt, \quad x > 0,$$

we have that $f \in \mathcal{C}^\lambda_N$.

A simple example in the case where $\lambda = 1$ is the following, see [23]. Let

$$h(s) = \begin{cases} 1, \ \ 0 < s < 1 \\ 2 - s, \ \ 1 < s < 2 \\ 0, \ \ 2 < s \end{cases}$$

An easy computation shows that

$$f(x) = \int_0^\infty e^{-xs}\, h(s)\, ds = \frac{1}{x} + \frac{e^{-2x} - e^{-x}}{x^2}.$$

Then $f \in \mathcal{C}^1_1 \setminus \mathcal{C}^1_2$.

The functions p appearing in the above Corollary have a name and they can be characterized in terms of integral representations.

Definition 2.11. A function $p : (0, \infty) \to \mathbb{R}$ is called *N-monotonic* if $p \in C^N((0, \infty))$ and satisfies

$$(-1)^k\, p^{(k)}(x) \geq 0, \ \text{ for } \ k = 0, 1, 2, \ldots N\,.$$

A characterization of N-monotonic functions is the following, see [9].

Theorem 2.12. *For a function $p : (0, \infty) \to \mathbb{R}$ the following statements are equivalent*
(i) p is N-monotonic.
(ii) There exist a unique constant $c \geq 0$ and a unique measure ν on $(0, \infty)$ such that

$$p(t) = c + \frac{1}{(N-1)!} \int_{(t,\infty)} (u - t)^{N-1}\, d\nu(u)\,.$$

(iii) There exists a unique measure ω_N on $[0,\infty)$ such that

$$p(t) = \int_{[0,\infty)} (1 - tu)_+^{N-1}\, d\omega_N(u)\,.$$

3. Absolutely monotonic functions

An important counterpart of completely monotonic functions are the absolutely monotonic functions. Let us recall the definition.

Definition 3.1. A function $\varphi : [0,\infty) \to \mathbb{R}$ is called absolutely monotonic if it is infinitely often differentiable on $[0,\infty)$ and $\varphi^{(k)}(x) \geq 0$ for all $k \geq 0$ and all $x \geq 0$.

An absolutely monotonic function φ on $[0,\infty)$ has an extension to an entire function with the power series expansion $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \geq 0$ for all $n \geq 0$.

The Laplace transform of φ is defined exactly when φ extends to an entire function of at most exponential type zero, meaning that φ has the following property. For any given $\epsilon > 0$ there exists a positive constant C_ϵ such that $|\varphi(z)| \leq C_\epsilon e^{\epsilon|z|}$ for all $z \in \mathbb{C}$.

There are some results for the Laplace transform of absolutely monotonic functions analogous to the ones given in the previous section. Let us begin with the following elementary example. The function $H(x) = x^{-1}e^{1/x}$ satisfies

$$H_k(x) \equiv (-1)^k \big(x^k\, H(x)\big)^{(k)} = x^{-(k+1)}\, e^{1/x}, \quad x > 0.$$

Hence H_k is completely monotonic for all $k \geq 0$, being a product of completely monotonic functions. We also have

$$H(x) = \int_0^\infty e^{-xt} h(t)\, dt,$$

where $h(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\, t^n$ is absolutely monotonic.

It turns out that a general characterization for the Laplace transforms of absolutely monotonic functions holds true. This is obtained in [21].

Theorem 3.2. *The following properties of a function $f : (0,\infty) \to \mathbb{R}$ are equivalent.*

(i) *There is an absolutely monotonic function $\varphi : [0,\infty) \to \mathbb{R}$ such that*

$$f(x) = \mathcal{L}(\varphi)(x) = \int_0^\infty e^{-xt} \varphi(t)\, dt, \quad x > 0.$$

(ii) *There is a sequence $\{a_n\}$, with $a_n \geq 0$, such that we have for all $n \geq 0$*

$$f(x) = \sum_{k=1}^{n} \frac{a_k}{x^k} + R_n(x), \quad x > 0$$

where R_n is a completely monotonic function of order n.

(iii) *The function $(-1)^k (x^k\, f(x))^{(k)}$ is completely monotonic for all $k \geq 0$.*

(iv) *The function $(-1)^k (x^k\, f(x))^{(k)}$ is non-negative for all $k \geq 0$.*

(v) *We have $f(x) \geq 0$ and $(x^k\, f(x))^{(2k-1)} \leq 0$ for all $k \geq 1$.*

For $\lambda > 0$ and k non-negative integer, we define

$$[d_k^\lambda(f)](x) := x^{\lambda-1}(-1)^k\,[c_k^\lambda(f)](x) = (-1)^k(x^{k+\lambda-1}\,f(x))^{(k)}.$$

We can obtain the following generalization.

Theorem 3.3. *Let $\lambda > 0$ be given. The following properties of a function $f : (0,\infty) \to \mathbb{R}$ are equivalent.*

(i) *There exists an absolutely monotonic function $\varphi : [0,\infty) \to \mathbb{R}$ such that*
$$f(x) = \int_0^\infty e^{-xt}t^{\lambda-1}\varphi(t)\,dt, \quad x > 0.$$
(ii) *The function $[d_k^\lambda(f)](x)$ is completely monotonic for all $k \geq 0$.*
(iii) *The function $[d_k^\lambda(f)](x)$ is non-negative for all $k \geq 0$.*

The proof of this theorem follows from Theorem 3.2 by noticing

$$f(x) = \int_0^\infty e^{-xt}t^{\lambda-1}\varphi(t)\,dt$$

for some absolutely monotonic function φ of exponential type zero if and only if

$$x^{\lambda-1}f(x) = \int_0^\infty e^{-xt}\psi(t)\,dt$$

for some absolutely monotonic function ψ of exponential type zero. Indeed, the relationship between the functions φ and ψ is:

$$\varphi(t) = \sum_{n=0}^\infty a_n t^n \Leftrightarrow \psi(t) = \sum_{n=0}^\infty \frac{a_n\Gamma(n+\lambda)}{n!}t^n.$$

There are various applications of this result in the context of special functions. Consider, for instance, the generalized hypergeometric series

$$\varphi(t) = {}_1F_2\big(a;b,c;t\big) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k(c)_k k!}t^k, \quad a > 0, b > 0, c > 0$$

defines an absolutely monotonic function on $[0,\infty)$. Its Laplace transform exists for all $x > 0$ and it is given by the formula

$$\begin{aligned} f(x) = \int_0^\infty e^{-xt}\,\varphi(t)\,dt &= \frac{1}{x}\,{}_2F_2\big(a,1;b,c;\frac{1}{x}\big) \\ &= \sum_{n=0}^\infty \frac{(a)_n}{(b)_n\,(c)_n}\,\frac{1}{x^{n+1}}. \end{aligned}$$

Moreover,

$$\int_0^\infty e^{-xt}\,t^{\lambda-1}\,{}_1F_2\big(a;b,c;t\big)\,dt = \frac{\Gamma(\lambda)}{x^\lambda}\,{}_2F_2\big(a,\lambda;b,c;\frac{1}{x}\big)\,,$$

for any $\lambda > 0$. Therefore the function $\frac{\Gamma(\lambda)}{x^\lambda}\,{}_2F_2\big(a,\lambda;b,c;\frac{1}{x}\big)$ has all the properties given in Theorem 3.3.

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MONOTONICITY THEOREMS FOR CONVEX CONFORMAL MAPPINGS

MARIA KOUROU

1 Euclidean Geometry in the Unit Disk

Let f be a holomorphic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Moreover, we denote by $r\,\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$ and $r\,\mathbb{T} := \partial(r\,\mathbb{D}) = \{z \in \mathbb{D} : |z| = r\}$ the open disk and circle of radius $r \in (0, 1)$, respectively.

According to G. Pólya and G. Szegő [18, p.165, Problem 309], the function

$$L(r) := \frac{\mathrm{L}\, f(r\,\mathbb{T})}{\mathrm{L}(r\,\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} \left| f'(re^{it}) \right| dt, \tag{1.1}$$

where L denotes the euclidean length of a curve, is increasing for $r \in (0, 1)$. In [1] and [7], the same was proved for the function

$$A(r) := \frac{\mathrm{A}\, f(r\,\mathbb{D})}{\mathrm{A}(r\,\mathbb{D})} = \frac{1}{\pi r^2} \mathrm{A}\, f(r\,\mathbb{D}), \quad 0 < r < 1 \tag{1.2}$$

where A denotes the euclidean area of a domain.

The above monotonicity results make a comparison between the size of $r\,\mathbb{D}$ or $r\,\mathbb{T}$ and their images, respectively, measuring them with the use of length and area. Several other geometric quantities have been used in order to compare $r\,\mathbb{D}$ or $r\,\mathbb{T}$ with their images under a holomorphic function. Such geometric quantities are logarithmic capacity, diameter, condenser capacity, inner radius, etc., as we can see in [3], [4], [6], [7] and [8]. Monotonicity results of this kind can be seen as geometric versions of the classical Schwarz's Lemma. In this way, information on the growth of the image is extracted that leads to a variety of distortion theorems.

If f is a conformal mapping, the curves $f(r\,\mathbb{T})$ possess stronger geometric properties. They are simple, smooth and closed curves for every $r \in (0, 1)$. At this point, let's recall that a univalent function f is called convex if $f(\mathbb{D})$ is convex. A holomorphic and locally univalent function f is convex if and only if

$$v_f(z) := \mathrm{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \tag{1.3}$$

for every $z \in \mathbb{D}$; more information on convex functions can be found in [9] and [19].

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Using F.R. Keogh's upper bound for the length of the image of $r\,\mathbb{T}$ under a convex conformal mapping, in [10], it is easily obtained that

$$L(r) \leq \frac{|f'(0)|}{1-r^2}, \tag{1.4}$$

with equality holding if and only if f maps $\mathbb{D}$ conformally onto a half-plane. We call *half-plane mapping* a conformal function that maps $\mathbb{D}$ onto a half-plane of $\mathbb{C}$.

With the use of the isoperimetric inequality, we can find an upper bound for the function $A(r)$, as well,

$$A(r) \leq \frac{|f'(0)|^2}{(1-r^2)^2}, \tag{1.5}$$

for every $r \in (0,1)$.

We set the function

$$\mathcal{A}(r) := \left(1-r^2\right)^2 A(r) = \frac{\left(1-r^2\right)^2}{\pi r^2} \int\int_{r\,\mathbb{D}} \left|f'(z)\right|^2 \, dA(z), \tag{1.6}$$

where dA is the Lebesgue measure on $\mathbb{D}$, and we obtain the following monotonicity result.

Theorem 1.1. *Let f be a convex mapping in $\mathbb{D}$. The function $\mathcal{A}(r)$ is decreasing for $r \in (0,1)$. Moreover, it is strictly decreasing if and only if f is not a half-plane mapping of $\mathbb{D}$. In this case, $\mathcal{A}(r)$ is constant and equal to $|f'(0)|^2$.*

The above result provides us with an estimate on how sharp the bounds of Keogh [10] are. The following isoperimetric-type inequality concerning the image of $r\,\mathbb{D}$ under a convex mapping is a consequence of Theorem 1.1.

Corollary 1.1. *Let f be a convex function in $\mathbb{D}$. Then*

$$\mathrm{L}^2\, f(r\,\mathbb{T}) < 4\pi \frac{1+r^2}{1-r^2}\, \mathrm{A}\, f(r\,\mathbb{D}),$$

for $r \in (0,1)$.

In addition, we are able to state a similar monotonicity theorem concerning the total absolute curvature. For any conformal mapping f, the curve $f(r\,\mathbb{T})$ is convex, when $r \leq 2-\sqrt{3}$; see [9, Theorem 2.13]. The number $2-\sqrt{3}$ is called radius of convexity and it is a sharp bound regarding the convexity of the domain $f(r\,\mathbb{D})$. The question that arises is what happens when r is greater than the radius of convexity. And what if the function f is not univalent but only locally univalent?

We need some kind of measurement to show us whether $f(r\,\mathbb{T})$ is convex or not and how much it diverges from being convex. The most suitable geometric quantity with this property is the total absolute curvature of $f(r\,\mathbb{T})$. The total absolute curvature of $f(r\,\mathbb{T})$ provides some kind of distance between a function and its convexity.

Let γ be a smooth curve in $\mathbb{D}$. We denote by $\kappa(z,\gamma)$ the signed euclidean curvature of γ at the point $z \in \gamma$. The *total absolute curvature* of γ is the quantity

$$\int_{\gamma} |\kappa(z,\gamma)||dz|.$$

It is known that the total absolute curvature of a smooth and closed curve is always greater than 2π, with equality holding if and only if the curve γ is convex; see [20, Corollary 6.18].

Let f be a holomorphic and locally univalent function on $\mathbb{D}$. By $\kappa(w, f(\gamma))$ we denote the euclidean curvature of $f(\gamma)$ at the point $w \in f(\gamma)$. Therefore, the quantity

$$\int_{f(\gamma)} |\kappa(w, f(\gamma))||dw|$$

is the total absolute curvature of $f(\gamma)$. The greater this quantity becomes, the less convex the function f is.

We should notice that the total absolute curvature of $r\,\mathbb{T}$ is constant. Since $r\,\mathbb{T}$ is a circle, and therefore a convex curve, its total absolute curvature is equal to 2π. Set

$$\Phi(r) := \frac{\int_{f(r\,\mathbb{T})} |\kappa(w, f(r\,\mathbb{T}))||dw|}{\int_{r\,\mathbb{T}} |\kappa(z, r\,\mathbb{T})||dz|} = \frac{1}{2\pi}\int_{f(r\,\mathbb{T})} |\kappa(w, f(r\,\mathbb{T}))||dw|, \tag{1.7}$$

which is the ratio of the total absolute curvature of $f(C_r)$ to the total absolute curvature of $r\,\mathbb{T}$.

Theorem 1.2. *Let f be a holomorphic and locally univalent function on $\mathbb{D}$. Then $\Phi(r)$ is a strictly increasing function of $r \in (0,1)$, except when f is convex. In this case, it is constant and equal to 2π.*

2 Hyperbolic Geometric Aspects of Schwarz's Lemma

As we stated above, monotonicity results concerning geometric quantities have been extensively examined when $f(\mathbb{D}) \subset \mathbb{C}$ and $\mathbb{C}$ is equipped with the euclidean metric. But what happens when $f(\mathbb{D})$ is seen from a hyperbolic perspective? Can the above geometric versions of Schwarz's Lemma be extended in the hyperbolic geometry of the unit disk?

Let f be a holomorphic function in the unit disk $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. We suppose that the unit disk is endowed with the hyperbolic metric

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2},$$

where $\lambda_{\mathbb{D}}(z)$ is the density of the hyperbolic metric.

Let Ω be a simply connected subdomain of $\mathbb{D}$ and f be a conformal mapping with $f(\Omega) = \mathbb{D}$. The hyperbolic metric $\lambda_{\Omega}(z)|dz|$ on Ω is defined to be

$$\lambda_{\Omega}(z) = \lambda_{\mathbb{D}}(f(z))\left|f'(z)\right|.$$

The hyperbolic distance between two points a, $b \in \Omega$ is defined by

$$d_{\Omega}(a,b) = \inf_{\gamma\subset\Omega}\int_{\gamma} \lambda_{\Omega}(z)|dz|,$$

where γ is any rectifiable curve that lies in Ω and joins a, b. If the infimum is attained for a curve $\gamma_0 \subset \Omega$, then γ_0 is called *hyperbolic geodesic*. In the unit disk, every pair of points is joined by a unique hyperbolic geodesic.

The hyperbolic geodesic curves of $\mathbb{D}$ are the arcs of euclidean circles in $\mathbb{D}$ that are perpendicular to the boundary. The hyperbolic distance in the unit disk, for a, $b \in \mathbb{D}$ is equal to

$$d_{\mathbb{D}}(a,b) = \operatorname{arctanh}\left|\frac{a-b}{1-\bar{a}b}\right|$$

and it is invariant under any conformal automorphism of $\mathbb{D}$. We denote the set of all conformal automorphisms of the unit disk $\mathbb{D}$ by $\operatorname{Aut}(\mathbb{D})$; this set consists of all the mappings

$$g(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},$$

where $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$. The hyperbolic metric and, in general, hyperbolic geometry of the unit disk are thoroughly examined in [2] and [14].

A subregion Ω of the unit disk is *hyperbolically convex* if for every pair of points in Ω the hyperbolic geodesic arc that joins them, lies in Ω; see [14]. Also from [14], a conformal map $f : \mathbb{D} \to \mathbb{D}$ is called *hyperbolically convex* if $f(\mathbb{D})$ is a hyperbolically convex subregion of $\mathbb{D}$. From the hyperbolic analogue of Study's Theorem, which is proved in [14], arises a basic property of hyperbolically convex functions. Suppose f is a conformal map with $f(\mathbb{D}) \subset \mathbb{D}$. If f is hyperbolically convex, then f maps every subdisk of $\mathbb{D}$ onto a hyperbolically convex region. More specifically, for $0 < r < 1$, the function $f(rz)$ is hyperbolically convex. Also in [14], we can see that a holomorphic and locally univalent function f in $\mathbb{D}$, with $f(\mathbb{D}) \subset \mathbb{D}$, is hyperbolically convex if and only if

$$u_f(z) := \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + \frac{2zf'(z)\overline{f(z)}}{1-|f(z)|^2}\right\} > 0, \tag{2.1}$$

for every $z \in \mathbb{D}$. For further information on hyperbolic convexity, the reader may refer to [13], [14], [15], [16] and [17].

The hyperbolic disk centered at the origin of radius ρ,

$$D_h(0,\rho) = \{z \in \mathbb{D} : \operatorname{arctanh}|z| < \rho\} = \{z \in \mathbb{D} : |z| < \tanh\rho\} \tag{2.2}$$

is a euclidean disk centered at the origin of radius $r := \tanh\rho$. Its boundary is

$$\partial D_h(0,\rho) = \{z \in \mathbb{D} : |z| = \tanh\rho\} = \partial D(0,r) = r\,\mathbb{T}, \tag{2.3}$$

where $r = \tanh\rho$.

For holomorphic functions $f : \mathbb{D} \to \mathbb{D}$, monotonicity results of the same kind as in the euclidean case have been proved, concerning some geometric quantities viewed in the hyperbolic geometry of the disk. In [5], it was proved that

$$r \mapsto \frac{\mathrm{R_h}\, f(r\,\mathbb{D})}{r} \quad \text{and} \quad r \mapsto \frac{\operatorname{caph} f(r\,\mathbb{D})}{r}$$

are increasing functions of $r \in (0,1)$, where $\mathrm{R_h}$ is the hyperbolic-area-radius of $f(r\,\mathbb{D})$ and caph denotes the hyperbolic capacity.

The monotonic behavior of the functions (1.1) and (1.2) plays a pivotal role in the euclidean geometry of the complex plane. However, there are not any similar results regarding length and area with respect to the hyperbolic geometry of the unit disk.

With the help of hyperbolic convexity, we present the hyperbolic analogues of the functions (1.1) and (1.2). Let's define the function

$$\mathcal{L}^h(r) := \frac{\mathrm{L}_h\, f(r\,\mathbb{T})}{\mathrm{L}_h(r\,\mathbb{T})}, \quad r \in (0,1), \tag{2.4}$$

where L_h is the hyperbolic length of a curve in the unit disk. We have the following outcome concerning the monotonicity of the function $\mathcal{L}^h(r)$.

Theorem 2.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then $\mathcal{L}^h$ is a decreasing function in $(0,1)$. Moreover, $\mathcal{L}^h$ is strictly decreasing if and only if f is not a conformal automorphism of the unit disk $\mathbb{D}$. In the case where $f \in \mathrm{Aut}(\mathbb{D})$, $\mathcal{L}^h$ is constant and equal to 1.*

Furthermore, we have a similar theorem for the hyperbolic area. Define the function

$$\mathcal{A}^h(r) := \frac{\mathrm{A}_h\, f(r\,\mathbb{D})}{\mathrm{A}_h(r\,\mathbb{D})}, \quad r \in (0,1), \tag{2.5}$$

where A_h is the hyperbolic area of a domain in $\mathbb{D}$.

Theorem 2.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then $\mathcal{A}^h$ is a decreasing function in $(0,1)$. Moreover, $\mathcal{A}^h$ is strictly decreasing if and only if f is not a conformal automorphism of the unit disk $\mathbb{D}$. In the case where $f \in \mathrm{Aut}(\mathbb{D})$, $\mathcal{A}^h$ is constant and equal to 1.*

An immediate consequence of Theorem 2.2 is the following isoperimetric-type inequality for the image of $r\,\mathbb{D}$.

Corollary 2.1. *For a hyperbolically convex mapping f in $\mathbb{D}$, it holds*

$$\mathrm{L}_h^2\, f(r\,\mathbb{T}) \le \frac{4\pi}{1-r^2}\, \mathrm{A}_h\, f(r\,\mathbb{D}),$$

for $r \in (0,1)$. Equality occurs if and only if f is a conformal automorphism of $\mathbb{D}$.

Corollary 2.1 provides an upper bound for the hyperbolic isoperimetric ratio of $f(r\,\mathbb{D})$. Furthermore, Theorems 2.1 and 2.2 lead to Schwarz-type inequalities involving hyperbolic length and hyperbolic area, as well as, information on their limiting behavior.

Corollary 2.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then*

$$\mathrm{L}_h\, f(r\,\mathbb{D}) \le \frac{|f'(0)|}{1-|f(0)|^2}\,\frac{2\pi r}{1-r^2} \quad \text{and} \quad \mathrm{L}_h\, f(r\,\mathbb{T}) = \mathcal{O}\left(\frac{1}{1-r^2}\right), \tag{2.6}$$

as $r \to 1^-$. If $\mathrm{A}_h\, f(\mathbb{D}) < +\infty$, then

$$\mathrm{L}_h\, f(r\,\mathbb{T}) = \mathcal{o}\left(\frac{1}{1-r^2}\right),$$

as $r \to 1^-$. Equality occurs in (2.6) if and only if $f \in \mathrm{Aut}(\mathbb{D})$.

A similar result holds for the hyperbolic area of $f(r\,\mathbb{D})$.

Corollary 2.3. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then*

$$\mathrm{A}_h\, f(r\,\mathbb{D}) \le \frac{|f'(0)|^2}{(1-|f(0)|^2)^2}\frac{\pi r^2}{1-r^2} \quad \text{and} \quad \mathrm{A}_h\, f(r\,\mathbb{D}) = \mathcal{O}\left(\frac{1}{1-r^2}\right), \tag{2.7}$$

as $r \to 1^-$. If $\mathrm{A}_h\, f(\mathbb{D}) < +\infty$, then

$$\mathrm{A}_h\, f(r\,\mathbb{D}) = o\left(\frac{1}{1-r^2}\right),$$

as $r \to 1^-$. Equality holds in (2.7) if and only if $f \in \mathrm{Aut}(\mathbb{D})$.

Moreover, the above outcomes lead to an integrated version of the classical Schwarz-Pick lemma for the class of hyperbolically convex functions.

Corollary 2.4. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then, for every $r \in (0,1)$,*

$$\frac{1}{2\pi}\int_{r\,\mathbb{T}} (1-|z|^2)\,\frac{|f'(z)|}{1-|f(z)|^2}|dz| \le \frac{|f'(0)|}{1-|f(0)|^2} r \le r,$$

where equality occurs if and only if $f \in \mathrm{Aut}(\mathbb{D})$.

To continue with, we will examine whether the hyperbolic analogue of Theorem 1.2 exists. Let γ be a smooth curve in the unit disk $\mathbb{D}$, with non-vanishing derivative, and f a holomorphic and locally univalent map with $f(\mathbb{D}) \subset \mathbb{D}$. The hyperbolic curvature of γ at the point $z \in \gamma$ is denoted by $\kappa_h(z,\gamma)$, whereas, the hyperbolic curvature of $f \circ \gamma$ at the point $f(z)$, $z \in \gamma$, is denoted by $\kappa_h(f(z), f \circ \gamma)$.

We should note that the hyperbolic curvature on the unit disk is invariant under conformal self-maps of $\mathbb{D}$. For the total hyperbolic curvature of the curve $f(r\,\mathbb{T})$ in the unit disk, the following monotonicity result holds.

Theorem 2.3. *Let f be a holomorphic and locally univalent function on $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$r \mapsto \int_{f(r\,\mathbb{T})} \kappa_h(w, f(r\,\mathbb{T}))\lambda_{\mathbb{D}}(w)|dw|, \quad 0 < r < 1, \tag{2.8}$$

is a strictly increasing function.

Let's define the function

$$\Phi_h(r) := \frac{\int_{f(r\,\mathbb{T})} |\kappa_h(w, f(r\,\mathbb{T}))|ds}{\int_{r\,\mathbb{T}} |\kappa_h(z, r\,\mathbb{T})|ds}, \quad 0 < r < 1, \tag{2.9}$$

which is the ratio of the hyperbolic total absolute curvature of $f(r\,\mathbb{T})$ to the hyperbolic total absolute curvature of $r\,\mathbb{T}$. The function $\Phi_h(r)$ is the hyperbolic analogue of the function $\Phi(r)$.

Theorem 2.4. *Let f be a hyperbolically convex function in $\mathbb{D}$, with $f(\mathbb{D}) \subset \mathbb{D}$. Then $\Phi_h(r)$ is a strictly decreasing function of $r \in (0,1)$, except when f is a conformal self-map of the unit disk. In that case, Φ_h is constant and equal to 1.*

3 Importance of Hyperbolic Convexity

A natural question is whether Theorems 2.1 and 2.2 can be generalized for all holomorphic functions of the unit disk into itself, or at least for conformal mappings. The answer is no and hyperbolic convexity is a property which cannot be omitted in the above results.

Consider the function

$$g(z) = k^{-1}\left(\frac{1}{2}k(z)\right),$$

where $k(z)$ is the Koebe function. The function g maps $\mathbb{D}$ conformally onto $\mathbb{D}\setminus(-1,-p]$, where $p = 3-2\sqrt{2}$. It is clear that $\mathbb{D}\setminus(-1,-p]$ is not a hyperbolically convex domain.

With the use of Mathematica®, we can see below the graphs of $\mathcal{L}^h(r)$ and $\mathcal{A}^h(r)$.

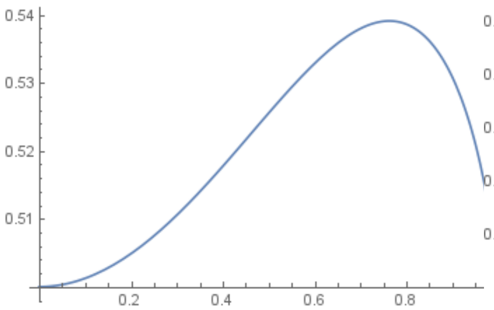

Figure 1. Graph of $\mathcal{L}^h(r) = \frac{\mathrm{L}_h(g(r\,\mathbb{T}))}{\mathrm{L}_h(r\,\mathbb{T})}$

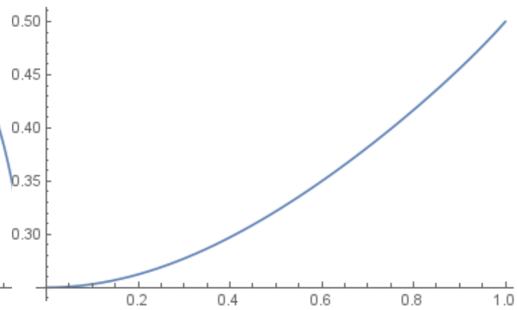

Figure 2. Graph of $\mathcal{A}^h(r) = \frac{\mathrm{A}_h(g(r\,\mathbb{D}))}{\mathrm{A}_h(r\,\mathbb{D})}$

Since for the mapping g, the functions $\mathcal{L}^h(r)$ and $\mathcal{A}^h(r)$ are not decreasing, we conclude that the assumption that f is hyperbolically convex in Theorems 2.1 and 2.2 cannot be omitted.

4 Acknowledgements and Remarks

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For the proofs of the Theorems and the Corollaries, the reader may refer to [11] and [12].

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NOTES ON ERGODIC AVERAGES WITH POLYNOMIAL ITERATES

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1. Introduction

In these notes we will present some recent results in ergodic theory that have to do with ergodic averages. More specifically, we will mainly deal with averages for a single transformation T with polynomial iterates, i.e., averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{[p_1(n)]}f_1\cdots T^{[p_k(n)]}f_k, \tag{1}$$

where p_i's are real polynomials (see below for quantifiers).

Our goal is to study the L^2-convergence (norm-convergence) of (1), with iterates coming from appropriate polynomial families, in order to obtain applications to other areas of mathematics as combinatorics, number theory, topological dynamics etc.

1.1. **History of the problem.** We always work on a *measure preserving system*, i.e., a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a set, $\mathcal{B}$ is a σ-algebra on X, μ is a probability measure on $\mathcal{B}$, and T is an invertible (this assumption sometimes can be skipped but for reasons of simplicity we will always assume it in these notes) measure preserving transformation, i.e., $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

The simplest average that one can study is

$$\frac{1}{N}\sum_{n=1}^{N} T^n f, \tag{2}$$

where $f \in L^2$ (note that in the following, whenever we deal with multiple terms, the assumption for our functions would be that these are elements of L^∞), $Tf(x) := f(Tx)$, while T^n is the composition of T with itself n times.

The first result in understanding (2) is due to von Neumann:

Theorem 1.1 (von Neumann, 1932)**.** *Under the previous assumptions, for every $f \in L^2$ we have that*

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \to Pf,$$

as $N \to \infty$, where P is the orthogonal projection to the space of the left T-invariant functions $\{f : Tf = f\}$ and the convergence takes place in L^2.

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The proof of this result is a simple, and nowadays classical, splitting of the Hilbert space L^2 and it is elementary. We also have that

$$Pf = \int f \, d\mu \quad \text{if and only if } \; T \; \text{ is } \textit{ergodic},$$

meaning that the only T-invariant sets of $\mathcal{B}$ are the ones with trivial measure, i.e., in $\{0, 1\}$.

Of course, someone can study more complicated expressions, like the following multiple average:

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k, \tag{3}$$

where $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in L^\infty$.

The study of the precise form of the L^2-limit of (3) proved to be a hard problem. Relatively recently, Host and Kra managed to show, in [13], the existence of the limit providing simultaneously a closed form of it which is rather complicated to be stated here (it has to do with conditional expectations on nilfactors which we won't cover in full detail in these short notes).

One may wonder why we are interested in studying the behavior of (3). The answer mainly lies in the following result that Furstenberg got, using ergodic theoretical methods, by studying (3):

Theorem 1.2 (Furstenberg, 1977, [11])**.** *Under the standard assumptions, for all $A \in \mathcal{B}$ with $\mu(A) > 0$, we have that*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-kn}A) > 0. \tag{4}$$

This breakthrough made the ergodic theory to blossom and gave numerous deep and interesting applications to many areas of Mathematics (see below its connection with Szemerédi's celebrated theorem).

We remark that the $\liminf$ that appears in (4) is actually a limit by [13]. Note that the connection of (4) with (3) can be reflected by the relations $T\mathbf{1}_A = \mathbf{1}_{T^{-1}A}$, and $\int T^n \mathbf{1}_A \, d\mu = \mu(T^{-n}A)$.

Via Furstenberg's *correspondence principle* (see below) we will get one of the most deep and interesting results about the set of natural numbers, namely Szemerédi's theorem, which states that every "large" subset of natural numbers is *AP rich*, i.e., contains arbitrarily long arithmetic progressions.

Before we state Furstenberg's correspondence principle (actually, we will state a reformulation of it which is due to Bergelson [1]) we define the motion of *upper density* of a subset of natural numbers:

Definition. Let $E \subseteq \mathbb{N}$. We define the *upper density* of E to be

$$\bar{d}(E) := \limsup_{N\to\infty} \frac{|E \cap \{1, \ldots, N\}|}{N}$$

which is a number between 0 and 1.

(The quantity $|E \cap \{1, \ldots, N\}|$ measures how many elements from the first N natural numbers the set E captures.)

Theorem 1.3 (Furstenberg correspondence principle, [11], [1])**.** *For any $E \subseteq \mathbb{N}$, there exists a system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E)$ such that*

$$\bar{d}(E \cap (E - n_1) \cap \ldots \cap (E - n_k)) \geq \mu(A \cap T^{-n_1} A \cap \ldots \cap T^{-n_k} A) \tag{5}$$

for all $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$.

Note that by starting with a set $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$, by Theorem 1.3 we can find a system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E) > 0$. For an arbitrary $k \in \mathbb{N}$, using Theorem 1.2, we can find $n_0 \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n_0} A \cap \ldots \cap T^{-kn_0} A) > 0,$$

so, by Theorem 1.3 we have that

$$\bar{d}(E \cap (E - n_0) \cap \ldots \cap (E - kn_0)) > 0.$$

In particular,

$$E \cap (E - n_0) \cap \ldots \cap (E - kn_0) \neq \emptyset,$$

hence, there exists $x_0 \in E$ such that

$$x_0, x_0 + n_0, \ldots, x_0 + kn_0 \in E.$$

Summing up the previous arguments, we have reproved the following:

Theorem 1.4 (Szemerédi, 1975)**.** *Every subset of natural numbers with positive upper density contains arbitrarily long arithmetic progressions.*

In order to state the next result, we have to recall the notion of a weakly mixing system:

Definition. Let $(X, \mathcal{B}, \mu, T)$ be a system. T is called *weakly mixing*, which we will denote with w.m., (and the whole system $(X, \mathcal{B}, \mu, T)$ is called *weakly mixing*) if for all $f, g \in L^2$ we have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int T^n f \cdot g \, d\mu - \int f \, d\mu \cdot \int g \, d\mu \right| = 0.$$

We remark at this point that a w.m. transformation is ergodic while the opposite is not in general true.

In the same paper, [11], under the weakly mixing assumption of T Furstenberg showed the following convergence result:

Theorem 1.5 (Furstenberg, [11])**.** *If $(X, \mathcal{B}, \mu, T)$ is a w.m. system, then for any $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in L^\infty$ we have*

$$\frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k \to \prod_{i=1}^{k} \int f_i \, d\mu, \tag{6}$$

as $N \to \infty$, where the convergence takes place in L^2.

1.2. **From linear iterates to polynomial ones.** Bergelson was the first to view the iterates $n, 2n, \ldots, kn$ as linear polynomials $p_1, \ldots, p_k$ with the property $p_i - p_j \neq$ constant for all $i \neq j$.

This naturally led him to the following definition for more general, than linear, polynomials.

Definition. The non-constant polynomials $p_1(t), \ldots, p_k(t)$ in $\mathbb{Z}[t]$ are called *essentially distinct* if $p_i - p_j \neq$ constant for all $i \neq j$.

Exploted the van der Corput trick, which we will see below, Bergelson showed the following:

Theorem 1.6 (Bergelson, [2]). *If $(X, \mathcal{B}, \mu, T)$ is a w.m. system, then for any $k \in \mathbb{N}$, $p_1(t), \ldots, p_k(t)$ essentially distinct polynomials in $\mathbb{Z}[t]$ and $f_1, \ldots, f_k \in L^\infty$ we have*

$$\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)}f_1 \cdots T^{p_k(n)}f_k \to \prod_{i=1}^{k}\int f_i \, d\mu, \tag{7}$$

as $N \to \infty$, where the convergence takes place in L^2.

To show this, Bergelson used the following reformulation, due to himself, of van der Corput trick:

Lemma 1.7 (van der Corput, Bergelson, [2]). *Let $(x_n)_n$ be a bounded sequence in a Hilbert space and suppose that for any $h \geq h_0$ (for some fixed, large, $h_0 \in \mathbb{N}$) we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle x_n, x_{n+h}\rangle = 0, \tag{8}$$

then

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} x_n \right\| = 0.$$

Remark. To study (7) via van der Corput trick, we assume without loss of generality that some $\int f_i \, d\mu = 0$, we set

$$x_n := T^{p_1(n)}f_1 \cdots T^{p_k(n)}f_k$$

and we try to show that for large enough h we have (8).

This is achieved by induction (which is formally called *PET induction*) on the "complexity" of the polynomial family since the quantity $\langle x_n, x_{n+h}\rangle$ leads to differences (i.e., derivatives) hence to reduction of the complexity.

Ten years after Theorem 1.6, studying ergodic averages with polynomial iterates, Bergelson and Leibman, in [4], obtained far-reaching polynomial multidimensional (in $\mathbb{Z}^d$) extensions of Szemerédi's theorem (showing existence of "polynomial progressions" in any "large" subset of $\mathbb{Z}^d$). They managed to show the corresponding polynomial relation to (4) and they also stated the following conjecture:

Conjecture ([4])**.** Let $k \in \mathbb{N}$. For any measure preserving system $(X, \mathcal{B}, \mu, T_1, \ldots, T_k)$, where each T_i is measure preserving and $T_iT_j = T_jT_i$, and all polynomials $p_1(t), \ldots, p_k(t)$ in $\mathbb{Z}[t]$ the following expression

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k$$

has a limit, in L^2, as $N \to \infty$ for all $f_1, \ldots, f_k \in L^\infty$.

This conjecture was answered positively in stages. Walsh was the one that eventually showed:

Theorem 1.8 (Walsh, [18])**.** *The conjecture of Bergelson-Leibman holds true.*

He actually showed that this is true for products of transformations, with any polynomial power, which more generally we don't have to assume that they commute but that they produce a nilpotent group.

Of course we have no information of the limit function in question.

In these notes we will solely deal with a single transformation. We will find a class of polynomials for which we have convergence in $\prod_{i=1}^k \int f_i \, d\mu$ with no assumption on the system. This lack of assumptions makes our result, even though it is stated with ergodic theory language, a combinatorial object. Also, the strong nature of the result will provide us with many interesting applications in other areas of mathematics.

2. Passing to integer parts - Main result

Someone of course can think of extending the polynomial iterates to iterates of the form $[p(n)]$, where $[x]$ denotes the integer part function, or floor function at $x \in \mathbb{R}$ which gives the closest integer which is less of equal to x and $p(t)$ is a real polynomial.

The transition from $p(t) \in \mathbb{Z}[t]$ to $[q(t)]$ for $q(t) \in \mathbb{R}[t]$ is not immediate, since PET induction is not immediately applicable. This is mainly due to error terms that appear in the corresponding differences $\langle x_n, x_{n+h}\rangle$ (recall that $[x] - [y] = [x - y] + e$, where $e \in \{0, 1\}$).

Following the work of Lesigne (for one term) and the work of Wierdl (for two terms) one can show that the expression

$$T^{[a_dn^d + \ldots + a_1n + a_0]}$$

"looks" like

$$S_{a_dn^d + \ldots + a_1n + a_0} = S_{a_d}^{n^d} \cdots S_{a_1}^{n} S_{a_0},$$

where S is the suspension flow (with respect to T). This aforementioned "looks like" statement that we mentioned above hides a periodic property that polynomials with no non-constant irrational coefficients and an equidistribution property that polynomials with some non-constant irrational coefficient have (for more details see [15]).

So, extending Lesigne's and Wierdl's argument for an arbitrary number of terms, using Walsh's result (the one about products of transformations) we get:

Theorem 2.1 (K, [15]). *Let $k \in \mathbb{N}$. For any measure preserving system $(X, \mathcal{B}, \mu, T_1, \dots, T_k)$, where each T_i is measure preserving and $T_iT_j = T_jT_i$, and all polynomials $p_1(t), \dots, p_k(t)$ in $\mathbb{R}[t]$ the expression*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{[p_1(n)]}f_1 \cdots T_k^{[p_k(n)]}f_k \tag{9}$$

has a limit as $N \to \infty$, in L^2, as $N \to \infty$ for all $f_1, \dots, f_k \in L^\infty$.

We remark at this point that we actually have the same result (i.e., L^2-convergence of the corresponding Equation (9)) for products of transformations for all real polynomials.

Hence, from now on we won't have to worry about existence of the limit of averages with integer parts of polynomial iterates.

2.1. **Results in general systems.** To study the existence of a limit and its precise value-expression are two different problems.

In case we knew for some specific polynomial families the precise expression of the limit of (9), we would be able to get deeper applications for the corresponding systems. In the special case where we could obtain a result like this for general systems under no assumptions, ergodicity etc, we would have a combinatorial result.

The first result ever in this direction for multiple averages with polynomial iterates of the form $[p(n)], 2[p(n)], \dots, k[p(n)]$ for some special polynomial $p(t) \in \mathbb{R}[t]$ (see below) is due to Frantzikinakis:

Theorem 2.2 (Frantzikinakis, [7]). *Let $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t) + d$, for all $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$. Then, for every $k \in \mathbb{N}$, system $(X, \mathcal{B}, \mu, T)$ and $f_1, \dots, f_k \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} T^{i[p(n)]}f_i = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} T^{in}f_i, \tag{10}$$

where the convergence takes place in L^2.

This result of Frantzikinakis is an intermediate result in order for him to study multiple averages with iterates coming from Hardy field functions (see [7]). This is the first result on polynomial non-linear iterates where we know the precise expression of the limit (via the work of Host-Kra, [13]).

Now, one can see that one out of the plethora of applications that Theorem 2.2 has, is a Szemerédi-type result implication. Indeed, using Theorems 1.2 and 1.3, Theorem 2.2 implies that any set $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic progressions of the form $\{[p(n)], 2[p(n)], \dots, k[p(n)]\}$ for any $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t) + d$, for all $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$, any $k \in \mathbb{N}$ and some n depending on k (for more applications check [7]).

Switching gears to multiple polynomials, one can generalize the condition that Frantzikinakis has for a single polynomial to the following:

Definition. For $k \in \mathbb{N}$, let $\{p_1, \dots, p_k\}$ be a family of real polynomials. We say that this family is *strongly independent* (or that the polynomials $p_1, \dots, p_k$ are *strongly independent*) if any non-trivial real linear combination of the polynomials p_i has a non-constant irrational coefficient.

Note that a family with one element, $\{p\}$, where $p \in \mathbb{R}[t]$, is strongly independent iff $p(t) \neq cq(t)+d$ for all $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$ (or $\mathbb{Z}[t]$ equivalently).

Examples. The family of polynomials $\{\sqrt{2}t^3+t^2, \sqrt{3}t^3-t\}$ is strongly independent while the families $\{\sqrt{5}t^3+t^2+\sqrt{6}t, t^2, \sqrt{7}t\}$ and $\{\sqrt{2}t^2+t, \sqrt{5}t^2-t\}$ are not.

At this point we also remark that a very nice family of polynomials, in the sense that we have many results for it, with integer coefficients of different degrees is trivially not strongly independent. This fact is natural though as it is known that polynomials of different degrees, while having nice properties and we know many converging results for averages with iterates such polynomials, don't behave in the expected way for general systems (see below for a more detailed clarification of this statement). Hence, someone, in order to get a general, for all systems result, has to restrict to a more special families of polynomials, as the strongly independent ones.

The main result of these notes is the following:

Theorem 2.3 (Karageorgos-K, [14])**.** *Let $k \in \mathbb{N}$, $p_1, \ldots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials, $(X, \mathcal{B}, \mu, T)$ be an ergodic system and $f_1, \ldots, f_k \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{[p_1(n)]}f_1 \cdots T^{[p_k(n)]}f_k = \prod_{i=1}^{k} \int f_i \, d\mu, \tag{11}$$

where the convergence takes place in $L^2(\mu)$.

So, not only we know the precise expression of the limit but it is also the expected one.

We note at this point that only for aesthetic reasons we stated Theorem 2.3 under the ergodicity assumption for if we deal with a general T, using the ergodic decomposition of μ (i.e., splitting the space into fibers where in each one the system is ergodic), we get that the limit of (11) is equal to the product of the conditional expectations of f_i's with respect to the σ-algebra of the T-invariant sets (which we denote with $\mathbb{E}(f_i|\mathcal{I}(T))$).

Remark. The assumption of Theorem 2.3 that the polynomials are strongly independent is necessary, since even for $k = 1$, $p(t) = \sqrt{2}t$ and ergodic rotations on the torus, (11) typically fails.

Hence, even for families of polynomials with integer coefficients it is not true in general that one has convergence as in (11), i.e., to the expected limit for a general ergodic system (see remark after Theorem 2.4). Such a result requires more assumptions on the system, as the total ergodicity one (see [8]). Also, one is "forced" to work with real polynomials in order to have this nice convergence behavior.

As a consequence of Theorem 2.3, via Hölder's inequality, we get the following recurrence result:

Theorem 2.4 (Karageorgos-K, [14])**.** *Let $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials. Then for every system $(X, \mathcal{B}, \mu, T)$*

and $A \in \mathcal{B}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\Big(A\cap T^{-[p_1(n)]}A\cap\ldots\cap T^{-[p_k(n)]}A\Big) \geq (\mu(A))^{k+1}. \tag{12}$$

Remark. The assumption that the polynomials are strongly independent is necessary since even for $k = 1$ and $p(t) = t^2$, (12) typically fails.

Hence, Theorem 2.4 is another indication that one has to work with real polynomials in order to have nice lower bounds as in (12) for general systems.

Note at this point that following the arguments of the proof of Theorem 2.3 we can show its uniform version, meaning that one can replace the standard Cesàro averages, $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N}$, with the respective uniform ones, $\lim_{N-M\to\infty}(N-M)^{-1}\sum_{n=M+1}^{N}$, and the natural upper density, $\bar{d}$, with the respective upper Banach density, d^* [1].

Then, one has that the uniform version of Theorem 2.4 implies that for any $A \in \mathcal{B}$ with $\mu(A) > 0$, and every $\varepsilon > 0$ the set

$$R_\varepsilon(A) = \Big\{n \in \mathbb{Z} : \ \mu\Big(A\cap T^{-[p_1(n)]}A\cap\ldots\cap T^{-[p_k(n)]}A\Big) > (\mu(A))^{k+1} - \varepsilon\Big\}$$

is syndetic (i.e., it has bounded gaps).

We note that this general result, which holds under no assumption on the system, implies that a family of strongly independent real polynomials has a much different behavior than a family of linear integer polynomials, since its stands in contrast with the Bergelson-Host-Kra-Ruzsa counterexample to the "higher-order Khintchine recurrence theorem". Indeed, in [3], the aforementioned authors found an ergodic system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ such that

$$\mu\Big(A\cap T^{-n}A\cap T^{-2n}A\cap T^{-3n}A\cap T^{-4n}A\Big) \leq \frac{\mu(A)^5}{2} \quad \text{for all} \quad n\neq 0$$

(so, for $p_i(t) = it$ we have that the syndeticity conclusion of the respective $R_\varepsilon(A)$ fails for certain ergodic systems when $k \geq 4$, while for certain non-ergodic systems it fails even when $k \geq 2$. For examples covering both cases, see [3]).

Using Theorem 2.4 and Furstenberg's corresponding principle, we have the following.

Theorem 2.5 (Karageorgos-K, [14])**.** *Let $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials. Then for every $E \subseteq \mathbb{N}$ we have*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \bar{d}(E\cap(E-[p_1(n)])\cap\ldots\cap(E-[p_k(n)])) \geq (\bar{d}(E))^{k+1}.$$

An immediate implication of the aforementioned result is the following.

[1] For a set $A \subseteq \mathbb{Z}$, we define its *upper Banach density*, $d^*(A)$, as

$$d^*(A) = \limsup_{N-M\to\infty} (N-M)^{-1}|A\cap\{M+1,\ldots,N\}|.$$

Theorem 2.6 (Karageorgos-K, [14]). *Let $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials. Then every $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic configurations of the form*

$$\{m, m + [p_1(n)], m + [p_2(n)], \ldots, m + [p_k(n)]\}$$

for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $[p_i(n)] \neq 0$, for all $1 \leq i \leq k$.

We note that one can get the aforementioned result for integer polynomials with no constant term from the polynomial Szemerédi theorem (Theorem A_0, [4]), but in the generality that we present it here it is not clear to us at all if the theorem follows from previous results in the literature.

3. Background material

3.1. Nilmanifolds.

In this subsection we recall some basic facts concerning nilmanifolds and equidistribution results on them.

3.1.1. *Definitions and basic properties.* Let G be a k-step nilpotent Lie group, meaning $G_{k+1} = \{e\}$ for some $k \in \mathbb{N}$, where $G_k = [G, G_{k-1}]$ denotes the k-th commutator subgroup, and let Γ be a discrete cocompact subgroup of G. The compact homogeneous space $X = G/\Gamma$ is called a *k-step nilmanifold* (or just *nilmanifold*).

The group G acts on G/Γ by left translations where the translation by an element $b \in G$ is given by $T_b(g\Gamma) = (bg)\Gamma$. We denote by m_X the normalized *Haar measure* on X, meaning the unique probability measure that is invariant under the action of G by left translations and $\mathcal{G}/\Gamma$ denotes the Borel σ-algebra of G/Γ. If $b \in G$, we call the system $(G/\Gamma, \mathcal{G}/\Gamma, m_X, T_b)$ a *k-step nilsystem* (or just *nilsystem*) and the elements of G *nilrotations*.

3.1.2. *Equidistribution on nilmanifolds.* Let $\exp : \mathfrak{g} \to G$ be the exponential map, where $\mathfrak{g}$ is the Lie algebra of G for a connected and simply connected Lie group G. For $b \in G$ and $s \in \mathbb{R}$ we define the element b^s of G as follows: If $Z \in \mathfrak{g}$ is such that $\exp(Z) = b$, then $b^s = \exp(sZ)$ (this is well defined since exp is a bijection).

If $(a(n))_n$ is a sequence of real numbers and $X = G/\Gamma$ is a nilmanifold with G connected and simply connected, we say that the sequence $(b^{a(n)}x)_n$ is *equidistributed* in a sub-nilmanifold Y of X, if for every $F \in C(Y)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(b^{a(n)}x) = \int F \, dm_Y.$$

If the sequence $(a(n))_n$ takes only integer values, we are not obliged to assume that G is connected and simply connected.

A nilrotation $b \in G$ is *ergodic*, or *acts ergodically* on X, if the sequence $(b^n\Gamma)_n$ is dense in X. If $b \in G$ is ergodic, then for every $x \in X$ the sequence $(b^n x)_n$ is equidistributed in X (a nontrivial fact which follows by unique ergodicity).

Let $X = G/\Gamma$ be a nilmanifold and $b \in G$. Then the orbit closure $\overline{(b^n\Gamma)}_n$ of b has the structure of a nilmanifold. Furthermore, the sequence $(b^n\Gamma)_n$ is equidistributed in $\overline{(b^n\Gamma)}_n$. If G is connected and simply connected and

$b \in G$, then $\overline{(b^s\Gamma)}_{s\in\mathbb{R}}$ is a nilmanifold. Furthermore, the nilflow $(b^s\Gamma)_{s\in\mathbb{R}}$ is equidistributed in $\overline{(b^s\Gamma)}_{s\in\mathbb{R}}$.

If G is a nilpotent group, then a sequence $g : \mathbb{N} \to G$ of the form $g(n) = b_1^{p_1(n)} \cdots b_k^{p_k(n)}$, where $b_i \in G$ and p_i are polynomials taking integer values at the integers for every $1 \leq i \leq k$ is called a *polynomial sequence* in G. A *polynomial sequence on the nilmanifold* $X = G/\Gamma$ is a sequence of the form $(g(n)\Gamma)_n$ where $g : \mathbb{N} \to G$ is a polynomial sequence in G.

The following qualitative equidistribution result was established by Leibman in [17]:

Theorem 3.1 (Theorems B, C, [17])**.** *Suppose that $X = G/\Gamma$ is a nilmanifold with G connected and simply connected and $(g(n))_n$ is a polynomial sequence in G. Let $Z = G/([G,G]\Gamma)$ and $\pi : X \to Z$ be the natural projection. Then the following statements hold:*

(i) *For every $x \in X$ the sequence $(g(n)x)_n$ is equidistributed in a finite union of subnilmanifolds of X.*

(ii) *For every $x \in X$ the sequence $(g(n)x)_n$ is equidistributed in X if and only if the sequence $(g(n)\pi(x))_n$ is equidistributed in Z.*

If $X = G/\Gamma$ is a nilmanifold with G connected and simply connected, then Z is a connected compact abelian Lie group, hence a torus, meaning $\mathbb{T}^s$ for some $s \in \mathbb{N}$, and as a consequence every nilrotation in Z is isomorphic to a rotation on $\mathbb{T}^s$.

3.2. **Ergodic Theory.** We gather below some basic notions and facts from ergodic theory that we use throughout the paper.

3.2.1. *Factors.* A *homomorphism* from a system $(X, \mathcal{X}, \mu, T)$ onto a system $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\pi : X \to Y$, such that $\mu \circ \pi^{-1} = \nu$ and $S \circ \pi(x) = \pi \circ T(x)$ for $x \in X$. When we have such a homomorphism we say that the system $(Y, \mathcal{Y}, \nu, S)$ is a *factor* of the system $(X, \mathcal{X}, \mu, T)$. If the factor map $\pi : X \to Y$ can be chosen to be injective, then we say that the systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are *isomorphic*. A factor can also be characterised by $\pi^{-1}(\mathcal{Y})$ which is a T-invariant sub-σ-algebra of $\mathcal{X}$. By a classical abuse of terminology we denote by the same letter the σ-algebras $\mathcal{Y}$ and $\pi^{-1}(\mathcal{Y})$.

3.2.2. *Characteristic Factors.* Let $(X, \mathcal{X}, \mu, T)$ be a system. We say that the σ-algebra $\mathcal{Y}$ of $\mathcal{X}$ is a *characteristic factor* for the family of integer sequences $\{(a_1(n))_n, \ldots, (a_k(n))_n\}$ if $\mathcal{Y}$ is T-invariant and

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^{a_1(n)} f_1 \cdots T^{a_k(n)} f_k - \frac{1}{N} \sum_{n=1}^{N} T^{a_1(n)} \tilde{f}_1 \cdots T^{a_k(n)} \tilde{f}_k \right\|_{L^2(\mu)} = 0$$

where $\tilde{f}_i = \mathbb{E}(f_i|\mathcal{Y})$, for $f_i \in L^\infty(\mu)$ for all $1 \leq i \leq k$ [2].

[2]Equivalently, if $\mathbb{E}(f_i|\mathcal{Y}) = 0$ for some $1 \leq i \leq k$, then $\left\| N^{-1} \sum_{n=1}^{N} \prod_{i=1}^{k} T^{a_i(n)} f_i \right\|_{L^2(\mu)}$ converges to 0 as $N \to \infty$.

3.2.3. *Seminorms and Nilfactors.* We follow [13] and [5] for the inductive definition of the seminorms $\|\!|\cdot|\!\|_k$. More specifically, the definition that we use here follows from [13] (in the ergodic case), [5] (in the general case) and the use of von Neumann's ergodic theorem.

Let $(X,\mathcal{B},\mu,T)$ be a system and $f\in L^\infty(\mu)$. We define inductively the seminorms $\|\!|f|\!\|_k$ as follows: For $k=1$ we set

$$\|\!|f|\!\|_1 := \|\mathbb{E}(f|\mathcal{I}(T))\|_{L^2(\mu)}\,.$$

For $k\geq 1$, we let

$$\|\!|f|\!\|_{k+1}^{2^{k+1}} := \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\|\!|\bar{f}\cdot T^n f|\!\|_k^{2^k}.$$

It was shown in [13] that for every integer $k\geq 1$ all these limits exist and $\|\!|\cdot|\!\|_k$ defines a seminorm on $L^\infty(\mu)$.

Using these seminorms we can construct factors $\mathcal{Z}_k=\mathcal{Z}_k(T)$ of X characterized by the property:

$$\text{for } f\in L^\infty(\mu),\quad \mathbb{E}(f|\mathcal{Z}_{k-1})=0 \text{ if and only if } \|\!|f|\!\|_k=0.$$

It was also shown in [13] that for every $k\in\mathbb{N}$ the factor $\mathcal{Z}_k$ has an algebraic structure, in fact we can assume that it is a k-step nilsystem. This is the content of the following Structure theorem, which we recall in the ergodic case and follows by Theorem 10.1 in [13]:

Theorem 3.2 (**Host & Kra**, [13]). *Let $(X,\mathcal{B},\mu,T)$ be an ergodic system and $k\in\mathbb{N}$. Then the factor $\mathcal{Z}_k(T)$ is an inverse limit of k-step nilsystems.* [3]

Because of this result we call $\mathcal{Z}_k$ the *k-step nilfactor* of the system. The smallest factor that is an extension of all finite step nilfactors is denoted by $\mathcal{Z}=\mathcal{Z}(T)$, meaning, $\mathcal{Z}=\bigvee_{k\in\mathbb{N}}\mathcal{Z}_k$, and is called the *nilfactor* of the system.

4. Proof of Theorem 2.3

In this section we will state the intermediate results that we used in order to prove Theorem 2.3.

The main argument is an equidistribution result involving nil-orbits of several sequences of strongly independent polynomials (first proved for Hardy field functions by Frantzikinakis, [6, Theorem 1.3]).

Theorem 4.1 (Karageorgos-K, [14]). *Let $k\in\mathbb{N}$ and $p_1,\ldots,p_k\in\mathbb{R}[t]$ be strongly independent real polynomials.*

(i) *If $X_i=G_i/\Gamma_i$, $1\leq i\leq k$, are nilmanifolds with G_i connected and simply connected, then for every $b_i\in G_i$ and $x_i\in X_i$ the sequence*

$$\left(b_1^{p_1(n)}x_1,\ldots,b_k^{p_k(n)}x_k\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^s x_1)}_{s\in\mathbb{R}}\times\cdots\times\overline{(b_k^s x_k)}_{s\in\mathbb{R}}.$$

[3] By this we mean that there exist T-invariant sub-σ-algebras $\mathcal{Z}_{k,i}$, $i\in\mathbb{N}$, of $\mathcal{B}$ such that $\mathcal{Z}_k=\bigcup_{i\in\mathbb{N}}\mathcal{Z}_{k,i}$ and for every $i\in\mathbb{N}$, the factors induced by the σ-algebras $\mathcal{Z}_{k,i}$ are isomorphic to k-step nilsystems.

(ii) *If $X_i = G_i/\Gamma_i$, $1 \leq i \leq k$, are nilmanifolds, then for every $b_i \in G_i$ and $x_i \in X_i$ the sequence*

$$\left(b_1^{[p_1(n)]}x_1, \dots, b_k^{[p_k(n)]}x_k\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^n x_1)}_n \times \cdots \times \overline{(b_k^n x_k)}_n.$$

Actually, by [6, Lemma 5.1], Part (ii) of Theorem 4.1 follows from Part (i).

Part (i) of Theorem 4.1 on the other hand, follows by the following two statements:

Proposition 4.2 (Karageorgos-K, [14])**.** *Let $k \in \mathbb{N}$ and $p_1, \dots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials. Let $X = G/\Gamma$ be a nilmanifold with G connected and simply connected and elements $b_i \in G$ acting ergodically on X. Then the sequence*

$$\left(b_1^{p_1(n)}\Gamma, \dots, b_k^{p_k(n)}\Gamma\right)_n$$

is equidistributed in the nilmanifold X^k.

This proposition uses the deep result of Leibman, Theorem 3.1, on the equidistribution of polynomial sequences in a nilmanifold. From the proof of this proposition, by Weyl's criterion, we get a first condition that our polynomials have to satisfy in order for Theorem 4.1 to hold.

The last ingredient in proving Part (i) of Theorem 4.1 is the following lemma:

Lemma 4.3 (Lemma 5.2, [6])**.** *Let $k \in \mathbb{N}$ and $X = G/\Gamma$ be a nilmanifold with G connected and simply connected. Then for every $b_1, \dots, b_k \in G$ there exists an $s_0 \in \mathbb{R}$ such that for all $1 \leq i \leq k$ the element $b_i^{s_0}$ acts ergodically on the nilmanifold $\overline{(b_i^s\Gamma)}_{s\in\mathbb{R}}$.*

The proof of this lemma (together with the proof of the precious proposition) gives us the precise condition, that of "strongly independence", that our polynomials have to satisfy for Theorem 4.1 to hold.

The last step before the proof of Theorem 2.3 is to show that the nilfactor is the characteristic factor for our "nice" polynomial iterates.

Definition ([7])**.** Let $k \in \mathbb{N}$ and for $N \in \mathbb{N}$, let $\mathcal{P}_N = \{p_{1,N}, \dots, p_{k,N}\}$ be a family of polynomials with real coefficients. We say that the collection $(P_N)_N$ is *nice* if for every $N \in \mathbb{N}$ the polynomials $p_{i,N}$ and $p_{i,N} - p_{j,N}$, $i \neq j$, are non-constant and their leading coefficients are independent of N.

Note that a strongly independent family of polynomials is nice.

Lemma 4.4 (Lemma 4.7, [7])**.** *Let $(\{p_{1,N}, \dots, p_{k,N}\})_N$ be a nice collection of polynomial families, $(X, \mathcal{B}, \mu, T)$ be a system and suppose that one of the functions $f_1, \dots, f_k \in L^\infty(\mu)$ is orthogonal to the nilfactor $\mathcal{Z}$. Then for any*

Følner sequence $(\Phi_N)_N$ *in* $\mathbb{Z}$ [4] *and any bounded two parameter sequence* $(c_{N,n})_{N,n}$ *of real numbers we have*

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} c_{N,n} T^{[p_{1,N}(n)]} f_1 \cdots T^{[p_{k,N}(n)]} f_k = 0, \tag{13}$$

where the convergence takes place in $L^2(\mu)$.

We close this subsection with the proof of Theorem 2.3:

Proof of Theorem 2.3, [14]. We start by using Lemma 4.4 in order to get that the nilfactor $\mathcal{Z}$ is characteristic for the corresponding multiple ergodic average. Via Theorem 3.2 we can assume without loss of generality that our system is an inverse limit of nilsystems. By a standard approximation argument, we can further assume that our system is a nilsystem.

Let $(X = G/\Gamma, \mathcal{G}/\Gamma, m_X, T_b)$ be a nilsystem, where $b \in G$ is ergodic, and $F_1, \dots, F_k \in L^\infty(m_X)$. Our objective now is to show that if $\{p_1, \dots, p_k\}$ is a strongly independent family of polynomials then

$$\lim_{N\to\infty} \sum_{n=1}^{N} F_1(b^{[p_1(n)]}x) \cdots F_\ell(b^{[p_k(n)]}x) = \int F_1 \, dm_X \cdots \int F_k \, dm_X \tag{14}$$

where the convergence takes place in $L^2(m_X)$. By density, we can assume that the functions $F_1, \dots, F_k$ are continuous. Then we can apply Theorem 4.1 to the nilmanifold X^k, the nilrotation $\tilde{b} = (b, \dots, b) \in G^k$, the point $\tilde{x} = (x, \dots, x) \in X^k$, and the continuous function $\tilde{F}(x_1, \dots, x_k) = F_1(x_1) \cdots F_k(x_k)$, to get that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \tilde{F}(b^{[p_1(n)]}x, \dots, b^{[p_k(n)]}x) = \int \tilde{F} \, dm_{X^k}$$

and this gives the desired limit in (14), completing the proof. □

5. From averages along natural to prime numbers

In this last section we will prove the corresponding expressions of Theorems 2.2 and 2.3, and so, their applications as well, along prime numbers. More specifically we will show:

Theorem 5.1 (Karageorgos-K., [14]). *Let* $q \in \mathbb{R}[t]$ *with* $q(t) \neq c\tilde{q}(t) + d$ *for all* $c, d \in \mathbb{R}$ *and* $\tilde{q} \in \mathbb{Q}[t]$*. Then for every* $k \in \mathbb{N}$*, system* $(X, \mathcal{B}, \mu, T)$ *and* $f_1, \dots, f_k \in L^\infty(\mu)$*, we have that*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} \prod_{i=1}^{k} T^{i[q(p)]} f_i = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} T^{in} f_i,$$

where the convergence takes place in $L^2(\mu)$ *and* $\pi(N) = |\mathbb{P} \cap [1, N]|$ *denotes the number of primes up to* N.

[4]A *Følner sequence in* $\mathbb{Z}$ is a sequence $(\Phi_n)_n$ of finite sunsets of $\mathbb{Z}$ such that for any $m \in \mathbb{Z}$ we have $\lim_{n\to\infty} |\Phi_n|^{-1} |(\Phi_n + m) \cap \Phi_n| = 1$.

Theorem 5.2 (Karageorgos-K., [14])**.** *Let $k \in \mathbb{N}$, $p_1, \dots, p_k \in \mathbb{R}[t]$ be strongly independent real polynomials, $(X, \mathcal{B}, \mu, T)$ be an ergodic system and $f_1, \dots, f_k \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} T^{[p_1(p)]}f_1 \cdots T^{[p_k(p)]}f_k = \prod_{i=1}^{k} \int f_i \, d\mu,$$

where the convergence takes place in $L^2(\mu)$.

We start by recalling the definition of the *von Mangoldt function*, $\Lambda : \mathbb{N} \to \mathbb{R}$, where $\Lambda(n) = \begin{cases} \log(p) & , \text{ if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and some } k \in \mathbb{N} \\ 0 & , \text{ elsewhere} \end{cases}$.

It is more natural though for us to work instead of Λ with the function $\Lambda' : \mathbb{N} \to \mathbb{R}$, where $\Lambda'(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \Lambda(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \log(n)$.

The function Λ', according to the following lemma, will allow us to relate averages along primes with weighted averages over the integers.

Lemma 5.3 ([9])**.** *If $a : \mathbb{N} \to \mathbb{C}$ is bounded, then*

$$\lim_{N\to\infty} \left| \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} a(p) - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) \cdot a(n) \right| = 0.$$

The proof of this lemma, which can be found in [9], uses the prime number theorem and it is relatively immediate.

Note that in order for someone to show convergence along primes of averages of the sequence $(a(n))_n$, according to this lemma, has to show convergence along natural numbers of averages of the sequence $(\Lambda'(n) \cdot a(n))_n$.

For $w > 2$, let

$$W = \prod_{p\in\mathbb{P}\cap[1,w-1]} p$$

be the product of primes bounded above by w. For $r \in \mathbb{N}$, let

$$\Lambda'_{w,r}(n) = \frac{\phi(W)}{W} \cdot \Lambda'(Wn + r),$$

where ϕ is the Euler function, be the *modified von Mangoldt function.*

The proposition below, the proof of which relies on a deep result due to Green and Tao ([12]) on the inverse conjecture for the Gowers norms, will provide us with a crucial intermediate step in order to prove Theorems 5.1 and 5.2, as well as as well as to get their implications (we will actually use a very weak version of it for all these results).

Proposition 5.4 (Proposition 3.2, [16])**.** *Let $k, m \in \mathbb{N}$, $(X, \mathcal{B}, \mu, T_1, \dots, T_m)$ be a system, where T_i's commute, $p_{i,j} \in \mathbb{R}[t]$ be real polynomials, $1 \le i \le m$, $1 \le j \le k$ and $f_1, \dots, f_k \in L^\infty(\mu)$. Then,*

$$\max_{1\le r\le W,\, (r,W)=1} \left\| \frac{1}{N} \sum_{n=1}^{N} (\Lambda'_{w,r}(n) - 1) \cdot \prod_{j=1}^{k} \left(\prod_{i=1}^{m} T_i^{[p_{i,j}(Wn+r)]} \right) f_j \right\|_{L^2(\mu)}$$

converges to 0 as $N \to \infty$ and then $w \to \infty$.

Proof of Theorem 5.1. We borrow the arguments from the proof of Theorem 1.3 from [10] (see also Theorem 1.3 in [16]). By Lemma 5.3 it suffices to show that the sequence

$$A(N) := \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) \cdot T^{[q(n)]}f_1 \cdot T^{2[q(n)]}f_2 \cdots T^{k[q(n)]}f_k$$

converges in $L^2(\mu)$ to the same limit as the sequence

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k$$

as $N \to \infty$. For w (which gives a corresponding W), $r \in \mathbb{N}$, we define

$$B_{w,r}(N) := \frac{1}{N}\sum_{n=1}^{N} T^{[q(Wn+r)]}f_1 \cdot T^{2[q(Wn+r)]}f_2 \cdots T^{k[q(Wn+r)]}f_k.$$

For any $\varepsilon > 0$, using Proposition 5.4 with $m = k$, $T_i = T$, $1 \leq i \leq k$ and

$$p_{i,j} = \begin{cases} 0, & \text{if } i \leq k-j \\ q, & \text{elsewhere,} \end{cases}$$

for sufficiently large N and some w_0 we have

$$\left\| A(W_0 N) - \frac{1}{\phi(W_0)} \sum_{1 \leq r \leq W_0,\, (r,W_0)=1} B_{w_0,r}(N) \right\|_{L^2(\mu)} < \varepsilon.$$

Note at this point that for all $W, r \in \mathbb{N}$ we have that $q(Wt+r) \notin c\mathbb{Q}[t] + d$ for $c, d \in \mathbb{R}$, for otherwise q would have the same property contradicting our assumption. By Theorem 2.2, we have that for any $1 \leq r \leq W_0$ the sequence $(B_{w_0,r}(N))_N$ converges to the same limit as the sequence $N^{-1}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k$, and since

$$\lim_{N\to\infty} \|A(W_0 N + r) - A(W_0 N)\|_{L^2(\mu)} = 0$$

for every $1 \leq r \leq W_0$, we get the result. □

Proof of Theorem 5.2. The proof is analogous to the previous one. In this case we define $A(N) := N^{-1}\sum_{n=1}^{N} \Lambda'(n) \cdot T^{[p_1(n)]}f_1 \cdots T^{[p_k(n)]}f_k$ and for $w, r \in \mathbb{N}$, $B_{w,r}(N) := N^{-1}\sum_{n=1}^{N} T^{[p_1(Wn+r)]}f_1 \cdots T^{[p_k(Wn+r)]}f_k$. We use Proposition 5.4 with $m = k$, $T_i = T$, $1 \leq i \leq k$,

$$p_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ p_i, & \text{if } i = j \end{cases}$$

and we note that the family $\{\tilde{p}_1, \ldots, \tilde{p}_k\}$, where $\tilde{p}_i(t) = p_i(Wt+r)$, is strongly independent for all $W, r \in \mathbb{N}$. (Indeed, if for some $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \setminus \{\vec{0}\}$, $d \in \mathbb{R}$, $q \in \mathbb{Q}[t]$ and $W, r \in \mathbb{N}$ we had $\sum_{i=1}^{k} \lambda_i p_i(Wt+r) = q(t) + d$, then $\sum_{i=1}^{k} \lambda_i p_i(t) = \tilde{q}(t) + d$, where $\tilde{q}(t) = q((t-r)/W) \in \mathbb{Q}[t]$, a contradiction to the strong independence assumption.) The result now follows similarly to the previous proof since by Theorem 2.3, we have that for any $1 \leq r \leq W_0$ the sequence $(B_{w_0,r}(N))_N$ converges, in $L^2(\mu)$, to $\prod_{i=1}^{k} \int f_i \, d\mu$. □

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QUASILINEAR ELLIPTIC EQUATIONS WITH LACK OF COMPACTNESS: SURVEY OF SOME RESULTS AND DISCUSSION OF OPEN PROBLEMS

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"*Ce que nous connaissons est peu de chose,*
ce que nous ignorons est immense"
Pierre-Simon Marquis de Laplace (*)

ABSTRACT. We discuss in the form of a survey, some open problems concerning the existence or absence of non-negative non-trivial (i.e. $\not\equiv 0$) weak solutions for quasilinear elliptic equations of the form

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \qquad x \in \mathbb{R}^N, \quad N \geqslant 2,$$

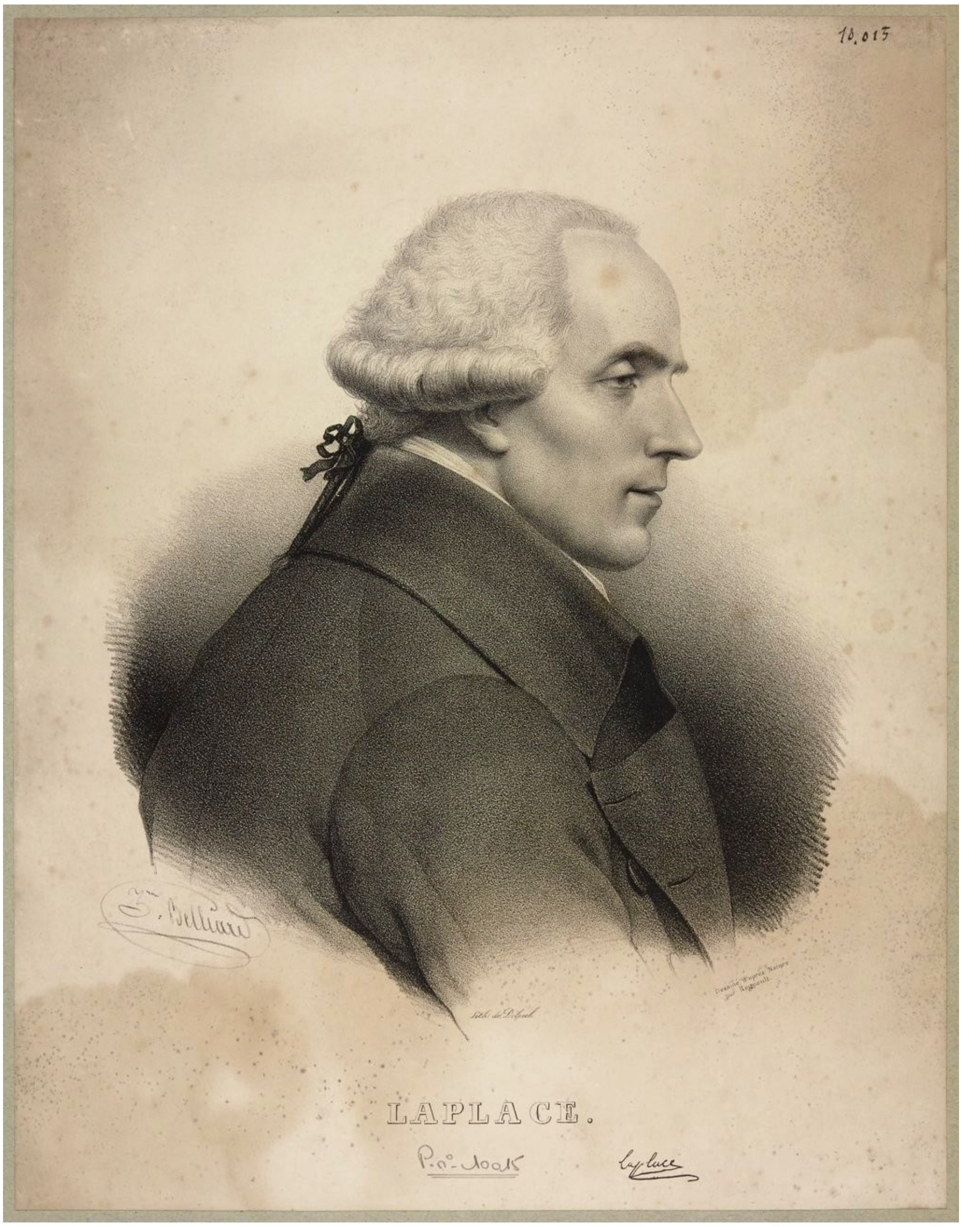
where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, $p \in (1,\infty)$, is the p-Laplace operator while $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ with $f(\cdot, 0) = 0$ are given continuous functions.

Keywords: p-Laplacian; nonlinear Schrödinger equation; subcritical, critical and supercritical Sobolev exponents; Palais-Smale condition

2010 Mathematics Subject Classification: 35J10, 35J92, 35Q35, 35Q55, 35Q74, 35B33, 35J20

CONTENTS

*) Allegedly his last words, reported in "*Éloge historique de M. le Marquis de Laplace*" with the comment: "C'est du moins, autant qu'on l'a pu saisir, les sens de ses dernières paroles à peine articulées"; delivered by Jean-Baptiste-Joseph Baron Fourier on June 15, 1829 before the *Académie Royale des Sciences* in Paris (published in: *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, Vol. 10, pp. lxxxi-cii, Gauthier-Villars, Paris, 1831).



PIERRE-SIMON MARQUIS DE LAPLACE

23 March 1749 – 5 March 1827

(Lithography by Delpech after a drawing by Zéphirin Belliard; dated probably during the period of *Directoire*)

1. Introduction

Consider the quasilinear elliptic equation

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{1}$$

where

$$\Delta_p u := div(|\nabla u|^{p-2}\nabla u), \qquad p \in (1,+\infty),$$

is the p-Laplace operator (which reduces to the classical Laplace operator when $p=2$) while $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$, $N \geqslant 2$, are given continuous functions.

Equations of this form not only exhibit unexpectedly rich mathematical structure but also are ubiquitous in many and diverse contexts of Mathematical Physics like non-relativistic quantum mechanics, field theory, nonlinear optics [18, 20, 21, 49, 113] and continuum mechanics [11, 22, 50, 59, 69, 78, 79, 124]; as well as in Astrophysics [35], Differential Geometry [13, 70, 118, 125], Geometric Function Theory [23, 85, 104] and elsewhere. As a consequence, their study has triggered an explosive development over the past four decades which, in return, has rendered a vast literature. The prototypical example is provided by the semilinear equation

$$-\varepsilon^2\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{2}$$

which arises when we seek *standing wave* solutions of the celebrated nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + W(x)\psi - g(x,|\psi|)\psi, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3}$$

i.e. solutions of the form

$$\psi(x,t) = \exp(-i\mathscr{E}t/\hbar)u(x), \qquad \mathscr{E} \in \mathbb{R}, \tag{4}$$

where $i=\sqrt{-1}$, $\hbar$ is Planck's constant, $m>0$ and $W(\cdot)$ is a real-valued potential. Such solutions, if they exist, have an important physical interpretation since they correspond to stable quantum states with "energy"$\mathscr{E}$. Clearly, (4) satisfies (3) if and only if $u(x)$ solves (2) with $V(x) = W(x) - \mathscr{E}$, $\varepsilon^2 = \hbar^2/2m$ and $f(x,u) = g(x,|u|)u$.

From a different perspective, one is also led to an equation of the above type (with $p=2$) when searching for *travelling waves* of the nonlinear Klein-Gordon equation

$$\varphi_{tt} - \Delta\varphi = g(|\varphi|)\varphi, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

i.e. solutions of the form

$$\varphi(x,t) = u(x-ct),$$

where c is a given vector in $\mathbb{R}^N$ with $|c|<1$; [109].

Eq. (2) has been studied extensively under various hypotheses on the potential $V(\cdot)$ and the nonlinearity $f(\cdot,\cdot)$. Much of the impetus for these studies seems to have originated from the seminal paper [60] by Floer and Weinstein in which the one-dimensional case ($N=1$) with $f(x,u)=u^3$ was considered. Actually, based on a Lyapunov-Schmidt-type reduction technique, it was shown there that if $V(\cdot)$ is a bounded potential having a single non-degenerate minimum point x_0 while $\inf_{\mathbb{R}} V > 0$ then (2) admits solutions which in the *semi-classical limit* (i.e. as $\varepsilon \downarrow 0$) concentrate around x_0;

see also [92, 93]. The extension of this important result to higher dimensions with $f(x,u) = |u|^{q-2}u$, $2 < q < 2^* := 2N/(N-2)$, $N \geqslant 3$, and $V(\cdot)$ having a finite set of non-degenerate critical points was achieved in [94]. A host of results regarding Eq. (2) appeared thereafter. A detailed recount, however, is beyond our present scope and the interested reader is referred to the excellent monograph [8], as well as to the copious bibliography cited therein.

Our aim here, instead, is to highlight in a concise state-of-the-art survey ([1]) some thorny open questions concerning the solvability of Eq. (1). Note, however, that no effort is made to be as general or comprehensive as possible. As a matter of fact, in order to achieve utmost clarity and simplicity, we have limited our discussion on the *existence* or *absence* of non-trivial non-negative weak solutions of the model equation

$$(\bigstar) \qquad -\Delta_p u + V(x)|u|^{p-2}u = |u|^{q-2}u, \qquad x \in \mathbb{R}^N, \qquad 1 < p < N,$$

by considering separately the following three very distinct cases:

A) Subcritical case:
$$p \leqslant q < p^*$$

B) Critical case:
$$q = p^*$$

and

C) Supercritical case:
$$q > p^*.$$

Here
$$p^* := \frac{Np}{N-p}, \qquad 1 < p < N,$$
is the well-known critical exponent in the classical Sobolev Embedding Theorem which asserts that if Ω is any C^1-smooth domain in $\mathbb{R}^N$ with bounded $\partial\Omega$ and
$$p \leqslant q \leqslant p^*$$
then the continuous injection
$$W^{1,p}(\Omega) \circlearrowleft L^q(\Omega)$$
holds, where

$$W^{1,p}(\Omega) := \{v \in L^p(\Omega) : \ \exists g_i \in L^p(\Omega),\ i = 1, ..., N,$$
$$\text{such that} \int_\Omega v \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_0^1(\Omega)\}.$$

In particular, if Ω is *bounded* then the Rellich-Kondrachov Theorem warrants that this injection is also *compact* when $p \leqslant q < p^*$, but *not* in the critical case $q = p^*$. On the other hand, if $\Omega = \mathbb{R}^N$ then this injection is *non-compact for all* $q \in [p, p^*]$.

[1]) Conceived, primarily, for the non-specialist in the field.

Notation:

- $C_0^1(\Omega)$, $\Omega \subseteq \mathbb{R}^N$, denotes the space of continuously differentiable functions with compact support in Ω; $C^k(\Omega)$, $k \in \mathbb{N}$, is the space of k times continuously differentiable functions in Ω.
- $C^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$, $\Omega \subseteq \mathbb{R}^N$, denotes the space of functions whose first order derivatives are Hölder continuous with exponent α.
- $L^s(\Omega)$, $1 \leqslant s \leqslant \infty$, $\Omega \subseteq \mathbb{R}^N$, are the usual Lebesgue spaces with norm denoted by $||\cdot||_{L^s(\Omega)}$; if $\Omega = \mathbb{R}^N$ we simply write $||\cdot||_s$.
- $\mathscr{D}^{1,p}(\mathbb{R}^N)$, $W^{1,p}(\Omega)$, $H^1(\Omega)$, $\Omega \subseteq \mathbb{R}^N$, are the usual Sobolev spaces.
- $\mathcal{S} := \mathcal{S}(p,N)$ denotes the best constant in the classical Sobolev inequality; that is

$$\mathcal{S}(p,N) := \inf_{v \in \mathscr{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^p dx}{\left(\int_{\mathbb{R}^N} |v|^{p^*} dx\right)^{p/p^*}}, \tag{5}$$

or, as is well-known (cf. [12, 114]),

$$\mathcal{S}(p,N) = N\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)\omega_N}{\Gamma(N+1)}\right)^{p/N},$$

where $\Gamma(\cdot)$ is Euler's gamma function and $\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of the unit sphere $\mathbb{S}^{N-1}$.

2. Lack of compactness

Many difficulties, which are deep-rooted and not just technical, make the analytical treatment of Eq. (★) very challenging, the most important ones being related to the severe lack of compactness, due essentially to:

i) the unboundedness of the domain
ii) the presence of critical or supercritical Sobolev exponents (case B or C, respectively)
iii) the non-standard functional setting which requires working with weighted Sobolev spaces when $V(\cdot) \not\equiv 0$.

In the *non-supercritical* case (i.e. when $q \in [p, p^*]$) the standard vehicle in proving existence of weak solutions is to search for non-trivial (i.e. $\neq 0$) stationary points in an appropriate function space E of the associated *action functional (Lagrangian)*, which is *formally* defined as follows:

$$\Phi(u) := \frac{1}{p}\int_{\mathbb{R}^N} \{|\nabla u|^p + V(x)|u|^p\}\, dx - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx. \tag{6}$$

Suppose $\Phi(\cdot)$ is well defined and of class $C^1(E)$. Then, by weak solution of (★) we mean a critical point $u \in E\setminus\{0\}$ of $\Phi(\cdot)$; i.e.

$$\left\langle \Phi'(u), \phi \right\rangle = 0, \quad \forall \phi \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of E^* and E. Moreover, any function $u \neq 0$ that minimizes $\Phi(\cdot)$ on the so-called *Nehari manifold* (cf. [89])

$$\mathscr{N} := \{w \in E\setminus\{0\} : \left\langle \Phi'(w), w \right\rangle = 0\},$$

is called a *ground state* or a *least-action solution* of (★). On the other hand, any critical point of $\Phi(\cdot)$ with finite energy (i.e. belonging to the Sobolev space $W^{1,p}(\mathbb{R}^N)$) is called a *bound state*. Apart from their own mathematical interest, solutions of the latter kind, if they exist, may also have important physical meaning. For instance, according to the well-known probabilistic interpretation of Quantum Mechanics, the most relevant standing wave solutions of the nonlinear Schrödinger equation (3) are these which are square-integrable in $\mathbb{R}^N$ since then, they correspond to localized elementary particles.

It is worth remarking here that if $1 < q \leqslant p^*$ and $V(\cdot) \in L^\infty_{loc}(\mathbb{R}^N)$ then it can be shown that any weak solution of (★) is in $L^\infty(\mathbb{R}^N)$ (cf. [97, 106]) and thus, by well-known regularity results, (equivalent to a function) of class $C^{1,\alpha}_{loc}(\mathbb{R}^N)$; cf. [51, 116]. This means that ∇u is locally Hölder continuous with exponent $\alpha = \alpha(N,p) \in (0,1)$ (2).

Observe also that if $u \in E$ is a critical point of $\Phi(\cdot)$ then $|u|$ is as well and so we can assume that u is non-negative. In particular, if $0 \leqslant u \in L^\infty(\mathbb{R}^N)$ and $u \not\equiv 0$ then Harnack's inequality [117] entails that $u > 0$ in $\mathbb{R}^N$.

In seeking stationary points of the action (6), the employed analysis involves several tools available from Calculus of Variations (i.e. Critical Point Theory [80, 111], Mountain-Pass Theorem [10, 68, 100, 102], Palais-Smale Condition [28, 81, 82, 83, 95, 110], Concentration-Compactness Method [73, 74], Morse Theory [36, 56, 80] etc.). However, as it is well-known (and should be strongly emphasized!), such an approach turns out to be very delicate in the *critical case* (due to the severe lack of compactness induced on Palais-Smale sequences) while in the *supercritical case* it is entirely inapplicable! As a matter of fact, a major technical obstacle in understanding the supercritical regime stems from the lack of local Sobolev embeddings suitably fit to a weak formulation.

2.1. **The Palais-Smale condition.** A sequence $\{u_n\}_{n\in\mathbb{N}} \in E$ is called a *Palais-Smale sequence* with respect to $\Phi(\cdot)$ at level c (or, $(PS)_c$-sequence for short) if

$$\Phi(u_n) \to c \quad \text{and} \quad ||\Phi'(u_n)||_{E^*} \to 0\ , \quad \text{as } n \to \infty.$$

Such functional sequences play a fundamental role in the employed variational arguments in order to *restore compactness*, while their existence is guaranteed by the classical Mountain-Pass Lemma whenever $\Phi(\cdot)$ enjoys a specific "geometric structure"; cf. [10, 26, 28].

If every $(PS)_c$-sequence is relatively compact in E then we say that the *Palais-Smale condition at level c* is satisfied. In some sense the $(PS)_c$-condition prevents critical points from "leaking at infinity". If it fails at some level c, this means, roughly speaking, that c is a critical value which

2) If $V(\cdot)$ is more regular, say $V(\cdot) \in C^{0,\alpha}(\mathbb{R}^N)$, one would expect the solution u to be more regular, as well. As it turns out, this is indeed true in the semilinear case $p = 2$ (cf. [65]) but in the singular ($1 < p < 2$) or the degenerate ($p > 2$) quasilinear case, where the uniform ellipticity of the p-Laplace operator is lost at the zeros of $|\nabla u|$, the best that one can hope for is $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$; cf. [51, 116, 119].

corresponds to the so-called "*critical points at infinity*"; a fundamental concept introduced by A. Bahri [15, 16]. For an excellent account of the historical evolution of the Palais-Smale condition in Critical Point Theory the interested reader is referred to [81].

3. $V(\cdot) \equiv 0$

3.1. Semilinear case ($p = 2$).

$$\left\{ \begin{array}{c} -\Delta u = |u|^{q-2}u \\ \\ u \geqslant 0 \end{array} \right\}, \quad x \in \mathbb{R}^N, \quad N \geqslant 3. \tag{7}$$

Despite its deceptive simplicity, Eq. (7) is analytically very difficult and its treatment requires a lot of sophisticated machinery. In the subcritical case the following beautiful and deep Liouville-type theorem of Gidas and Spruck holds true.

Theorem 1. [64]. *Let* $2 \leqslant q < 2^*$. *Then the only solution of* Eq. (7) *is* $u \equiv 0$.

Remark 2. The original (and lengthy!) proof in [64] relies on some remarkable functional identities while it is a direct consequence of a much more general theorem involving nonlinear elliptic equations on complete Riemannian manifolds. A very short and ingenious proof in the Euclidean case (which is actually valid for $1 < q < 2^*$) was later discovered by Chen and Li [37] by using the Kelvin transform [115],

$$v := v(x) = |x|^{2-N} u\left(\frac{x}{|x|^2}\right), \qquad x \neq 0. \tag{8}$$

As it can be directly verified, v satisfies the equation

$$-\Delta v = |x|^{-\gamma} v^{q-1}, \quad x \in \mathbb{R}^N \backslash \{0\}, \quad N \geqslant 3,$$

where $\gamma := (N+2) - (q-1)(N-2) > 0$. By applying the Alexandrov-Serrin method of moving planes [1, 107], it is then shown that v is necessarily radially symmetric about the origin and hence, so is u. But due to the invariance of Eq. (7) under rotations and translations, if x_1 and x_2 are two arbitrary points in $\mathbb{R}^N$, the origin can be chosen to be the mid point of the line segment joining them. Therefore, we must have $u(x_1) = u(x_2)$; i.e. u is constant. Hence, by (7), $u \equiv 0$.

Remark 3. Theorem 1 is characterized by two striking features:

i) nothing is assumed about the behavior of $u(x)$ as $|x| \to \infty$, ([3])
ii) it fails for all $q \geqslant 2^*$.

In fact, we have:

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[3]) e.g. the same conclusion can be readily obtained when $q \in (2, 2^*)$ via a variational identity [99] by imposing the decay-assumption

$$u(x) \leqslant C|x|^{-2/(q-2)}, \quad x \neq 0.$$

Theorem 4. *Let $q \geqslant 2^*$. Then* Eq. (7) *admits a continuum of (positive) radial solutions which tend to zero as $|x| \to \infty$.*

Remark 5. Theorem 4 can be demonstrated by applying the transformation (due to Fowler [61])

$$v(s) = r^{2/(q-2)}u(r), \qquad r = |x| = e^s,$$

which transforms (7) into the autonomous ordinary differential equation

$$\ddot{v} + \alpha\dot{v} - \beta v + v^{q-1} = 0,$$

where

$$\alpha = N - 2 - \frac{4}{q-2} \geqslant 0, \quad \beta = \frac{2}{q-2}\left(N - 2 - \frac{2}{q-2}\right) > 0.$$

As it can then be shown via phase-plane analysis, all non-trivial radial solutions of (7) are given by the two-parameter family of functions

$$u(x) = U_{\lambda,x_0}(x) := \lambda^{2/(q-2)}U(\lambda|x - x_0|), \quad \lambda > 0,$$

where $U(\cdot)$ is the unique positive radial solution satisfying $U(0) = 1$, $U'(0) = 0$. Moreover, $U(\cdot)$ is monotone decreasing and if $q > 2^*$ then

$$\lim_{|x|\to\infty} \frac{U(|x|)}{|x|^{-2/(q-2)}} = \beta^{1/(q-2)}.$$

Incidentally, if $q > 2(N-1)/(N-2)$ then (7) admits the explicit singular radial solution

$$\widetilde{U}(x) := \beta^{1/(q-2)}|x|^{-2/(q-2)}, \qquad x \neq 0,$$

which is also a weak solution in $H^1_{loc}(\mathbb{R}^N)$ when $q > 2^*$.

In particular, in the critical case the following very elegant and celebrated "uniqueness" holds.

Theorem 6. [32, 37, 62, 63]. *If $q = 2^*$ then the only non-trivial C^2-solutions of* Eq. (7) *are given explicitly by the formula*

$$u(x) = \left(\frac{\lambda\sqrt{N(N-2)}}{|x - x_0|^2 + \lambda^2}\right)^{(N-2)/2}, \tag{9}$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^N$ are arbitrary.

Remark 7. The functions given by (9) are called Talenti instantons and achive equality in Sobolev's inequality [12, 114]; that is, they are minimizers of the Sobolev quotient

$$\inf_{v\in\mathscr{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx}{\left(\int_{\mathbb{R}^N} |v|^{2^*} dx\right)^{2/2^*}},$$

where

$$\mathscr{D}^{1,2}(\mathbb{R}^N) := \left\{v \in L^{2^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla v|^2 dx < \infty\right\}.$$

Remark 8. Theorem 6 was originally proved by Gidas, Ni and Nirenberg [63] by showing first that all positive solutions with reasonable behavior at infinity (i.e. $u = O(|x|^{2-N})$) are necessarily radially symmetric and monotone decreasing about some point x_0. Hence, (9) follows directly from the fact that u satisfies the ordinary differential equation

$$u'' + \frac{N-1}{r}u' + u^{2^*-1} = 0,$$

with $u'(0) = 0$, where $u = u(r)$, $r = |x - x_0|$. The removal of any decay assumptions on u was achieved in [32]. A short and somewhat elementary proof was provided later in [37]. It is worth pointing out that the arguments in all these proofs employ the moving-plane technique, as well as the invariance properties of (7) when $q = 2^*$; namely, under conformal transformations $x \longmapsto y$ ([4]) in [63] or under the Kelvin transform (8) in [32, 37]. The reader should be warned, on the other hand, that the non-negativity of u is crucial for the validity of Theorem 6; cf. [52].

3.2. **Quasilinear case $(1 < p < N)$.**

$$\left\{ \begin{array}{c} -\Delta_p u = |u|^{q-2}u \\ \\ u \geqslant 0 \end{array} \right\}, \quad x \in \mathbb{R}^N, \quad 1 < p < N. \tag{10}$$

For comparison purposes with the aforementioned results, let us first state the precise relationship between the quasilinear Δ_p-operator and the classical Laplacian when they are applied to any $v \in C^2(\mathbb{R}^N)$ with $\nabla v \neq 0$:

$$\Delta_p v = |\nabla v|^{p-4} \left\{ |\nabla v|^2 \Delta v + (p-2) \sum_{i,j=1}^{N} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} \right\},$$

or, equivalently,

$$\Delta_p v = |\nabla v|^{p-2} \cdot \mathrm{Trace} \left\{ \left(\mathbf{I} + (p-2)\frac{\nabla v \otimes \nabla v}{|\nabla v|^2} \right) \mathrm{Hessian}(v) \right\},$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the dyadic matrix with components $a_i b_j$.

In the quasilinear case $(p \neq 2)$ problem (10) is considerably more difficult due to the nonlinear nature of the p-Laplace operator, the lack of C^2-regularity of the solutions and the fact that comparison principles are no longer equivalent to maximum principles; cf. [101]. Moreover, a Kelvin-type transform is, unfortunately, not available in this case; cf. [72].

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[4]) i.e. if u is a solution then the function

$$v(y) := [J(x)]^{-1/2^*} u(x),$$

where $J(\cdot)$ is the Jacobian of the transformation, is also a solution. In particular, if $u(x)$ is a solution, then for any $\lambda > 0$ and $x_0 \in \mathbb{R}^N$, $v_{\lambda,x_0}(x) := \lambda^{-(N-2)/2} u((x - x_0)/\lambda)$ is also a solution.

Nevertheless, the following spectacular generalization of the Gidas-Spruck result [64] holds true in the *entire* subcritical regime ([5]).

Theorem 9. [108]. *Let* $1 < q < p^*$. *Then the only weak solution* ([6]) *of* Eq. (10) *is* $u \equiv 0$.

Remark 10. Like in the semilinear case, Theorem 9 is also optimal since no assumptions on the behavior of $u(x)$ at infinity are imposed while the conclusion fails for all $q \geqslant p^*$.

As a matter of fact, we have:

Theorem 11. [90]. *If* $q \geqslant p^*$ *then* Eq. (10) *admits a continuum of* (*positive*) *radial solutions which tend to zero as* $|x| \to \infty$.

In particular, the following magnificent "uniqueness" result holds in the critical case (Its long and arduous proof was very recently completed by the combined efforts of several researchers).

Theorem 12. [42, 67, 105, 120]. *If* $q = p^*$ *then the only non-trivial solutions of* Eq. (10) *in* $\mathscr{D}^{1,p}(R^N)$ *are given explicity by the formula*

$$u(x) = \left(\frac{\lambda^{1/(p-1)} \left(N^{1/p} \left(\frac{N-p}{p-1} \right)^{(p-1)/p} \right)}{|x - x_0|^{p/(p-1)} + \lambda^{p/(p-1)}} \right)^{(N-p)/p},$$

where $\lambda > 0$ *and* $x_0 \in \mathbb{R}^N$ *are arbitrary.*

Remark 13. Like the semilinear case, these special functions (called again Talenti instantons) are classical solutions while they achive equality in Sobolev's inequality in $\mathscr{D}^{1,p}(R^N)$, [12, 114]; that is, they are minimizers of the Sobolev quotient

$$\inf_{v \in \mathscr{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^p dx}{\left(\int_{\mathbb{R}^N} |v|^{p^*} dx\right)^{p/p^*}},$$

where

$$\mathscr{D}^{1,p}(\mathbb{R}^N) := \left\{ v \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla v|^p dx < \infty \right\}.$$

It should be pointed out, though, that the non-negativity of u is crucial (as when $p = 2$) for the validity of Theorem 12; cf. [40].

Theorem 11 raises directly the following problem.

[5]) Actually, the salient complexity of the proof by Gidas and Spruck in the semilinear case is so deterring that no one could have dreamed of extending it to a more general situation.

[6]) A function $u \in C^1(\mathbb{R}^N)$ is called a *weak solution* of Eq. (10) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx = \int_{\mathbb{R}^N} |u|^{q-2} u \phi dx \quad \text{for all } \phi \in C_0^1(\mathbb{R}^N).$$

Note, however, that due to the classical regularity results mentioned earlier, if $1 < q \leqslant p^*$ then one can require less smoothness for u; i.e. $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$.

Open Problem 1

Do there exist non-radial solutions of Eq. (10) *when* $q > p^*$ with $p \neq 2$ *? Note that the question regarding the existence of non-symmetric solutions in the supercritical semilinear case* ($i.e.$ *when* $p = 2$, $q > 2^*$) *has been answered affirmatively in* [132].

4. $V(\cdot) \not\equiv 0$

As it may be anticipated, one encounters here a strikingly different situation with a much greater variety of phenomena arising.

4.1. **Semilinear case ($p = 2$).**

$$\left\{ \begin{array}{c} -\Delta u + V(x)u = |u|^{q-2}u \\ \\ u \geqslant 0 \end{array} \right\}, \quad x \in \mathbb{R}^N, \quad N \geqslant 3. \tag{11}$$

In the subcritical case ($2 < q < 2^*$) a host of results has been recorded in the literature thus far; mostly, under the assumption

$$\inf_{x \in \mathbb{R}^N} V(x) > 0, \tag{12}$$

which renders a more tractable functional analytic framework. As a relevant sample, we single out the following two important results.

Theorem 14. [103]. *Let* $2 < q < 2^*$. *If* $V(\cdot) \in C^1(\mathbb{R}^N)$ *and*

$$0 < V_0 \leqslant V(x) \leqslant \liminf_{|x| \to \infty} V(x),$$

with strict inequality on a set of positive measure, then Eq. (11) *has a (positive) solution in* $H^1(\mathbb{R}^N)$.

Theorem 15. [34]. *Let* $2 < q < 2^*$. *Assume further that*

$$V(\cdot) \in L^{N/2}_{loc}(\mathbb{R}^N),$$

$$V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0,$$

$$V_\infty := \lim_{|x| \to \infty} V(x) > 0,$$

while the following "slow-decay" condition at infinity holds

$$\exists\ \delta \in (0, \sqrt{V_\infty}) : \quad \lim_{|x| \to \infty} (V(x) - V_\infty)\, e^{\delta |x|} = +\infty.$$

Then, there exists a positive constant $\Lambda = \Lambda(N, V_0, V_\infty, \delta)$ *such that if*

$$\sup_{y \in \mathbb{R}^N} ||V(x) - V_\infty||_{L^{N/2}(B_1(y))} < \Lambda,$$

Eq. (11) *has infinitely many (positive) solutions in* $H^1(\mathbb{R}^N)$.

Remark 16. For several other results obtained in the subcritical case, with $V(\cdot)$ satisfying condition (12), we refer to [17, 33, 41, 48, 123]. On the other hand, for results concerning the singularly perturbed equation

$$-\varepsilon^2\Delta u+V(x)u=|u|^{q-2}u \quad x\in\mathbb{R}^N, \quad N\geqslant 3, \tag{13}$$

where $\varepsilon>0$ is a small parameter, the interested reader may consult [6, 44, 45, 46, 47, 94], as well as the monograph [8]. Let us note here that all these works postulate the validity of (12) as well. The case in which $V(\cdot)$ changes sign but

$$\textit{meas}\left\{x\in\mathbb{R}^N : V(x)<V_0\right\}<\infty \qquad \text{for some } V_0>0, \tag{14}$$

is treated in [54]. The so-called critical frequency case; i.e. when

$$\liminf_{|x|\to\infty} V(x)>\inf_{\mathbb{R}^N} V=0,$$

is studied in [29, 30, 31]. Finally, the case where $V(\cdot)\geqslant 0$, $V(\cdot)\not\equiv 0$, but

$$\liminf_{|x|\to\infty} V(x)=0,$$

while the right-hand side of (13) has the "weighted" form $K(x)|u|^{q-2}u$, is investigated in [7, 9, 14, 24, 25, 87, 112, 128].

By contrast, the following non-existence result can be shown in the critical case.

Theorem 17. [99, 133]. *Let $q=2^*$. If $V(\cdot)\in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is such that $(|x|^2V(x))'$ maintains non-zero constant sign (where $'$ denotes derivative in the direction of the radial unit vector $x/|x|$) then the only solution in $H^1(\mathbb{R}^N)$ of* Eq. (11) *is $u\equiv 0$.*

In particular, we have the following.

Corollary 18. *Let $q=2^*$. If $V(x)\equiv V_0\neq 0$ (const.) then the only solution in $H^1(\mathbb{R}^N)$ of* Eq. (11) *is $u\equiv 0$.*

Remark 19. Theorem 17 should be contrasted with Theorem 6.

On the other hand, the following very intriguing existence result due to Benci and Cerami holds.

Theorem 20. [19]. *Let $q=2^*$. Suppose also that $V(\cdot)\geqslant 0$ but is bounded away from zero in the vicinity of a point, while there exist $s_1<N/2$ and $s_2>N/2$ (with $s_2<3$ if $N=3$) such that*

$$V(\cdot)\in L^s(\mathbb{R}^N) \quad \textit{for all} \quad s\in[s_1,s_2],$$

and

$$||V||_{L^{N/2}}<\mathcal{S}(2^{2/N}-1),$$

where $\mathcal{S}$ is the best constant in Sobolev's inequality; cf. (5) with $p=2$. Then Eq. (11) *has at least one (positive) solution in $\mathscr{D}^{1,2}(\mathbb{R}^N)$.*

Remark 21. Actually, Theorem 20 is the earliest and at the same time the only general existence result available to date in the critical semilinear case. Yet, it excludes potentials which obey the, weaker than (12), condition

$$V(x)\geqslant V_0>0 \qquad \text{for } |x| \text{ large.} \tag{15}$$

In fact, we have the following long-standing and elusive problem.

Open Problem 2

Prove the existence of a non-trivial solution to Eq. (11) *if* $q = 2^*$ *while* $V(\cdot)$ *satisfies* (15).

It appears as a scandal that such a result is not yet available in the general situation (i.e. without imposing further conditions on $V(\cdot)$). An important reason which partially accounts for this lacking is that a direct variational approach based on the concentration-compactness method cannot be applied in the critical case with $V(\cdot) \geqslant 0$, $V(\cdot) \not\equiv 0$, since the corresponding mountain-pass level is not, as it can be shown, a critical value.

Nonetheless, in view of Theorem 17, we have the following very remarkable result of Chen, Wei and Yan that was obtained 22 years (!) after Benci and Cerami [19] and which, at the expense of postulating *radial symmetry* for $V(\cdot)$, gives via a finite-dimensional reduction technique, a rather surprising (due to the *symmetry-breaking*) answer.

Theorem 22. [39]. *Let $q = 2^*$ and $N \geqslant 5$. If $V(\cdot) \in C^1(\mathbb{R}^N)$ is non-negative and radially symmetric while the function $|x|^2 V(|x|)$ has an isolated positive local maximum (or minimum) at $x_0 \neq 0$ with $V(|x_0|) > 0$ then Eq.* (11) *admits infinitely many non-radial* (*positive*) *solutions in $H^1(\mathbb{R}^N)$.*

Remark 23. Theorem 22 (which, in particular, does not require condition (15)) was very recently improved in [96] by imposing a weaker symmetry hypothesis on $V(\cdot)$. Moreover, it was extended to dimension $N = 4$ in [121]. It is worth pointing out here that the critical case with a radially symmetric $V(\cdot)$ satisfying (12) while the function $|x|^2 V(|x|)$ has a non-degenerate critical point, was studied earlier in [122]; albeit only for $N = 3, 4$ or 5.

In view of Theorems 20 and 22, a more general problem than Open Problem 2 is the following.

Open Problem 3

Prove the existence of a non-trivial solution to Eq. (11) *when* $q = 2^*$ *while the potential* $V(\cdot)$ *does not conform with any kind of integrability or symmetry assumptions. Furthermore, study the issue of multiplicity of solutions in that case.*

Turning now to the supercritical case, very few results are available up to the time of this writing. As a matter of fact, the nearly critical case where $q = 2^* + \delta$ with $\delta > 0$ small was investigated via a Lyapunov-Schmidt reduction in [84, 91] assuming that $N \geqslant 7$, $V(\cdot) \in L^\infty(\mathbb{R}^N) \cap L^{N/2}(\mathbb{R}^N)$ while $||V||_{L^{N/2}}$ is less than a certain constant. The only other result recorded in the literature (which, in fact, does not require q to be close to 2^*) is the following.

Theorem 24. [43]. *If $N \geqslant 4$, $q > \frac{2(N-1)}{N-3} > 2^*$ and*

$$0 \leqslant V(x) = o(|x|^{-2}), \quad \textit{as } |x| \to \infty,$$

then Eq. (11) *admits a continuum of (positive) solutions* u_λ *such that*

$$\lim_{\lambda\to 0} u_\lambda(x) = 0,$$

uniformly in $\mathbb{R}^N$. *Furthermore, if*

$$0 \leqslant V(x) = O(|x|^{-\alpha}), \quad \text{with} \ \ \alpha > N \geqslant 3,$$

then the same conclusion holds for any $q > 2^*$.

Remark 25. Theorem 24 is established by a linearization and a perturbation technique in suitable function spaces. Actually, by replacing u with $\lambda^{\frac{2}{q-2}}u(\lambda x)$, where λ is a positive parameter, and exploiting the decay properties of $V(\cdot)$, it is shown that solutions of (11) lie close to the radial solution $U_{\lambda,0}(x)$ of (7) for any small $\lambda > 0$; cf. Remark 5. In particular, the central role in the analysis is played by the invertibility properties of the linear operator $\Delta + qU_{\lambda,0}^{q-1}$.

Remark 26. It is worth noting that the existence of radially symmetric decaying solutions of Eq. (11) when $q > 2^*$ and $V(\cdot)$ is the singular Hardy potential $V(x) = c/|x|^2$, $c \in \mathbb{R}$, was very recently studied in [88].

On account of [84, 91] and Theorem 24, we are thus immediately led to the following problem.

Open Problem 4

Prove the existence of a non-trivial solution to Eq. (11) *when* $q > 2^*$ and $V(x)$ *does not decay to zero as* $|x| \to \infty$ ([7]). *Under which conditions do there exist infinitely many solutions in that case?*

4.2. **Quasilinear case** $(1 < p < N)$.

$$\left\{ \begin{array}{c} -\Delta_p u + V(x)|u|^{p-2}u = |u|^{q-2}u \\ \\ u \geqslant 0 \end{array} \right\}, \quad x \in \mathbb{R}^N, \quad 1 < p < N. \tag{16}$$

By contrast to the semilinear case, a systematic development of a theory encompassing Eq. (16) with $p \neq 2$ is still a desideratum. To some extent, this can be attributed to the fact that a principal obstacle which makes the analytical treatment of the quasilinear case quite different from the semilinear one, is the lack of the Hilbertian structure of the "natural" solution space $W^{1,p}_{loc}(\mathbb{R}^N)$ that arises when $V(\cdot) \in L^\infty_{loc}(\mathbb{R}^N)$. For instance, perturbative or finite-dimensional reduction techniques, which were proven very effective when $p = 2$ (e.g. Lyapunov-Schmidt-type reduction; cf. [6, 8, 9, 39]), are here no longer applicable since they cannot "decompose" the nonlinear Δ_p

[7]) Note that if $q \geqslant 2^*$ and $V(x) \equiv V_0 > 0$ (const.) then (11) does not admit a non-trivial solution in $H^1(\mathbb{R}^N)$; cf. [20].

operator to *invariant* subspaces ([8]). Furthermore, as one would expect, several other complicating factors also emerge; e.g. the understanding of the spectrum of the p-Laplacian when $p \neq 2$ is as yet in an embryonic stage; cf. [71]. Consequently, very few results are known thus far. As a matter of fact, to the best of the author's knowledge, [66, 126] are the only papers dealing with the subcritical case wherein, under the validity of (12) and some other technical hypotheses, the existence of a solution is established by variational methods. On the other hand, it can be readily checked that the proof of Theorem 14 covers, mutatis mutandis, Eq. (16) as well. For the case where $V(\cdot) \geqslant 0$, $V(\cdot) \not\equiv 0$, but

$$\liminf_{|x|\to\infty} V(x) = 0,$$

while the right-hand side of (16) has the "weighted" form $K(x)|u|^{q-2}u$, the interested reader is referred to [38, 75, 76].

On account of the above discussion, it becomes therefore very interesting, as well as very challenging, to find out which are the proper extensions of the results obtained in [6, 8, 29, 30, 31, 33, 34, 39, 41, 48, 54, 96, 123] to the quasilinear situation. Even though we are still very far from this goal, it is likely that a great variety of new techniques will be uncovered by a thorough investigation. Actually, the success already achieved in the semilinear case fully justifies a strong interest in that research direction.

Towards that end, it would be very interesting to answer, in particular, the following problem (cf. Theorem 15 for the semilinear case).

Open Problem 5

Let $p < q < p^*$ with $p \neq 2$. Find conditions on $V(\cdot)$ which ensure the existence of multiple solutions to Eq. (16)*.*

With regard now to the critical case, we only have the following generalization of Theorem 20, due to Alves.

Theorem 27. [2]. *Let $q = p^*$ with $2 \leqslant p < N$. Suppose also that $V(\cdot) \geqslant 0$ but is bounded away from zero in the vicinity of a point, while*

$$V(\cdot) \in L^s(\mathbb{R}^N) \quad \text{for all} \quad s \in [s_1, s_2],$$

where $1 < s_1 < N/p < s_2$ (with $s_2 < N(p-1)/(p^2-N)$ if $N < p^2$) and

$$||V||_{L^{N/p}} < \mathcal{S}(2^{p/N} - 1),$$

where $\mathcal{S} := \mathcal{S}(p, N)$ is the best constant in Sobolev's inequality; cf. (5). Then Eq. (16) *has at least a (positive) solution in $\mathscr{D}^{1,p}(\mathbb{R}^N)$.*

On the other hand, the following analogue of Theorem 17 holds.

[8]) For its interest, the "linearization" of the weak form of Δ_p at u is formally given by the operator

$$\mathcal{L}_u(\varphi)[\psi] := \int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla\varphi \cdot \nabla\psi)dx + (p-2)\int_{\mathbb{R}^N} |\nabla u|^{p-4}(\nabla u \cdot \nabla\varphi)(\nabla u \cdot \nabla\psi)dx \quad \forall \varphi, \psi \in C_0^1(\mathbb{R}^N).$$

Theorem 28. [133]. *Let $q = p^*$. If $V(\cdot) \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is such that $(|x|^p V(x))'$ maintains non-zero constant sign in $\mathbb{R}^N$ (where $'$ denotes derivative in the direction of the radial unit vector $x/|x|$) then the only C^1-weak solution of Eq. (16) is $u \equiv 0$.*

Hence, as in the semilinear case, the following problem naturally arises.

Open Problem 6

Prove the existence of a non-trivial solution to Eq. (16) when $q = p^*$ while $V(\cdot)$ does not comply with any integrability assumptions. In particular, prove the existence of a non-trivial solution when (14) or (15) holds.

4.2.1. **The critical quasilinear case under the effect of a subcritical perturbation.**

In an effort to shed some light on the intricacies associated with Open Problem 6, it seems natural to consider the following "perturbed" critical quasilinear elliptic equation

$$(\spadesuit) \qquad -\Delta_p u + V(x)|u|^{p-2}u = |u|^{p^*-2}u + K(x)|u|^{q-2}u, \qquad x \in \mathbb{R}^N,$$

where

Σ_0: $p < q < p^*$ (i.e. the last term in ($\spadesuit$) has subcritical growth in u).

Perturbations of this type were first considered by Brézis and Nirenberg in their pioneering study [27] of the semilinear elliptic problem

$$\begin{aligned} -\Delta u &= |u|^{2^*-2}u + f(x,u), && x \in \Omega, \\ u &> 0, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}$$

where

$$\lim_{u\to\infty} \frac{f(x,u)}{|u|^{2^*-1}} = 0,$$

and Ω is a bounded domain in $\mathbb{R}^N$. Their celebrated paper became the incentive for a great number of related investigations afterwards. As a matter of fact, Eq. ($\spadesuit$) has been considered already by several researchers motivated by [27]; cf. [3, 4, 5, 53, 86, 130] when $p = 2$ and [55, 58, 127, 129, 131, 133] when $1 < p < N$. However, a thorough discussion of the results obtained therein lies beyond the scope of the present article and the interested reader should consult the above references. It is worth remarking, though, that all these works share a common feature: they exclude the physically very important class of potentials $V(\cdot)$ which may decay to zero at infinity since the imposed assumptions conform either with (15) or its relaxed variant (14).

On the other hand, by postulating the more general structural hypotheses:

Σ_1: $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and non-negative. Furthermore, there exist $\alpha \geqslant 0$, $\Lambda > 0$ and $r_0 > 0$ such that

$$\inf_{|x|\geqslant r_0} |x|^\alpha V(x) \geqslant \Lambda,$$

Σ_2: $K:\mathbb{R}^N \to \mathbb{R}$ is continuous, non-negative and bounded. Moreover, there exists $\rho_0 > 0$ such that

$$\kappa_0 := \min_{|x|\leqslant \rho_0} K(x) > 0,$$

the following result can be proved.

Theorem 29. [77]. *Let the conditions* (Σ_0)-(Σ_2) *hold with* $\rho_0 \geqslant r_0$. *Assume further that for some* $\lambda > 0$ *and* $\rho_1 \geqslant \rho_0$,

$$K(x) \leqslant \lambda V(x) \qquad \text{if} \quad |x| \geqslant \rho_1.$$

Then, for any fixed $\theta \in (p,q)$ *and* $\tau \in (0, r_0)$ *there exists a constant* $\gamma = \gamma(N,p,q,\theta,\lambda,\tau,V_\tau,K) > 0$, *where* $V_\tau := ||V||_{L^\infty(|x|\leqslant\tau)}$, *such that* Eq. (♠) *admits a positive weak solution*

$$u \in E := \left\{ v \in \mathscr{D}^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|v|^p dx < \infty \right\},$$

for any $\Lambda \geqslant \gamma {\rho_0}^{\alpha}$, *provided*

$$0 \leqslant \alpha \leqslant \frac{(q-p)(N-p)}{p-1},$$

and any one of the following conditions is satisfied:

i) $N \geqslant p^2$

ii) $p < N < p^2$ *and* $p^* - p/(p-1) < q$

iii) $p < N < p^2$, $p^* - p/(p-1) \geqslant q$ *and* κ_0 *is sufficiently large.*

In particular, $u(x) = O(|x|^{-(N-p)/(p-1)})$, *as* $|x| \to \infty$, *and thus if* $p^2 < N$ *then* $u \in W^{1,p}(\mathbb{R}^N)$; *i.e.* u *is a bound state for* (♠).

With regard to Theorem 29, the following important remarks should be pointed out:

i) In comparison with the results obtained in [2, 19] (cf. Theorems 20 and 27), it guarantees the existence of a positive weak solution to Eq. (♠) also for potentials $V(\cdot)$ which do not belong to $L^{N/p}(\mathbb{R}^N)$. For instance, such a case arises when

$$0 \leqslant \alpha \leqslant \min\left\{ p, \frac{(q-p)(N-p)}{p-1} \right\}.$$

It is worth emphasizing here that it is still not known whether the $L^{N/2}$- integrability of $V(\cdot)$, which was postulated in [19], is actually essential for establishing, in general, existence of solutions to Eq. (11) when $q = 2^*$.

ii) It allows for potentials $V(\cdot)$ which vanish at infinity with a *fast* decay-rate (i.e. with $\alpha \geqslant p$) whenever $p_* \leqslant q < p^*$ where $p_* := p(N-1)/(N-p)$ is the so-called *Serrin's critical exponent*. Actually, this should be contrasted with a non-existence result of Zou [133] obtained via a sophisticated Gidas-Spruck-type integral inequality [108] and a Pohozaev-type identity [98, 99], according to which Eq. (♠)

does not admit any non-trivial non-negative C^1-solutions if $V(\cdot)$ is such that

$$(|x|^p V(x))' \leqslant 0 \quad \text{in } \mathbb{R}^N,$$
$$V(x) = O(|x|^{-p}), \ \nabla V(x) = O(|x|^{-p-1}), \quad \text{as } |x| \to \infty,$$

while $K(x) \equiv K_0 > 0$ and $p_* < q < p^*$ (as before, $'$ denotes derivative in the direction of the radial unit vector $x/|x|$).

iii) It holds also for potentials $V(\cdot)$ obeying condition (15).

To close our discussion, it remains to consider the supercritical quasilinear case. At the present time, however, the situation here (if $p \neq 2$) is *terra incognita*. Therefore, we confine ourselves to plainly stating

Open Problem 7

Prove the existence of a non-trivial solution to Eq. (16) when $q > p^*$ and *$p \neq 2$. In particular, prove the existence of a non-trivial solution when $V(x)$ decays to zero as $|x| \to \infty$ or is radially symmetric.*

Epilogue

On January 10, 2016 the eminent Tunisian mathematician Abbas Bahri passed away at the age of 61, after four years of courageous battle against cancer.

He was, for more than three decades, cruising at the highest altitude in the mathematical sky of "understanding non-compact phenomena".

Thanks to his unique talent, he was able to infuse fresh and extremely innovative perspectives into nonlinear PDEs by a fascinating combination of Analysis and Algebraic Topology.

For his truly outstanding achievements he was awarded in 1989 the first *Fermat prize* together with the *Langevin prize* of the *French Academy of Sciences*.

Researchers will certainly be kept busy for many decades to come on the legacy he left to the mathematical community.

It almost goes without saying that, had he lived, he could offer many seminal ideas and strategies on how to tackle *all* the open problems discussed earlier in the article.

Abbas Bahri

1 January 1955 – 10 January 2016

Professor at Rutgers University (1987 – 2016)
Maître de Conférences at the École Polytechnique (1984 – 1992)

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ASPECTS OF HARMONIC ANALYSIS ON LOCALLY SYMMETRIC SPACES

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1. Introduction

Let G be a real *semisimple* Lie group, connected, noncompact, with finite center and K be a maximal compact subgroup of G. We denote by X the Riemannian symmetric space G/K. Recall that the real hyperbolic space $\mathbb{H}^{n+1} = SO(n+1,1)/SO(n+1)$ is a symmetric space.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. The Killing form is defined by

$$B(Y,Z) = \mathrm{Tr}\left(ad_Y \circ ad_Z\right), \quad Y, Z \in \mathfrak{g},$$

where

$$ad_Y(Z) = [Y,Z]$$

is the Lie bracket. Let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that the Killing form, by the assumption of semisiplicity, is non-degenerate on $\mathfrak{p}$ and therefore it defines a (canonical) Riemannian metric on X. For example, in the upper-half space model of $\mathbb{H}^{n+1} = \{x \in \mathbb{R}^n, y > 0\}$ the hyperbolic metric is given by

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right).$$

Note also that $\mathfrak{p}$ is isomorphic to $T_0(X)$, the tangent space of X at the origin.

Denote by $d(x,y)$ and by dx the Riemannian distance and measure on X and bear in mind that X has exponetial volume growth:

$$|B(x,r)| \leq ce^{c'(n-1)r}, \quad n = \dim X.$$

This is a sacrée difference with the case of $\mathbb{R}^n$.

1.1. Some "classical" problems of (Harmonic) Analysis on non-compact symmetric spaces.

1.1.1. *Problem (H): estimates of the heat kernel.* Denote by Δ_X the Laplace-Beltrami operator on X and consider the heat operator $e^{-t\Delta_X}, t > 0$, and denote by $p_t^X(x,y)$ its kernel, i.e. the fundamental solution of the parabolic equation

$$\partial_t p_t^X = \Delta_X p_t^X, \quad p_t^X(x,y) \underset{t\to 0}{\to} \delta_y(x).$$

In [9] Davies and Mandouvalos obtained the following precise estimates of $p_t^{\mathbb{H}^{n+1}}(x,y)$:

$$p_t^{\mathbb{H}^{n+1}}(x,y) \sim t^{-\frac{n+1}{2}}(1+r)(1+t+r)^{\frac{n-2}{2}} e^{-\frac{n^2}{4t}-\frac{nr}{2}-\frac{r^2}{4t}},$$

where $r = d(x,y)$.

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Eleven years later, Anker and Ji [3] generalized the above estimate in the case of non-compact symmetric spaces. Its expression is fairly complicated to be presented here.

In any case, the above estimates play a very important role in the treatment of some very classical problems of Harmonic Analysis on symmetric and locally symmetric spaces. For example,

1.1.2. *Problem (R): L^p-boundedness of the Riesz transform.* Recall that the Riesz transform R_X on the symmetric space X is given by

$$R_X = \nabla \Delta_X^{-1/2} = c \int_0^\infty \nabla e^{-t\Delta_X} \frac{dt}{\sqrt{t}}.$$

For example, in [2], Anker, using the estimates of the heat kernel p_t^X, proves that R_X is bounded from L^1 to L^1_w.

1.1.3. *Problem (M): L^p-boundedness of convolution operators (multipliers).* To state the problem we need to introduce some notation. Let $\mathfrak{a}$ be an abelian maximal subspace of $\mathfrak{p}$, $\mathfrak{a}^*$ its dual. We say that $\alpha \in \mathfrak{a}^*$ is a root of the pair $(\mathfrak{g}, \mathfrak{a})$ if the root space

$$g_\alpha := \{Y \in \mathfrak{g} : [Y, H] = \alpha(H) Y, \ \forall H \in \mathfrak{a}\}$$

is non-trivial.

Recall that the *Weyl group* W is the finite group of reflections about the hyperplanes perpendicular to the roots.

For $1 \leq p < \infty$, denote by $\mathcal{CO}_p(X)$ the Banach algebra of bounded linear operators T on $L^p(X)$, which are translation invariant under G. Then, [8, 1], $\mathcal{CO}_2(X)$ is isomorphic to the algebra $L^\infty(\mathfrak{a}^*)^W$ of all W-invariant bounded measurable functions on $\mathfrak{a}^*$. The isomorphism $T \leftrightarrow m$ is given by

$$\mathcal{H}(Tf)(\lambda) = m(\lambda)\mathcal{H}f(\lambda), \quad f \in L^2(X), \ \lambda \in \mathfrak{a}^*, \tag{1}$$

where $\mathcal{H}f$ is the spherical Fourier transform of f:

$$\mathcal{H}f(\lambda) = \int_G f(x)\varphi_\lambda(x)\,dx, \quad \lambda \in \mathfrak{a}^*, \quad f \in S(K\backslash G/K),$$

where $\varphi_\lambda(x)$ are the elementary spherical functions, the analogue of the imaginary exponentials $e^{i\langle\lambda,x\rangle}$.

We denote by T_m the operators associated to m by (1). Note that (1) is equivalent to

$$T_m f(x) = (f * \kappa)(x) = \int_G f(g)\kappa\left(g^{-1}x\right)dg, \quad x \in G, \ f \in C_0^\infty(X),$$

where

$$\kappa(x) = \left(\mathcal{H}^{-1}m\right)(x) = c\int_{\mathfrak{a}^*} m(\lambda)\varphi_{-\lambda}(x)\frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G, \tag{2}$$

is the *inverse spherical Fourier transform* of m in the sense of tempered distributions. $\mathbf{c}(\lambda)$ is the Harish-Chandra function.

Further, $\mathcal{CO}_p(X) \subset \mathcal{CO}_2(X)$, $1 \leq p < \infty$, and so every $T_m \in \mathcal{CO}_p(X)$ is of the form (1), i.e. a Fourier multiplier. The corresponding functions $m : \mathfrak{a}^* \longrightarrow \mathbb{C}$, are called Fourier multipliers of $L^p(X)$ and they are denoted by $\mathcal{M}_p$.

Multiplier's Problem: Find the optimal assumptions on a bounded and W-invariant function $m : \mathfrak{a}^* \longrightarrow \mathbb{C}$ that insure that m is an $\mathcal{M}_p$ multiplier for some $p \geq 1$.

Let us recall that in case of $\mathbb{R}^n$, the *Hörmander-Mikhlin multiplier theorem*, [16, 27], asserts that if κ is a tempered distribution, then the convolution operator $Tf = f * \kappa$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, provided that the Fourier transform m of κ satisfies the symbol estimates

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |\partial^\alpha m(\xi)| < +\infty,$$

for all multi-indices α with $|\alpha| \leq [n/2] + 1$, where $[t]$ is the integer part of $t \in \mathbb{R}$.

In the present case of symmetric spaces we need something more and this is due to the exponential volume growth. Let us proceed. Let ρ be the half-sum of positive roots counted with their multiplicity. ρ is an important geometric invariant of X. For example, the bottom of the spectrum of $-\Delta_X$ is equal to $-\|\rho\|^2$.

Denote by C_ρ the convex hull in $\mathfrak{a}^*$ generated by $w.\rho$, $w \in W$. Then, Clerc and Stein in their pioneer work, [8], observed that in the case of symmetric spaces of noncompact type, every $m \in \mathcal{M}_p$ extends to a W-invariant bounded holomorphic function inside the tube $\mathcal{T}^v = \mathfrak{a}^* + ivC_\rho$, where $v = |(2/p) - 1|$, $p \in (1, \infty)$.

Anker's class of multipliers: We say that

$$m \in \mathcal{M}(v, N), \quad v \in \mathbb{R}_+, \quad N \in \mathbb{N},$$

if

(i) m is analytic inside the tube $\mathcal{T}^v$ and

(ii) for all multi-indices α, with $|\alpha| \leq N$, $\partial^\alpha m(\lambda)$ extends continuously to the whole of $\mathcal{T}^v$ with

$$\left(1 + |\lambda|^2\right)^{|\alpha|/2} |\partial^\alpha m(\lambda)| < \infty, \quad \lambda \in \mathcal{T}^v. \tag{3}$$

In [1], Anker proved that if $m \in \mathcal{M}(v, N)$ with $v = |(2/p) - 1|$, $p \in (1, \infty)$ and $N = [v \dim X] + 1$, then T_m is bounded on $L^p(X)$.

Note that if $m \in \mathcal{M}_p$, then as it is mentioned above, by [8], m is a holomorphic function inside the tube $\mathcal{T}^v$. Thus, Anker obtained the optimal width of the tube $\mathcal{T}^v$ of analyticity.

1.1.4. *Problem (S): Dispersive estimates of the Schrödinger operator and applications.* Let M be a Riemannian manifold and denote by Δ its Laplace-Beltrami operator. The nonlinear Schrödinger equation (NLS) on M

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = F(u(t,x)), \\ u(0,x) = f(x), \end{cases} \tag{4}$$

has been extensively studied the last thirty years. Its study relies on precise estimates of the kernel s_t of the Schrödinger operator $e^{it\Delta}$, *the heat kernel of pure imaginary time*. The estimates of s_t allow us to obtain *dispersive estimates* of the operator $e^{it\Delta}$ of the form

$$\left\| e^{it\Delta} \right\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c\psi(t), \quad t \in \mathbb{R}, \tag{5}$$

for all $q, \widetilde{q} \in (2, \infty]$, where ψ is a positive function and $\widetilde{q}'$ is the conjugate of $\widetilde{q}$.

Dispersive estimates of $e^{it\Delta}$ as above, allow us to obtain *Strichartz estimates* of the solutions $u(t,x)$ of (4) of the form

$$\|u\|_{L^p(\mathbb{R};L^q(M))} \leq c\left\{\|f\|_{L^2(M)} + \|F\|_{L^{\widetilde{p}'}\left(\mathbb{R};L^{\widetilde{q}'}(M)\right)}\right\}, \tag{6}$$

for all pairs $\left(\frac{1}{p}, \frac{1}{q}\right)$ and $\left(\frac{1}{\widetilde{p}}, \frac{1}{\widetilde{q}}\right)$ which lie in a certain interval or triangle.

Strichartz estimates have applications to *well-posedness* and *scattering* theory for the NLS equation.

In the case of $\mathbb{R}^n$, the first such estimate was obtained by Strichartz himself [31] in a special case. Then, Ginibre and Velo [12] obtained the complete range of estimates except the case of endpoints which were proved by Keel and Tao [19].

In view of the important applications to nonlinear problems, many attempts have been made to study the dispersive properties for the corresponding equations on various Riemannian manifolds. In particular, dispersive and Strichartz estimates for the Schrödinger equation on real hyperbolic spaces have been stated by Banica, Pierfelice, Anker, Ionescu and Staffilani, [6, 7, 29, 30, 4, 17]. In a recent paper Anker, Pierfelice and Vallarino [5] treat NLS in the context of Damek-Ricci spaces, which include all *rank one symmetric spaces of noncompact type.*

2. Locally symmetric spaces

We shall now present some results related to the above problems we obtained in the context of locally symmetric spaces. They appeared in a series of papers [24, 23, 10, 25, 22, 11] written in collaboration with **Noël Lohoué**, **Nikolaos Mandouvalos** and **Anestis Fotiadis**.

Let Γ be a *discrete, torsion free subgroup* of G and let us denote by M the *locally symmetric space* $\Gamma \backslash X = \Gamma \backslash G/K$. The quotient M equipped with the projection of the canonical Riemannian structure of X, becomes a Riemannian manifold.

It is important to note that in general, locally symmetric spaces do not have bounded geometry, since the injectivity radius of M is not in general strictly positive, [9], (we have the presence of cusps).

Let us now present what has been done for the solution of the 4 problems we mentioned above in the context of locally symmetric spaces.

2.1. Problem (H): estimates of the heat kernel.

Recall that by Weyl's formula, the heat kernel $p_t^M(x,y)$ on M is given by the series

$$p_t^M(x,y) = \sum_{\gamma \in \Gamma} p_t^X(x, \gamma y).$$

In [9] Davies and Mandouvalos obtain upper estimates for the heat kernel on Kleinian groups $\Gamma \backslash \mathbb{H}^{n+1}$. For that, recall that the *counting function* $N_\Gamma(x,y,R)$ is given by

$$N_\Gamma(x,y,R) = \#\{\gamma \in \Gamma : d(x, \gamma y) \leq R\}, \ x, y \in X, R > 0,$$

where $\#(A)$ is the cardinal of the set A. If the exponent of convergence

$$\delta(\Gamma) = \limsup_{R\to\infty} \frac{\log N_\Gamma(x,y,R)}{R},$$

satisfies $\delta(\Gamma) < \alpha < n/2$, then, [9],

$$p_t^M(x,y) \le c(n)\, t^{-(n+1)/2}(1+t)^{(n/2)-1} e^{-n^2t/4-d(x,y)^2/4t} P_\alpha(x,y),$$

where

$$P_\alpha(x,y) = \sum_{\gamma\in\Gamma} e^{-\alpha d(x,\gamma y)}$$

are the Poincaré series, which converges for $\alpha > \delta(\Gamma)$.

Similar estimates of $p_t^M(x,y)$ are obtained by Weber [33] in the general case of locally symmetric, but in both cases, the estimates of the Poincaré series are not really good enough. Note that estimates of the Poincaré series are obtained by using the estimates of the counting function $N_\Gamma(x,y,R)$. Note also, that optimal uniform estimates of N_Γ are available only in the *rank one case* and for some classes of Γ, as well as for quotients of *Cartan-Hadamard manifolds* and $CAT(-1)$ *spaces*. As we shall see below this is due to the fact that to obtain the optimal estimates of the counting function we make use of the Patterson-Sullivan measures, [28, 32].

2.2. **Problem (R): L^p boundedness of the Riesz transform.** Recall that the Riesz transform R_M on the locally symmetric space $M = \Gamma\backslash X$ is given by

$$R_M = \nabla\Delta_M^{-1/2} = c\int_0^\infty \nabla e^{-t\Delta_M}\frac{dt}{\sqrt{t}}. \tag{7}$$

It is clear from (7) that the proof of the L^p-boundedness of the Riesz transform depends heavily on the heat kernel of M. As we shall see it depends also on the L^2-spectrum of Δ_M. Let us explain the situation:

We first recall that the L^2-spectrum of on a non-compact locally symmetric space is in general unknown. In the present work we shall assume that it is equal to

$$\{\lambda_0,...,\lambda_m\}\cup[2,\infty), \tag{8}$$

where the eigenvalues $\lambda_0,...,\lambda_m$ are of finite multiplicity. This is the case if M is the quotient of the hyperbolic space $\mathbb{H}^{n+1}$ by a geometrically finite Kleinian group Γ, i.e. when $M = \Gamma\backslash\mathbb{H}^{n+1}$, [20]. Note that in this case $\rho = n/2$.

Theorem 1. *(i) If the discrete L^2-spectrum is empty, then R_M is bounded on $L^p(M)$ for all $p\in(1,\infty)$.*

(ii) If the discrete L^2-spectrum is non-empty and $\lambda_0 = 0$, then R_M is bounded on $L^p(M)$ for all p in an interval (r_1,r_2) around 2.

(iii) If the discrete L^2-spectrum is non-empty and $\lambda_0 \neq 0$, then R_M is bounded on $L^p(M)$ for all $p\in(r_1,r_2)$ and for all $f\in C_0^\infty(M)$ such that

$$\int_M u_0^j(x) f(x)\,dx = 0,$$

where u_0^j, $j\le k_0$ are the eigenfunctions associated to $\lambda_0\neq 0$.

Remark 2. *The results above have been improved by Ji, Kunstmann and Weber in* [18].

2.2.1. *Problem (M): L^p boundedness of convolution operators (multipliers).* Let $m \in L^\infty(\mathfrak{a}^*)^W$, and let us denote by κ its inverse spherical transform. Consider the convolution operator

$$T_m u(x) = \int_G u(\overline{g})\, \kappa\left(g^{-1}x\right) dg, \quad x \in G,\ u \in C_0^\infty(M),$$

where $\overline{g} = \{\gamma g k : \gamma \in \Gamma, k \in K\}$ is the class of $g \in G$ in M. Note that to show that T_m is a well defined operator on $C_0^\infty(M)$, we have to verify that if $u \in C_0^\infty(M)$, then $T_m u$ is left Γ-invariant and right K-invariant. This follows directly from the fact that u is left Γ-invariant and κ is K-bi-invariant.

Multiplier's Problem: *Find the minimal conditions on the multiplier $m \in L^\infty(\mathfrak{a}^*)^W$, in order to have that T_m is bounded on $L^p(M)$, for some individual $p \in (1,\infty)$.*

The strategy of the proof of the multiplier theorem on M: As usual, we perform the splitting of the kernel $\kappa = \mathcal{H}^{-1}m$:

$$\kappa = \kappa^0 + \kappa^\infty,$$

with $\operatorname{supp}\left(\kappa^0\right) \subset B(0,2)$ and $\operatorname{supp}(\kappa^\infty) \subset B(0,1)^c$. Denote by T_m^0 and by T_m^∞ the corresponding operators.

The **continuity of** T_m^0 for all $p \in (1,\infty)$, is proved in [23]. This is carried out by observing that T_m^0 can be defined as an operator on the group G, and then, by the local result of [1], we conclude its boundedness on $L^p(G)$, $p \in (1,\infty)$. The continuity of T_m^0 on $L^p(M)$, follows by applying Herz's Theorem A, [15], (it is a hard piece of Functional Analysis).

The crucial step for the proof of the **boundedness of** T_m^∞, is to obtain an estimate of the norm $\|T_m^\infty\|_{p\to p}$. In the case of symmetric spaces, to prove the finiteness of $\|T_m^\infty\|_{p\to p}$, we make use of *Kunze and Stein phenomenon,* [1]. For the case of locally symmetric spaces, we proved an analogue of Kunze and Stein phenomenon in [22].

2.2.2. *Kunze and Stein phenomenon.* Let us recall that a central result in the theory of convolution operators on semisimple Lie groups is the Kunze-Stein phenomenon, which, in the case when κ is K-bi-invariant, states that if $p \geq 1$, then

$$\begin{aligned} || * |\kappa| \, ||_{L^p(G)\to L^p(G)} &= C \int_G |\kappa(g)| \, \varphi_{-i\rho_p}(g) dg \\ &= C \int_{\mathfrak{a}_+} |\kappa(\exp H)| \, \varphi_{-i\rho_p}(\exp H) \delta(H) dH, \end{aligned} \tag{9}$$

where $\rho_p = |2/p - 1|\rho$, $p \geq 1$.

In [22], we proved an analogue of this phenomenon for a class of locally symmetric spaces. More precisely, let λ_0 be the bottom of the L^2-spectrum of $-\Delta$ on M. We say that M *possesses property (KS)* if there exists a vector $\eta_\Gamma \in C_\rho \cap S\left(0, \left(\rho^2 - \lambda_0\right)^{1/2}\right)$, such that for all $p \in (1,\infty)$,

$$\|* |\kappa|\|_{L^p(M)\to L^p(M)} \leq \int_G |\kappa(g)| \, \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg, \tag{10}$$

where

$$s(p) = 2\min\left((1/p), (1/p')\right). \tag{11}$$

Note that M possesses property (KS) if it is contained in the following classes:

(i). Γ *is a lattice* i.e. $\mathrm{vol}(\Gamma\backslash G) < \infty$,

(ii). *G possesses Kazhdan's property (T).* Recall that G has property (T) iff G has no simple factors locally isomorphic to $SO(n,1)$ or $SU(n,1)$, [13, ch. 2]. In this case $\Gamma\backslash G/K$ possesses property (KS) for all discrete subgroups Γ of G. Recall that $H^n(\mathbb{H}) = Sp(n,1)/Sp(n)$ and $H^2(\mathbb{O}) = F_4^{-20}/Spin(9)$. So, $\Gamma\backslash H^n(\mathbb{H})$ and $\Gamma\backslash H^2(\mathbb{O})$ have property (KS) for *all* discrete subgroups Γ of $Sp(n,1)$ and F_4^{-20} respectively.

Thus, from cases (i) and (ii) we deduce that *all* locally symmetric spaces $\Gamma\backslash H^n(\mathbb{H})$ and $\Gamma\backslash H^2(\mathbb{O})$ have property (KS).

On the contrary, the isometry groups $SO(n,1)$ and $SU(n,1)$ of real and complex hyperbolic spaces do not have property (T) and consequently the quotients $\Gamma\backslash H^n(\mathbb{R})$ and $\Gamma\backslash H^n(\mathbb{C})$ of infinite volume do not in general belong in the class (ii). The class (iii) below covers also this case.

(iii) *$\Gamma\backslash G$ is non-amenable.* Note that since G is non-amenable, then $\Gamma\backslash G$ is non-amenable if Γ is amenable. So, if Γ is amenable, then the quotients $\Gamma\backslash H^n(\mathbb{R})$ and $\Gamma\backslash H^n(\mathbb{C})$ possesses property (KS) even if they have infinite volume.

For $p \in (1,\infty)$ we set

$$v_\Gamma(p) = 2\min\left((1/p), (1/p')\right)\frac{\|\eta_\Gamma\|}{\|\rho\|} + |(2/p) - 1|, \tag{12}$$

where p' is the conjugate of p.

If $n = \dim X$ and $a = \dim \mathfrak{a}$ is the rank of X, set $b = n - a$. Let b' be the smallest integer $\geq b/2$, and set

$$N = [a/2] + b' + 1.$$

Finally recall that the multiplier m belongs in the class $\mathcal{M}(v,N)$ if

(i) m is analytic inside the tube $\mathcal{T}^v = \mathfrak{a}^* + ivC_\rho$ and

(ii) for all multi-indices α, with $|\alpha| \leq N$,

$$\left(1 + |\lambda|^2\right)^{|\alpha|/2} |\partial^\alpha m(\lambda)| < \infty, \quad \lambda \in \mathcal{T}^v.$$

Theorem 3. *Assume that M satisfies property (KS). Let $v_\Gamma(p)$, $p \in (1,\infty)$ and N be as above. If $m \in \mathcal{M}(v,N)$, with $v > v_\Gamma(p)$, then the operator T_m is bounded on $L^p(M)$.*

Note that $N = [n/2] + 2$, if a is even and b odd and $N = [n/2] + 1$, otherwise. So, in the case when $N = [n/2] + 1$, the number of derivatives of the multiplier $m(\lambda)$ we need to control in Theorem 3, is the same as in the version of the Hörmander-Mikhlin theorem, mentioned in the Introduction .

It is important to note that if $\lambda_0 = \|\rho\|^2$, then the width v of the tube $\mathcal{T}^v$ of analyticity satisfies $v > |(1/p) - (1/2)|$. Note that in the case of symmetric spaces, by [8, 1], the optimal width is $|(1/p) - (1/2)|$.

Next, let us comment on the number N of the derivatives of the multiplier $m(\lambda)$ we need to control. Anker in [1, Theorem 1], obtained $N = [vn] + 1 = [|(1/p) - (1/2)|\, n] + 1$, which is smaller than $[a/2] + b' + 1$, obtained in Theorem 3 above. Note that this sharp result of [1, Theorem 1], is obtained by using a heavy machinery of function space theory, while the best that one can get by a direct kernel estimate, is the number N appearing in Theorem 3 above, (cf. [1, Proposition 7]).

3. Problem (S): Dispersive estimates of the Schrödinger operator and applications

Let M be a Riemannian manifold and denote by Δ its Laplace-Beltrami operator. The nonlinear Schrödinger equation (NLS) on M

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = F(u(t,x)), \\ u(0,x) = f(x), \end{cases}$$

has been extensively studied the last thirty years. In [11] we treat NLS equations on a class on *rank one* locally symmetric spaces.

3.1. **The class (S) of locally symmetric spaces.** We shall first describe the class (S) of rank one locally symmetric spaces on which we shall treat NLS equations.

Denote by s_t the fundamental solution of the Schrödinger equation on the symmetric space X:

$$-i\partial_t s_t(x,y) = \Delta_X s_t(x,y), \quad t \in \mathbb{R}, \quad x,y \in X.$$

Then s_t is a K-bi-invariant function and the Schrödinger operator $S_t = e^{it\Delta_X}$ on X is defined as a convolution operator:

$$S_t f(x) = \int_G f(y) s_t(y^{-1}x)dy = (f * s_t)(x), \quad f \in C_0^\infty(X). \tag{13}$$

Using that s_t is K-bi-invariant, we deduce that if $f \in C_0^\infty(M)$, then $S_t f$ is right K-invariant and left Γ-invariant, i.e. a function on the locally symmetric space M. Thus the Schrödinger operator $\widehat{S}_t$ on M is also defined by formula (13).

The *first ingredient* for the proof of the dispersive estimate (5) are precise estimates of the kernel s_t. In the context of rank one symmetric spaces they are obtained in [5].

The *second ingredient* is the analogue of Kunze-Stein phenomenon:

$$\| * |\kappa| \|_{L^p(M) \to L^p(M)} \leq \int_G |\kappa(g)|\, \varphi_{-i\eta_\Gamma}(g)^{s(p)}\, dg, \tag{14}$$

we have already presented in the previous section. Note that in the rank one case, the class of locally symmetric spaces where (14) is valid contains the following classes:

(i) the quotients $\Gamma\backslash H^n(\mathbb{H})$ and $\Gamma\backslash H^2(\mathbb{O})$ for *all* discrete subgroups Γ of $Sp(n,1)$ and F_4^{-20} respectively.

(ii) the infinite volume quotients $\Gamma\backslash H^n(\mathbb{R})$, $\Gamma\backslash H^n(\mathbb{C})$ with Γ amenable.

The *third ingredient* are norm estimates of the kernel $\widehat{s}_t$ of the Schrödinger kernel on M, which is given by

$$\widehat{s}_t(x,y) = \sum_{\gamma\in\Gamma} s_t(x,\gamma y). \tag{15}$$

One can prove that the series above converges when $\delta(\Gamma) < \rho$.

The *last ingredient,* are uniform asymptotics of the *counting function* N_Γ of Γ. The *asymptotic properties of the counting function* in various geometric contexts have been a subject of many investigations since Margulis [26]. In [32], they are obtained in the context of Hadamard manifolds with pinched negative sectional curvature and in [28] in the more general context of $CAT(-1)$ spaces. Note that rank one symmetric spaces have pinched negative sectional curvature and consequently they are contained in the above mentioned classes of spaces.

In [28, 32] it is proved, under some precise conditions on Γ (for example when Γ is *convex co-compact*) that N_Γ satisfies the following uniform asymptotics: there is a constant $C > 0$, such that for all $x, y \in X$,

$$\lim_{R\to\infty} \frac{N_\Gamma(x,y,R)}{e^{\delta(\Gamma)R}} = C. \tag{16}$$

It is important to say that for the proof of (16) we make use of the *Patterson-Sullivan densities* which, in the context of symmetric spaces, they are known only in the rank one case.

Definition 4. *We say that a rank one locally symmetric space $M = \Gamma\backslash G/K$ belongs in the class (S) if*

(i) for every K-bi-invariant function κ the estimate (14) is satisfied, (Kunze and Stein),

(ii) $\delta(\Gamma) < \rho$, and

(iii) the counting function $N_\Gamma(x,y,R)$ satisfies (16).

Note that if $\delta(\Gamma) < \rho$, then $\lambda_0 = \rho^2$, [21]. So, if $M \in (S)$, then the vector η_Γ appearing in (14) equals 0. Note also that if $\mathrm{vol}(M) < \infty$, i.e., if M is a lattice, then $\lambda_0 = 0$. So, condition (ii) of Definition 4 implies that if $M \in (S)$, then $\mathrm{vol}(M) = \infty$.

If $M \in (S)$, then using the expression (15) of the Schrödinger kernel $\widehat{s}_t(x,y)$ of M and under the condition that $N_\Gamma(x,y,R)$ satisfies (16), we deduce estimates of the norm

$$\|\widehat{s}_t(x,.)\|_{L^q(M)}, \quad q > 2,$$

from the corresponding ones on the symmetric space $X = G/K$. This is the crucial step for the proof of the dispersive estimate (5) of the operator $\widehat{S}_t$ for $M \in (S)$.

Finally, it is important to note that if $M \in (S)$, then we are able to prove the same results as in the case of the hyperbolic spaces [4].

3.2. **Dispersive and Strichartz estimates on locally symmetric spaces.**

We have the following dispersive estimate.

Theorem 5. *Assume that $M \in (S)$. Then for all $q, \tilde{q} \in (2, \infty]$, there is a constant $c > 0$ such that*

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c|t|^{-n \max\{(1/2)-(1/q),(1/2)-(1/\tilde{q})\}}, \quad |t| < 1,$$

and

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c|t|^{-3/2}, \quad |t| \geq 1.$$

Consider the following Cauchy problem for the linear inhomogeneous Schrödinger equation on M:

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = F(t,x), \\ u(0,x) = f(x). \end{cases} \tag{17}$$

Combining the above dispersive estimate with the classical TT^* method [12] we obtain Strichartz estimates for the solutions $u(t,x)$ of (17). Consider the triangle

$$T_n = \left\{ \left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \left(0, \tfrac{1}{2}\right] \times \left(0, \tfrac{1}{2}\right) : \tfrac{2}{p} + \tfrac{n}{q} \geq \tfrac{n}{2} \right\} \cup \left\{ \left(0, \tfrac{1}{2}\right) \right\}. \tag{18}$$

We say that the pair (p,q) is admissible if $\left(\frac{1}{p}, \frac{1}{q}\right) \in T_n$.

Theorem 6. *Assume that $M \in (S)$. Then the solutions $u(t,x)$ of the Cauchy problem (17) satisfy the Strichartz estimate*

$$\|u\|_{L_t^p L_x^q} \leq c \left\{ \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \right\}, \tag{19}$$

for all admissible pairs (p,q) and $(\tilde{p}, \tilde{q})$ corresponding to the triangle T_n.

As it is noticed in [4, 5] the above set T_n of admissible pairs is much wider that the admissible set in the case of $\mathbb{R}^n$, which is just the lower edge of the triangle. This phenomenon was already observed for hyperbolic spaces in [7, 17].

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PHASE TRANSITION AND GINZBURG-LANDAU MODELS OCCURING IN THE PHYSICS OF LIQUID CRYSTALS

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ABSTRACT. We study global minimizers of an energy functional arising as a thin sample limit in the theory of light-matter interaction in nematic liquid crystals. We show that depending on the parameters various defects are predicted by the model. In particular we show existence of a new type of topological defect which we call the *shadow vortex*. Finally, we discovered that at the boundary of the illuminated region, the profile of the minimizers is given by the universal equation of Painlevé.

1. PHYSICAL MOTIVATION

In a suitable experimental set up [5, 6, 3, 4, 7] involving a liquid crystal sample, a laser and a photoconducting cell one can observe light defects such as kinks, domain walls and vortices (cf. Figure 1). A concrete example of formation of optical vortices is presented in [7].

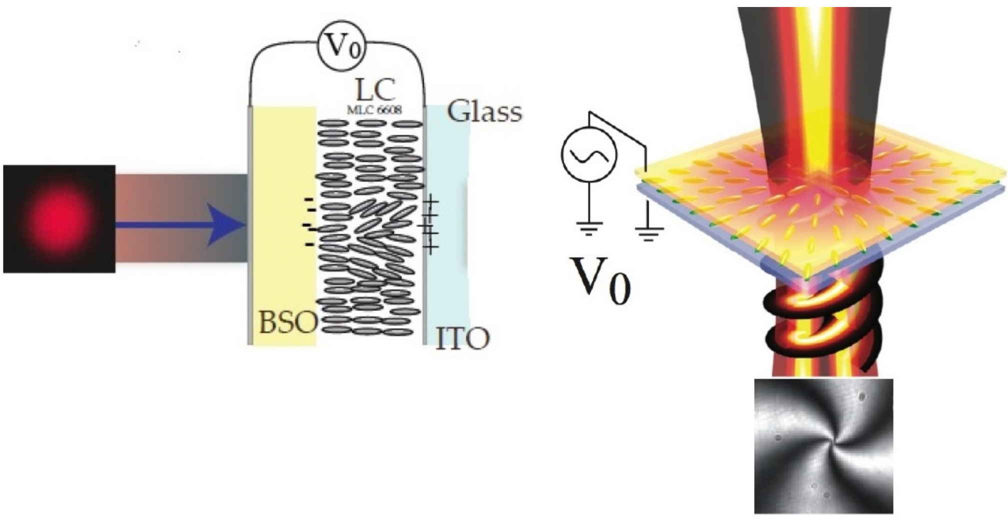

FIGURE 1. The experimental set up on the left: a laser light is applied to a thin layer of liquid crystals. As a consequence, the orientation of the molecules changes. On the right, a vortex is induced by the laser light.

To describe the energy of the illuminated liquid crystal light valve (LCLV) filled with a negative dielectric nematic liquid crystal which is homeotropically anchored, we consider the Oseen-Frank model in the vicinity of the

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Fréedericksz transition. Denoting the molecular director by $\vec{n}$ the Oseen-Frank energy is given by [13]

(1.1)

$$\mathcal{F} = \int \frac{K_1}{2}(\nabla\cdot\vec{n})^2 + \frac{K_2}{2}\left(\vec{n}\cdot(\nabla\times\vec{n})\right)^2 + \frac{K_3}{2}\left(\vec{n}\times(\nabla\times\vec{n})\right)^2 - \frac{\varepsilon_a}{2}(\vec{E}\cdot\vec{n})^2,$$

where $\{K_1, K_2, K_3\}$ are, respectively, the splay, twist, and bend elastic constants of the nematic liquid crystal and ε_a anisotropic dielectric constant ($\varepsilon_a < 0$). We will neglect the anisotropy i.e. we will assume that $K_1 = K_2 = K_3 = K$. Under uniform illumination $\vec{E} = [V_0 + aI]/d\ \hat{z}$, where V_0 is the voltage applied to the LCLV, d thickness of the cell, I intensity of the illuminating light beam, and a is a phenomenological dimensional parameter that describes the linear response of the photosensitive wall [19]. The homeotropic state, $\vec{n} = \hat{z}$, undergoes a stationary instability for critical values of the voltage which match the Fréedericksz transition threshold $V_{FT} = \sqrt{-K\pi^2/\varepsilon_a} - aI$.

Illuminating the liquid crystal light valve with a Gaussian beam induces a voltage drop with a bell-shaped profile across the liquid crystal layer, higher in the center of the illuminated area. The electric field within the thin sample takes the form [3]

$$\vec{E} = E_z\hat{z} + E_r\hat{r} \equiv \frac{[V_0 + aI(r)]}{d}\hat{z} + \frac{za}{d\omega}I'(r)\hat{r}, \tag{1.2}$$

where r is the radial coordinate centered on the beam, $\hat{r}$ the unitary radial vector, $I(r)$ the intensity of Gaussian light beam, $I(r) = I_0 e^{-r^2/2\omega^2}$, I_0 the peak intensity, and ω the width of the light beam.

If the intensity of the light beam is sufficiently close to the Fréedericksz transition the director is slightly tilted from the $\hat{z}$ direction and one can use the following ansatz

$$\vec{n}(x,y,z) \approx \begin{pmatrix} n_1(x,y,\pi z/d) \\ n_2(x,y,\pi z/d) \\ 1 - \frac{(n_1^2+n_2^2)}{2} \end{pmatrix}. \tag{1.3}$$

Introducing the above ansatz in the energy functional $\mathcal{F}$ and taking the limit of the thickness of the sample $d \to 0$ one obtains the following problem (written here for simplicity in a non dimensional form) [14, 5, 3]

(1.4)

$$G(u) = \int_{\mathbb{R}^2} \frac{\epsilon}{2}|\nabla u|^2 - \frac{1}{2\epsilon}\mu(x,y)|u|^2 + \frac{1}{4\epsilon}|u|^4 - a\left(f_1(x,y)u_1 + f_2(x,y)u_2\right),$$

where $u = (u_1, u_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an order parameter describing the tilt of $\vec{n}$ from the $\hat{z}$ direction in the thin sample limit, $\epsilon \ll 1$ is proportional to the width of the Gaussian beam and in radial co-ordinates

(1.5)

$$\mu(x,y) = e^{-r^2} - \chi, \qquad f(x,y) = -\frac{1}{2}\nabla\mu(x,y) = e^{i\theta} r e^{-r^2}, \qquad (x,y) = re^{i\theta},$$

and $\chi \in (0,1)$ is a fixed constant. The function μ describes light intensity and is sign changing due to the fact that the light is applied to the sample locally and areas where $\mu < 0$ are interpreted as shadow zones while areas where $\mu > 0$ correspond to illuminated zones. The function f describes the electric field induced by the light due to the photo conducting bluewall mounted on top of the sample [3]. Experiments show that as the intensity of the applied laser light represented here explicitly by the parameter a increases, defects such as light vortices appear first on the border of the illuminated zone and then in its center. This transition takes places suddenly once a threshold value of a is attained. At large values of a vortices have local profiles resembling the profile of the standard vortex of degree $+1$ in the Ginzburg-Landau theory. At low values of a vortices are located in the shadow area (we call them shadow vortices) and their local profiles are very different than that of the standard ones. In particular while the amplitude of the standard vortex is of order $\mathcal{O}(1)$ in ϵ the amplitude of the shadow vortex is of order $\mathcal{O}(\epsilon^{1/3})$. This picture is confirmed experimentally, numerically and by formal calculations [7]. Currently new experiments are being designed in order to realize experimentally other types of defects, such as kinks or domain walls. In the context of the model energy (1.4) this amounts to assuming that $u_2 \equiv 0$ (domain walls) or $u = u(x)$ and $u_2 \equiv 0$ (kinks). In the latter case the energy takes form

$$E(u) = \int_{\mathbb{R}} \frac{\epsilon}{2}|u'|^2 - \frac{1}{2\epsilon}\mu(x)u^2 + \frac{1}{4\epsilon}u^4 - af(x)u, \tag{1.6}$$

with $\mu(x)$ and $f(x)$ given by:

$$\mu(x) = e^{-x^2} - \chi, \quad \chi \in (0,1), \qquad f(x) = -\frac{1}{2}\mu'(x) = xe^{-x^2}, \tag{1.7}$$

where $\chi \in (0,1)$ is fixed.

For the sake of simplicity we shall first examine the one dimensional functional (1.6). The energy $E(u)$ is a real valued, one dimensional version of $G(u)$, yet both show a remarkable qualitative agreement. This is not surprising in view of the fact that both of them come from taking the thin sample and small tilt of the director limit of the Oseen-Frank energy (1.1). The theoretical value of our study lies in understanding and explaining the basic mechanism of formation of the various types of defects on the basis of the analogous mechanism for the energy functionals $E(u)$ and $G(u)$. In particular we will show existence of a new type of defect, the shadow kink, appearing in the one dimensional case, at the points where μ changes sign i.e. in the shadow area of the one dimensional model. Its analog for the energy G is the shadow vortex [7] and here we make a first step in understanding its local profile via the universal Painlevé equation.

2. The one dimensional model [10]

In the physical context described previously the functions μ and f are specific (cf. (1.7)). However, in the mathematical model presented below our results hold under the following more general hypothesis on μ and f:

$$\begin{cases} \mu \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is even, } \mu' < 0 \text{ in } (0,\infty), \\ \qquad\qquad \text{and } \mu(\rho) = 0 \text{ for a unique } \rho > 0, \\ \\ f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \text{ is odd, } f(x) > 0, \ \forall x > 0, \\ \qquad\qquad \text{and } f'(0) > 0. \end{cases} \tag{2.1}$$

We consider the energy

$$E(u)=\int_{\mathbb{R}}\Big(\frac{\epsilon}{2}|u'(x)|^2-\frac{1}{2\epsilon}\mu(x)u^2(x)+\frac{1}{4\epsilon}u^4(x)-af(x)u(x)\Big)\,\mathrm{d}x, \tag{2.2}$$

$u\in H^1(\mathbb{R})$, where $\epsilon>0$, and $a\geq 0$ are real parameters. Under assumptions (2.1), it is easy to establish by the direct method that functional (2.2) admits a global minimizer $v\in H^1(\mathbb{R})$ i.e. $E(v)=\min_{H^1(\mathbb{R})}E$. In addition, $v\in C^2(\mathbb{R})$ is a classical solution of the Euler- Lagrange equation

$$\epsilon^2 v''(x)+\mu(x)v(x)-v^3(x)+\epsilon a f(x)=0,\qquad \forall x\in\mathbb{R}. \tag{2.3}$$

We also note that due to the symmetries in (2.1), the energy (2.2) and equation (2.3) are invariant under the odd symmetry $v(x)\mapsto -v(-x)$.

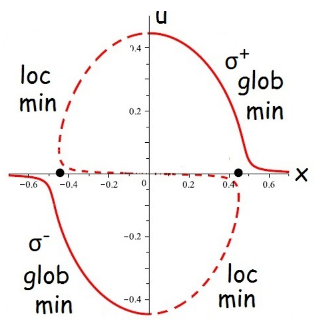

FIGURE 2. Assuming that $a>0$, the phase transition occurs between the two branches $\sigma^{\pm}(x)$ where the potential $W(x,u)$ admits a global minimum.

To understand the qualitative properties of the global minimizers, it helps to write alternatively equation (2.3) and functional (2.2) as

$$\epsilon^2 u''=\frac{\partial W}{\partial u}(x,u),$$

and

$$E(u)=\int_{\mathbb{R}}\Big(\frac{\epsilon}{2}|u'|^2+W(x,u)\Big)\mathrm{d}x,$$

with a potential $W(x,u)=\frac{1}{4\epsilon}u^4-\frac{\mu(x)}{2\epsilon}u^2-af(x)u$ depending on x and u. In the case where $a>0$, the function $W(x,u)$ admits when x is fixed, a global minimum on a branch $u=\sigma^+(x)$ located in the upper half-plane when $x\geq 0$, and on a branch $u=\sigma^-(x)$ located in the lower half-plane when $x\leq 0$ (cf. Figure 2). At $x=0$ there is a discontinuity since $\sigma^{\pm}(0)=\pm\sqrt{\mu(0)}$. To minimize the energy (2.2) the right balance between the contributions of the kinetic energy $\frac{\epsilon}{2}|u'|^2$ and the potential should be achieved. On the one hand the term $\frac{\epsilon}{2}|u'|^2$ penalizes high variations of u, while on the other hand the potential term forces the minimizer to be close to $\sigma^{\pm}$.

In Theorem 2.1 we will study the behaviour of the global minimizers as $\epsilon\to 0$, for fixed a. According to the value of a, we will see that transitions of the global minimizers as $\epsilon\to 0$ connect the branches $\sigma_{\pm}$ either near $x=\pm\rho$

(the shadow kink, cf. Figure 3 right) or at $x = 0$ (the standard kink, cf. Figure 3 left).

One can understand intuitively why the transition occurs near the origin for $a > a^*$ by considering the term $-\int_{\mathbb{R}} af(x)u(x)\mathrm{d}x$, whose contribution in (2.2) increases with a. When u vanishes at 0, the value of $-\int_{\mathbb{R}} fu$ is minimal, since u and f have the same sign. This gain of energy compensates the cost of a transition near the origin for $a > a^*$. When $a > a^*$ the global minimizer has a profile of suitably re-scaled and modulated hyperbolic tangent. This is not surprising since $u(x) = \tanh(x/\sqrt{2})$ is a solution of the Allen-Cahn equation

$$u'' = u^3 - u, \qquad \text{in } \mathbb{R}, \tag{2.4}$$

and it is a standard, local profile of topological defects such as kinks or domain walls appearing in many phase transition problems.

On the other hand, when $a \in (0, a_*)$ the name *shadow kink* for the global minimizer is justified by the fact that the zero of the global minimizer occurs near the points where μ changes its sign i.e. between the illuminated zone and the dark zone in the nematic liquid crystal experiment. Because of this, unlike in the case of the standard kink, the shadow kink is hard to detect experimentally.

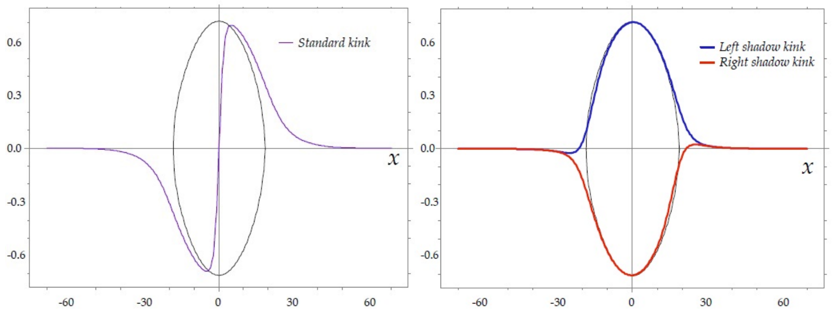

FIGURE 3. On the right, the standard kink, and on the left the shadow kink. As $\epsilon \to 0$, the minimizers approach the curves $\pm\sqrt{\mu^+}$ (cf. Theorem 2.1).

From the preceding discussion we see that when the parameter a changes from $a < a_*$ to $a > a^*$ the global minimizer changes its character very significantly and in the particular case $a_* = a^*$ occuring in the physical context, an abrupt transition between the shadow kink and the standard kink takes place.

Finally, in the case where $a = 0$, the potential $W(x, u) = \frac{1}{4\epsilon}u^4 - \frac{\mu(x)}{2\epsilon}u^2$ admits a global minimum on two branches $\sigma^{\pm} = \pm\sqrt{\mu^+(x)}$, and the minimizer up to change of sign, interpolates σ^+ (cf. Figure 4).

Theorem 2.1. [10] *The following statements hold.*

(i) *When $a = 0$ the global minimizer v is even, and positive up to change of v by $-v$.*

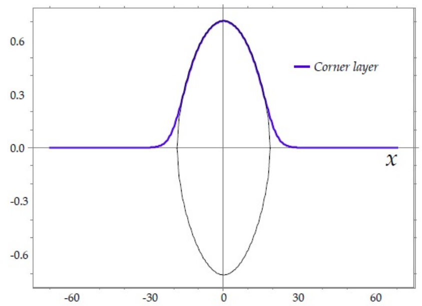

FIGURE 4. When $a = 0$, the global minimizer v which is even, and positive up to change of v by $-v$, interpolates the curve $\sigma^+ = \sqrt{\mu^+(x)}$ (cf. Theorem 2.1 (i)).

(ii) *For $a > 0$, the global minimizer v has a unique zero $\bar{x}$ such that*

$$|\bar{x}| \leq \xi + \mathcal{O}(\sqrt{\epsilon}), \text{ and } v(x) > 0, \forall x > \bar{x}, \text{ while } v(x) < 0, \forall x < \bar{x}. \tag{2.5}$$

(iii) *Let*

$$a^* := \sup_{x \in [-\xi, 0)} \frac{\sqrt{2}\big((\mu(0))^{3/2} - (\mu(x))^{3/2}\big)}{3 \int_x^0 |f| \sqrt{\mu}}, \tag{2.6}$$

and note that $a^* < \infty$. For all $a > a^*$, $\bar{x} \to 0$ as $\epsilon \to 0$, and the global minimizer v satisfies

$$\begin{aligned} &\lim_{\epsilon \to 0} v(\bar{x} + \epsilon s) = \sqrt{\mu(0)} \tanh(s \sqrt{\mu(0)/2}), \\ &\lim_{\epsilon \to 0} v(x + \epsilon s) = \begin{cases} \sqrt{\mu(x)} & \text{for } 0 < x < \rho, \\ -\sqrt{\mu(x)} & \text{for } -\rho < x < 0, \\ 0 & \text{for } |x| \geq \rho, \end{cases} \end{aligned} \tag{2.7}$$

in the $C^1_{\mathrm{loc}}(\mathbb{R})$ sense.

(iv) *Let*

$$a_* := \inf_{x \in (-\xi, 0]} \frac{\sqrt{2}(\mu(x))^{3/2}}{3 \int_{-\xi}^x |f| \sqrt{\mu}} \in (0, \infty), \text{ and note that } a_* \leq a^*.$$

Up to change of $v(x)$ by $-v(-x)$, for all $a \in (0, a_*)$, $\bar{x} \to -\rho$ as $\epsilon \to 0$, and

$$\lim_{\epsilon \to 0} v(x + s\epsilon) = \begin{cases} \sqrt{\mu(x)} & \text{for } |x| < \rho, \\ 0 & \text{for } |x| \geq \rho, \end{cases} \tag{2.8}$$

in the $C^1_{\mathrm{loc}}(\mathbb{R})$ sense. The above asymptotic formula holds as well when $a = 0$. Moreover, when $f = -\frac{\mu'}{2}$ we have $a_* = a^* = \sqrt{2}$.

3. THE TWO DIMENSIONAL MODEL [11]

In the two dimensional model, we suppose that the function

$$\mu \in C^\infty(\mathbb{R}^2, \mathbb{R})$$

is radial i.e. $\mu(x) = \mu_{\mathrm{rad}}(|x|)$, with $\mu_{\mathrm{rad}} \in C^\infty(\mathbb{R}, \mathbb{R})$ an even function, and the vector field $f = (f_1, f_2) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is also radial i.e. $f(x) =$

$f_{\mathrm{rad}}(|x|)\frac{x}{|x|}$, with $f_{\mathrm{rad}} \in C^\infty(\mathbb{R},\mathbb{R})$ an odd function. In addition we assume that

(3.1)
$$\begin{cases} \mu \in L^\infty(\mathbb{R}^2,\mathbb{R}), \ \mu'_{\mathrm{rad}} < 0 \text{ in } (0,\infty), \text{ and } \mu_{\mathrm{rad}}(\rho) = 0 \text{ for a unique } \rho > 0, \\ f \in L^1(\mathbb{R}^2,\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2,\mathbb{R}^2), \text{ and } f_{\mathrm{rad}} > 0 \text{ on } (0,\infty). \end{cases}$$

Next, we consider the Ginzburg-Landau type energy functional

$$G(u) = \int_{\mathbb{R}^2} \frac{\epsilon}{2}|\nabla u|^2 - \frac{1}{2\epsilon}\mu(x)|u|^2 + \frac{1}{4\epsilon}|u|^4 - af(x)\cdot u, \tag{3.2}$$

where $u = (u_1,u_2) \in H^1(\mathbb{R}^2,\mathbb{R}^2)$ and $\epsilon > 0$, $a \geq 0$ are real parameters. As in the one dimensional model, it is easy to establish that under assumptions (3.1), functional (3.2) admits a global minimizer $v \in H^1(\mathbb{R}^2,\mathbb{R}^2)$ i.e. $G(v) = \min_{H^1(\mathbb{R}^2,\mathbb{R}^2)} G$. In addition, $v \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ is a classical solution of the Euler- Lagrange equation

$$\epsilon^2\Delta v + \mu(x)v - |v|^2 v + \epsilon a f(x) = 0, \qquad x \in \mathbb{R}^2. \tag{3.3}$$

We also note that due to the radial symmetry of μ and f, the energy (3.2) and equation (3.3) are invariant under the transformations $v(x) \mapsto g^{-1}v(gx)$, $\forall g \in O(2)$.

Our main purpose is to study qualitative properties of the global minimizers of G as the parameters a and ϵ vary. In general we will assume that $\epsilon > 0$ is small and $a \geq 0$ is bounded uniformly in ϵ. As we will see critical phenomena such as symmetry breaking and restoration occur along curves of the form $a = a(\epsilon)$ in the (ϵ,a) plane.

To determine the limit of the minimizers $v_{\epsilon,a(\epsilon)}$ as $\epsilon \to 0$, we apply the following method. First, we rescale the minimizers according to the region we are studying. The appropriate rescaling is the one providing uniform bounds up to the second derivatives. Next, we apply the theorem of Ascoli, and obtain the convergence of the rescaled minimizers to a solution of the limiting equation. Since the convergence is in C^2_{loc}, the solution obtained at the limit is minimal in the sense that any perturbation with compact support has greater or equal energy. Finally, we utilize the classification of minimal solutions of the limiting equation, and the properties of the minimizers to determine their limit.

More precisely for $x \in D(0;\rho)$ the relevant rescaling is $\tilde{v}_\epsilon(s) = v_\epsilon(x+\epsilon s)$, and as $\epsilon \to 0$, we obtain the convergence of $\tilde{v}_\epsilon$ to a minimal solution of the Ginzburg-Landau system:

$$\Delta\eta = (|\eta|^2 - 1)\eta, \qquad \eta : \mathbb{R}^2 \to \mathbb{R}^2, \tag{3.4}$$

which is the Euler-Lagrange equation of the energy functional

$$E_{\mathrm{GL}}(u,\Omega) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-|u|^2).$$

We recall that a solution u of (3.4) is a critical point of E_{GL}, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0} E_{\mathrm{GL}}(u+\lambda\phi, \operatorname{supp}\phi) = 0,$$

$\forall \phi \in C_0^\infty(\mathbb{R}^2,\mathbb{R}^2)$, while a minimal solution η satisfies the stronger condition:

$$E_{\mathrm{GL}}(\eta+\phi, \operatorname{supp}\phi) \geq E_{\mathrm{GL}}(\eta, \operatorname{supp}\phi), \ \forall\phi \in C_0^\infty(\mathbb{R}^2,\mathbb{R}^2).$$

This notion of minimality is standard for many problems in which the energy of a localized solution is actually infinite due to non compactness of the domain. It is known [18] that any minimal solution of (3.4) is either a constant of modulus 1 or (up to orthogonal transformation in the range and translation in the domain) the radial solution $\eta(s) = \eta_{\mathrm{rad}}(|s|)\frac{s}{|s|}$. We also mention some properties of η:

(i) $\eta'_{\mathrm{rad}} > 0$ on $(0,\infty)$, $\eta_{\mathrm{rad}}(0) = 0$, $\lim_{r\to\infty}\eta_{\mathrm{rad}}(r) = 1$,
(ii) $\int_{\mathbb{R}^2}|\nabla\eta|^2 = \infty$.

On the other hand, for $x \in \mathbb{R}^2 \setminus \overline{D(0;\rho)}$ we rescale the minimizers by setting $\tilde{v}_\epsilon(s) = \frac{v_\epsilon(x+\epsilon s)}{\epsilon}$, and obtain as $\epsilon \to 0$ the convergence of $\tilde{v}_\epsilon(s)$ in C^2_{loc} to the constant $\tilde{V}(s) \equiv -\frac{a_0}{\mu(x)}f(x)$, with $a_0 = \lim_{\epsilon\to 0} a(\epsilon)$. This case is easy to study, since the limiting equation has a convex potential, and thus a unique solution $\tilde{V}$ bounded in $\mathbb{R}^2$.

The convergence of the minimizers in a neighbourhood of the points where μ vanishes will be discussed in the next section. As a consequence of what precedes we establish

Theorem 3.1. [11] *Let $v_{\epsilon,a}$ be a global minimizer of G, let $a \geq 0$ be bounded (possibly dependent on ϵ), and let $\rho > 0$ be the zero of μ_{rad}. The following statements hold:*

(i) *Let $\Omega \subset D(0;\rho)$ be an open set such that $v_{\epsilon,a} \neq 0$ on Ω, for every $\epsilon \ll 1$. Then $|v_{\epsilon,a}| \to \sqrt{\mu}$ in $C^0_{\mathrm{loc}}(\Omega)$.*
(ii) *Assuming that $\lim_{\epsilon\to 0} a(\epsilon) = a_0$, then we have*

$$\lim_{\epsilon\to 0}\frac{v_{\epsilon,a}(x)}{\epsilon} = -\frac{a_0}{\mu(x)}f(x)$$

uniformly on compact subsets of $\{|x| > \rho\}$.

From Theorem 3.1 (ii), it follows that when $a_0 > 0$ and $\epsilon \ll 1$, the minimizers have the same topological degree as f in the region where $\mu < 0$. This is the idea to establish the existence of vortices, i.e. points where the vector field of the minimizer v vanishes, even when $a(\epsilon)$ converges to 0 (but not exponentially fast, cf. Theorem 3.2 (i) below) . Next, in Theorem 3.2 (ii) and (iii), we locate the vortices of v according to the regime of the parameters (ϵ, a). We observe for instance that as a corollary of Theorem 3.1 (i) and Theorem 3.2 (ii) below, we obtain when $a = o(\epsilon|\ln\epsilon|)$ the convergence $|v_{\epsilon,a}| \to \sqrt{\mu}$ in $C^0_{\mathrm{loc}}(D(0;\rho))$, thus $\Omega = D(0;\rho)$ in this case.

Theorem 3.2. [11] *Let $v_{\epsilon,a}$ be a global minimizer. Assume that $a(\epsilon) > 0$, a is bounded and $\lim_{\epsilon\to 0}\epsilon^{1-\frac{3\gamma}{2}}\ln a = 0$ for some $\gamma \in [0, 2/3)$.*

(i) *For $\epsilon \ll 1$, $v_{\epsilon,a}$ has at least one zero $\bar{x}_\epsilon$ such that*

$$|\bar{x}_\epsilon| \leq \rho + o(\epsilon^\gamma). \tag{3.5}$$

In addition, any sequence of zeros of $v_{\epsilon,a}$, either satisfies (3.5) or it diverges to ∞.

(ii) *For every $\rho_0 \in (0,\rho)$, there exists $b_* > 0$ such that when*

$$\limsup_{\epsilon\to 0}\frac{a}{\epsilon|\ln\epsilon|} < b_*$$

then any limit point $l \in \mathbb{R}^2$ of the set of zeros of $v_{\epsilon,a}$ satisfies

$$\rho_0 \leq |l| \leq \rho. \tag{3.6}$$

In addition if $a = o(\epsilon|\ln\epsilon|)$ then $|l| = \rho$.

(iii) *On the other hand, for every $\rho_0 \in (0, \rho)$, there exists $b^* > 0$ such that when $\limsup_{\epsilon\to 0} \frac{a}{\epsilon|\ln\epsilon|^2} > b^*$, the set of zeros of $v_{\epsilon,a}$ has a limit point l such that*

$$|l| \leq \rho_0. \tag{3.7}$$

If $v_{\epsilon,a}(\bar{x}_\epsilon) = 0$ and $\bar{x}_\epsilon \to l$ then up to a subsequence

$$\lim_{\epsilon\to 0} v_{\epsilon,a}(\bar{x}_\epsilon + \epsilon s) \to \sqrt{\mu(l)}(g \circ \eta)(\sqrt{\mu(l)}s),$$

in $C^2_{\text{loc}}(\mathbb{R}^2)$, for some $g \in O(2)$. In addition if $\limsup_{\epsilon\to 0} \frac{a}{\epsilon|\ln\epsilon|^2} = \infty$ then $l = 0$.

Our proof to locate the vortices is based on energy considerations. The idea is to compute an upper bound of the energy, and then proceed by contradiction. Assuming for each regime, the existence of a vortex in a region where it cannot occur, we compute an increase of energy that contradicts the upper bound. In contrast with the one dimensional model, the uniqueness of the vortex in the two dimensional model is a difficult open question. Indeed, the existence of vortices close to the boundary of the disc $D(0; \rho)$ induce an infinitesimal variation of the total energy that make them difficult to detect.

When a satisfies the hypothesis of Theorem 3.2 (ii) the global minimizer has a vortex appearing at the boundary of the disc $D(0; \rho)$ corresponding to the illuminated region. For this reason we call it *shadow vortex*. Based on numerical simulations we conjecture that, rather than coming from the equation (3.4), its rescaled local profile comes from the generalized second Painlevé equation (cf. (4.8) in Section 4). Despite the radial forcing term $\epsilon a f$ in (3.3), Theorem 3.2 shows that the radial symmetry of the global minimizer is broken for $a < b_*\epsilon|\ln\epsilon|$. However, numerical simulations and experiments suggest that a weaker symmetry with respect to a reflection line should be preserved (cf. Figure 5).

Theorem 3.2 (iii) states further increase of the value of a leads to the restoration of the symmetry at least locally. Indeed, when $a > b^*\epsilon|\ln\epsilon|^2$ Theorem 3.2 (iii) ensures the existence of at least one vortex in the interior of $D(0; \rho)$, whose local profile is given by the radial vortex η of the Ginzburg-Landau equation (3.4). This is the reason why we call it *standard vortex*. Finally, one can see in [11] that given $\epsilon > 0$, the radial symmetry is completely restored provided that a is large enough.

We also mention (cf. [11]) that when $a = 0$ the global minimizer inherits the one dimensional radial profile of μ. It is unique up to rotations and can be written as $v(x) = (v_{\text{rad}}(|x|), 0)$, with v_{rad} a positive and even function.

The results of Theorem 3.2 show an excellent agreement with the experiments performed with physical parameters [7].

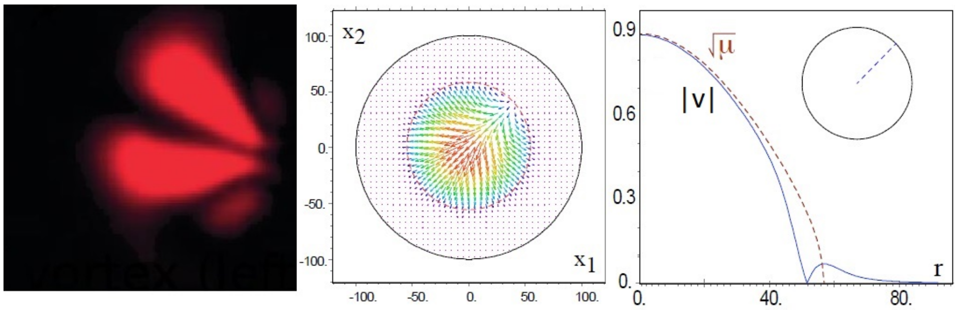

FIGURE 5. On the left, a picture from an experiment performed when $a(\epsilon) = o(\epsilon|\ln\epsilon|)$. The black colour corresponds to the region where the product of the two components of the vector field is small. The *shadow vortex* is located at the boundary of the illuminated region, and since the vector field diverges from it we observe the two lobes separated by the black colour. On the right, a simulation of the vector field v and a radial section of its modulus at an angle indicated in the upper right corner.

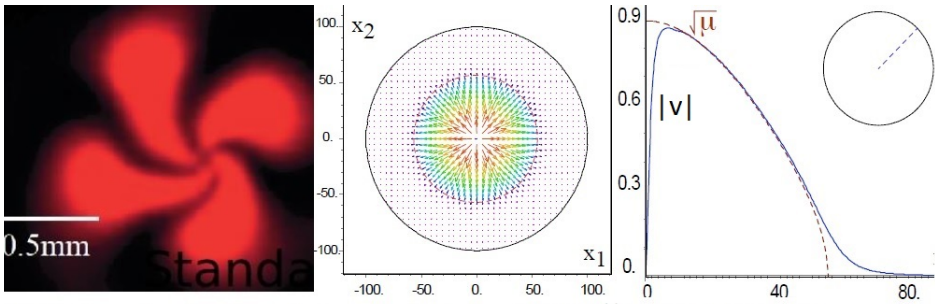

FIGURE 6. On the left, a picture from an experiment performed when $\frac{a(\epsilon)}{\epsilon|\ln\epsilon|^2} \gg 1$. Again, the black colour corresponds to the region where the product of the two components of the vector field is small. The cross shape is due to the fact that the *standard vortex* is close to the origin, and the vector field is almost radial. On the right, a simulation of the vector field v and a radial section of its modulus at an angle indicated in the upper right corner.

4. The connection of the liquid crystal models with the Painlevé equation (cf. [10], [12])

We have seen in Theorem 2.1 that in the homogeneous case $a = 0$, the global minimizer $v_\epsilon > 0$ of E converges as $\epsilon \to 0$, to the curve $\sigma^+(x)$ where the potential $W(x,u)$ attains its global minima. Since σ^+ is not smooth, the minimizer exhibits a boundary layer behaviour near the zero level set of μ and its local profile, after suitable scaling, is given by the homogeneous second Painlevé ODE:

$$y'' - sy - 2y^3 = 0, \qquad \text{in } \mathbb{R}. \tag{4.1}$$

This phenomenon is also known as the corner layer and it is present in the context of the Bose-Einstein condensates [1, 17], as well as in many other problems (see for example [2]).

More precisely, following the method detailed in Section 3, we rescale the minimizer $v_{\epsilon,a}$ of E in a neighbourhood of the point ρ where μ vanishes, setting

$$w_{\epsilon,a}(s) = 2^{-1/2}(-\mu_1\epsilon)^{-1/3} v_{\epsilon,a}\Big(\rho + \epsilon^{2/3}\frac{s}{(-\mu_1)^{1/3}}\Big),$$

where $\mu_1 = \mu'(\rho) < 0$. Then, as $\epsilon \to 0$, the functions w_ϵ converge in $C^2_{\mathrm{loc}}(\mathbb{R})$ to a bounded at ∞ minimal solution of (4.1). To explain what this means, let

$$E_{\mathrm{P_{II}}}(y, I) = \int_I \left(\frac{1}{2}|y'|^2 + \frac{1}{2}sy^2 + \frac{1}{2}y^4\right),$$

be the energy functional associated to (4.1). By definition a solution of (4.1) is minimal if

$$E_{\mathrm{P_{II}}}(y, \operatorname{supp}\phi) \le E_{\mathrm{P_{II}}}(y+\phi, \operatorname{supp}\phi)$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Finally, in view of the description of solutions of (4.1) in [15], we deduce that w_ϵ converges as $\epsilon \to 0$ to the Hastings-McLeod solution h of (4.1) behaving asymptotically as in (4.3). Moreover, we prove in Theorem 4.1 that h and $-h$ are the only minimal solutions of (4.1) bounded at ∞. The asympotic behaviour of h is determined by the location of the global minima of the potential $H(s,y) = \frac{1}{2}sy^2 + \frac{1}{2}y^4$ associated to equation (4.1) that can alternatively be written $y'' - H_y(s,y) = 0$. Indeed, for s fixed H attains its global minimum when $y = 0$ and $s \ge 0$, and when $y = \pm\sqrt{|s|/2}$ and $s < 0$. Thus, the global minima of H bifurcate from the origin, and the two minimal solutions $\pm h$ of (4.1) interpolate these two branches of minima.

In the nonhomogeneous case where $a > 0$, the rescaled minimizers w_ϵ converge to a minimal solution of the nonhomogeneous Painlevé ODE

$$y''(s) - sy(s) - 2y^3(s) - \alpha = 0, \qquad \forall s \in \mathbb{R}, \tag{4.2}$$

with $\alpha = \frac{af(\rho)}{\sqrt{2}\mu_1} < 0$. If $v \ge 0$ on $(\rho - \delta, \rho + \delta)$, for small $\delta > 0$ (in particular if v is the left shadow kink, cf. Figure 3 left), then w_ϵ converges as $\epsilon \to 0$ to a minimal solution of (4.2) behaving asymptotically as in (4.4). On the other hand, in the case of the right shadow kink occuring when the zero $\bar{x}_\epsilon$ of v converges to ρ (cf. Figure 3 left), we are not aware if the limit of w_ϵ is a sign changing solution of (4.2) (cf. [8] for the existence of such a solution). This is equivalent to establishing the bound $|\bar{x}_\epsilon - \rho| = O(\epsilon^{2/3})$.

Theorem 4.1. [10] *The following statements hold. For any $\alpha \le 0$*[1] *the second Painlevé equation (4.2) has a positive minimal solution h, which is strictly decreasing ($h' < 0$) and such that*

(a) *When $\alpha = 0$*

$$\begin{aligned} &h(s) \sim Ai(s), && s \to \infty \\ &h(s) \sim \sqrt{|s|/2}, && s \to -\infty \end{aligned} \tag{4.3}$$

Moreover, h and $-h$ are the only minimal solutions, bounded at ∞.

[1] By changing h by $-h$, we obtain the solutions of (4.2) corresponding to $\alpha \ge 0$

(b) *When* $\alpha < 0$

$$h(s) \sim \frac{|\alpha|}{s}, \qquad s \to \infty$$
$$h(s) \sim \sqrt{|s|/2}, \qquad s \to -\infty \tag{4.4}$$

A natural generalization of the Painlevé equation (4.1) is

$$\Delta y - x_1 y - 2y^3 = 0, \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2. \tag{4.5}$$

In Theorem 4.2 below (cf. [12]) we constructed the first to our knowledge example of solutions of this PDE, both relevant from the applications point of view and mathematically interesting. It has a form of a quadruple connection between the Airy function Ai, two one dimensional Hastings-McLeod solutions $\pm h$ (cf. Theorem 4.1 (a)), and the heteroclinic solution $t \mapsto \tanh(t/\sqrt{2})$ of the one dimensional Allen-Cahn ODE (2.4). This construction resulted form the study of minimizers $v : \mathbb{R}^2 \to \mathbb{R}$ of a functional similar to E.

Theorem 4.2. [12] *There exists a solution* $y : \mathbb{R}^2 \to \mathbb{R}$ *to* (4.5)*, such that*

(i) y *is positive in the upper-half plane and odd with respect to* x_2 *i.e.* $y(x_1, x_2) = -y(x_1, -x_2)$.
(ii) y *and its derivatives are bounded in the half-planes* $[s_0, \infty) \times \mathbb{R}$, $\forall s_0 \in \mathbb{R}$.
(iii) y *is minimal with respect to perturbations* $\phi \in C_0^\infty(\mathbb{R}^2)$ *such that* $\phi(x_1, x_2) = -\phi(x_1, -x_2)$.
(iv) $\frac{|y(x_1,x_2)|}{Ai(x_1)} = O(1)$*, as* $x_1 \to \infty$ *(uniformly in* x_2*).*
(v) *For every* $x_2 \in \mathbb{R}$ *fixed, let* $\tilde{y}(t_1, t_2) := \frac{\sqrt{2}}{(-\frac{3}{2}t_1)^{\frac{1}{3}}}\, y\big(-(-\frac{3}{2}t_1)^{\frac{2}{3}}, x_2 + t_2(-\frac{3}{2}t_1)^{-\frac{1}{3}}\big)$. *Then*

$$\lim_{l \to -\infty} \tilde{y}(t_1 + l, t_2) = \begin{cases} \tanh(t_2/\sqrt{2}) & \text{when } x_2 = 0, \\ 1 & \text{when } x_2 > 0, \\ -1 & \text{when } x_2 < 0, \end{cases} \tag{4.6}$$

for the $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ *convergence.*
(vi) $y_{x_1}(x_1, x_2) < 0$, $\forall x_1 \in \mathbb{R}$, $\forall x_2 > 0$.
(vii) $y_{x_2}(x_1, x_2) > 0$, $\forall x_1, x_2 \in \mathbb{R}$, *and* $\lim_{l \to \pm\infty} y(x_1, x_2 + l) = \pm h(x_1)$ *in* $C^2_{\mathrm{loc}}(\mathbb{R}^2)$*, where* h *is the Hastings-McLeod solution of* (4.1) (cf. Theorem 4.1 (a)).

Comparing (iv) with (4.3) we see that as $x_1 \to \infty$ the function $y(x_1, x_2)$ behaves similarly as the Hastings-McLeod solution $h(x_1)$. At the same time, as $x_2 \to \pm\infty$ we have $y(x_1, x_2) \to \pm h(x_1)$, $x_2 \to \pm\infty$. Perhaps the most interesting aspect of the above solution y is stated in property (v), since after rescaling we obtain as $x_1 \to -\infty$, the convergence to the heteroclinic orbit $t \mapsto \tanh(t/\sqrt{2})$ of the Allen-Cahn ODE (2.4). We recall that this orbit connecting the two minima ± 1 of the corresponding potential $W(u) = \frac{1}{4}(1 - u^2)^2$, plays a crucial role (cf. [20]) in the study of minimal solutions of the Allen-Cahn equation

$$\Delta u = u^3 - u, \quad u \colon \mathbb{R}^n \to \mathbb{R}. \tag{4.7}$$

Again, we say that u is a minimal solution of (4.7) if

$$E_{\mathrm{AC}}(u, \operatorname{supp}\phi) \leq E_{\mathrm{AC}}(u+\phi, \operatorname{supp}\phi),$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$, where

$$E_{\mathrm{AC}}(u,\Omega) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2$$

is the Allen-Cahn energy associated to (4.7). In the proof of Theorem 4.2 it is shown that a minimal solution of (4.5) rescaled as in (v), converges as $x_1 \to -\infty$ to a minimal solution of (4.7). This deep connection of the structure of the Painlevé equation with the Allen-Cahn PDE, suggests that several properties of the Allen-Cahn equation should be transfered to the Painlevé equation. Although by construction the solution y is only minimal for odd perturbations, we expect that y is actually minimal for general perturbations, and plays a similar role that the heteroclinic orbit for the Allen-Cahn equation. What's more the two global minimizers ± 1 of the functional E_{AC} have their counterparts in the two minimal solutions $\pm h$ of the Painlevé equation. Indeed, property (vii) establishes that y connects monotonically along the vertical direction x_2, the two minimal solutions $\pm h(x_1)$, in the same way that the heteroclinic orbit $t \mapsto \tanh(t/\sqrt{2})$ connects monotonically the two global minimizers ± 1. This analogy between the Painlevé and the Allen-Cahn equation is natural if seen from the point of view of the potential $H(x_1,y) = \frac{1}{2}x_1 y^2 + \frac{1}{2}y^4$ corresponding to (4.5) (cf. the expression of the functional $E_{\mathrm{P_{II}}}(y,\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla y|^2 + \frac{1}{2}x_1 y^2 + \frac{1}{2}y^4\right)$ associated to (4.5)) compared with $W(u) = \frac{1}{4}(1-u^2)^2$. For the latter the phase transition connects two minima ± 1 while for the former the phase transition connects the two branches $\pm\sqrt{(-x_1)^+/2}$ of minima of the potential H parametrized by x_1.

The study of the vector Painlevé PDE

$$\Delta y(s) - s_1 y(s) - 2|y(s)|^2 y(s) - \alpha = 0, \qquad \forall s = (s_1,s_2) \in \mathbb{R}^2, \tag{4.8}$$

with $y : \mathbb{R}^2 \to \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$, is a completely open problem.

In the context of Section 3 where we described the minimizers $v_{\epsilon,a} : \mathbb{R}^2 \to \mathbb{R}^2$ of the energy functional G, we can show that in a neighbourhood of a point ξ where μ vanishes, the local profile of v is given by a minimal solution of (4.8). More precisely, for every $\xi = \rho e^{i\theta}$, we consider the local coordinates $s = (s_1,s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$. Then, if we rescale the minimizers $v_{\epsilon,a} : \mathbb{R}^2 \to \mathbb{R}^2$ of G by setting:

$$w_{\epsilon,a}(s) = 2^{-1/2}(-\mu_1\epsilon)^{-1/3} v_{\epsilon,a}\Big(\xi + \epsilon^{2/3}\frac{s}{(-\mu_1)^{1/3}}\Big),$$

and if we assume that $\lim_{\epsilon\to 0} a(\epsilon) = a_0$, we obtain as $\epsilon \to 0$ the convergence of $w_{\epsilon,a}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ up to subsequence, to a minimal solution y of (4.8), with $\alpha := \frac{a_0 f(\xi)}{\sqrt{2}\mu_1}$. We expect that when $a(\epsilon) = o(\epsilon|\ln\epsilon|)$, the limit y is a minimal solution of (4.8) with an isolated zero. The availability of such a result would imply that the shadow vortex has a profile coming from (4.8) and is located at a distance of order $O(\epsilon^{2/3})$ from the boundary $\partial D(0;\rho)$ of the illuminated region.

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A NEW MONOTONICITY FORMULA FOR SOLUTIONS TO THE ELLIPTIC SYSTEM $\Delta u = \nabla W(u)$

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ABSTRACT. *Using the physically motivated stress energy tensor, we prove variants of the well known weak and strong monotonicity formulas for solutions to the semilinear elliptic system $\Delta u = \nabla W(u)$ with W nonnegative.*

Consider the semilinear elliptic system

$$\Delta u = \nabla W(u) \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 1, \tag{1}$$

where $W \in C^3(\mathbb{R}^m;\mathbb{R})$, $m \geq 1$, is *nonnegative*.

In the scalar case, namely $m = 1$, Modica [7] used the maximum principle to show that every bounded solution to (1) satisfies the pointwise gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \quad \text{in} \quad \mathbb{R}^n, \tag{2}$$

(see also [3]). Using this, together with Pohozaev identities, it was shown in [8] that the following strong monotonicity property holds:

$$\frac{d}{dR}\left(\frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) \geq 0, \quad R > 0, \tag{3}$$

where B_R stands for the n-dimensional ball of radius R that is centered at 0 (keep in mind that (1) is translation invariant).

In the vectorial case, that is $m \geq 2$, in the absence of the maximum principle, it is not true in general that the gradient bound (2) holds (see [11] for a counterexample). Nevertheless, it was shown in [1], using a physically motivated stress energy tensor, that every solution to (1) satisfies the weak monotonicity property:

$$\frac{d}{dR}\left(\frac{1}{R^{n-2}}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) \geq 0, \quad R > 0, \quad n \geq 2, \tag{4}$$

where $|\nabla u|^2 = \sum_{i=1}^n |u_{x_i}|^2$ (for related results, obtained via Pohozaev identities, see [2], [4] and [9]). In fact, as was observed in [1], if u additionally satisfies the vector analog of Modica's gradient bound (2), we have the strong monotonicity property (3).

Interesting applications of these formulas can be found in the aforementioned references. The importance of monotonicity formulas in the study of nonlinear partial differential equations is also highlighted in the recent article [5].

Recently, the following interesting result was proven in [11]. If u is a solution to (1) with $n = 2$ satisfying a generalization of Modica's gradient estimate (an implication of it in fact), then it holds

$$\frac{d}{dR}\left(\frac{1}{R}\int_{B_R} W(u)\,dx\right) \geq 0, \quad R > 0. \tag{5}$$

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This was accomplished by deriving an alternative form of the stress energy tensor for solutions defined in planar domains, and by giving a geometric interpretation of Modica's estimate (2). We emphasize that the interesting techniques in [11] are intrinsically two dimensional and seem hard to generalize to higher dimensions.

Interestingly enough, in the vector case, it is stated in [6] (without proof) that Pohozaev identities imply that solutions to the Ginzburg-Landau system

$$\Delta u = \left(|u|^2 - 1\right) u, \quad u : \mathbb{R}^n \to \mathbb{R}^m, \quad \left(\text{here } W(u) = \frac{\left(1-|u|^2\right)^2}{4}\right),$$

with $n \geq 2, m \geq 2$, satisfy the weak monotonicity property

$$\frac{d}{dR}\left(\frac{1}{R^{n-2}}\int_{B_R}\left\{\frac{n-2}{2}|\nabla u|^2 + n\frac{\left(1-|u|^2\right)^2}{4}\right\}dx\right) \geq 0, \quad R > 0. \tag{6}$$

It is tempting to wonder whether there is a strong version of (6), that is with R^{n-1} in place of R^{n-2}, in the scalar case (for any smooth $W \geq 0$), which for $n = 2$ gives (5). In this note, by appropriately modifying the systematic approach of [1], we prove the following general result which, in particular, confirms this connection.

Theorem 1. *If $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n \geq 2, m \geq 1$, solves (1) with $W \in C^1(\mathbb{R}^m;\mathbb{R})$ nonnegative, we have the weak monotonicity formula:*

$$\frac{d}{dR}\left(\frac{1}{R^{n-2}}\int_{B_R}\left\{\frac{n-2}{2}|\nabla u|^2 + nW(u)\right\}dx\right) \geq 0, \quad R > 0. \tag{7}$$

In addition, if u satisfies Modica's gradient bound, that is

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \quad in \quad \mathbb{R}^n, \tag{8}$$

we have the strong monotonicity formula:

$$\frac{d}{dR}\left(\frac{1}{R^{n-1}}\int_{B_R}\left\{\frac{n-2}{2}|\nabla u|^2 + nW(u)\right\}dx\right) \geq 0, \quad R > 0. \tag{9}$$

Proof. By means of a direct calculation, it was shown in [1] that, for solutions u to (1), the stress energy tensor $T(u)$, which is defined as the $n \times n$ matrix with entries

$$T_{ij} = u_{,i} \cdot u_{,j} - \delta_{ij}\left(\frac{1}{2}|\nabla u|^2 + W\left(u\right)\right), \quad i,j = 1,\cdots,n, \ (\text{where } u_{,i} = u_{x_i}),$$

satisfies

$$\text{div}T(u) = 0, \tag{10}$$

using the notation $T = (T_1, T_2, \cdots, T_n)^\top$ and

$$\text{div}T = (\text{div}T_1, \text{div}T_2, \cdots, \text{div}T_n)^\top,$$

(see also [10]). Observe that

$$\text{tr}T = -\left(\frac{n-2}{2}|\nabla u|^2 + nW(u)\right), \tag{11}$$

and that

$$T+\left(\frac{1}{2}|\nabla u|^2+W\left(u\right)\right)I_n=(\nabla u)^{\top}(\nabla u)\geq 0 \quad \text{(in the matrix sense)}, \quad (12)$$

where I_n stands for the $n\times n$ identity matrix.

As in [1], writing $x=(x_1,\cdots,x_n)$, and making use of (10), we calculate that

$$\sum_{i,j=1}^{n}\int_{B_R}\left(x_iT_{ij}\right)_{,j}dx=\sum_{i,j=1}^{n}\int_{B_R}\left\{\delta_{ij}T_{ij}+x_i(T_{ij})_{,j}\right\}dx=\sum_{i=1}^{n}\int_{B_R}T_{ii}dx. \quad (13)$$

On the other side, from the divergence theorem, denoting $\nu=x/R$, and making use of (12), we find that

$$\begin{aligned}&\sum_{i,j=1}^{n}\int_{B_R}\left(x_iT_{ij}\right)_{,j}dx\\ &\qquad =R\sum_{i,j=1}^{n}\int_{\partial B_R}\nu_iT_{ij}\nu_jdS\geq -R\int_{\partial B_R}\left(\frac{1}{2}|\nabla u|^2+W\left(u\right)\right)dS. \quad (14)\end{aligned}$$

Since W is nonnegative, if $n\geq 3$, we have that

$$\frac{1}{2}|\nabla u|^2+W\left(u\right)\leq\frac{1}{n-2}\left(\frac{n-2}{2}|\nabla u|^2+nW\left(u\right)\right). \quad (15)$$

Let

$$f(R)=\int_{B_R}\left(\frac{n-2}{2}|\nabla u|^2+nW\left(u\right)\right)dx, \quad R>0.$$

By combining (11), (13), (14) and (15), for $n\geq 3$, we arrive at

$$-f(R)\geq-\frac{R}{n-2}\frac{d}{dR}f(R), \quad R>0,$$

which implies that

$$\frac{d}{dR}\left(R^{2-n}f(R)\right)\geq 0, \quad R>0,$$

(clearly this also holds for $n=2$). We have thus shown the first assertion of the theorem.

Suppose that u additionally satisfies Modica's gradient bound (8). Then, we can strengthen (15), for $n\geq 2$, by noting that

$$\begin{aligned}\tfrac{1}{2}|\nabla u|^2+W\left(u\right) &= \tfrac{1}{n-1}\left(\tfrac{n-2}{2}|\nabla u|^2+\tfrac{1}{2}|\nabla u|^2+(n-1)W\left(u\right)\right)\\ &\leq \tfrac{1}{n-1}\left(\tfrac{n-2}{2}|\nabla u|^2+nW\left(u\right)\right).\end{aligned}$$

Now, by combining (11), (13), (14) and the above relation, we arrive at

$$-f(R)\geq-\frac{R}{n-1}\frac{d}{dR}f(R), \quad R>0,$$

which implies that

$$\frac{d}{dR}\left(R^{1-n}f(R)\right)\geq 0, \quad R>0,$$

as desired. □

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