Ο όγκος της B_p^n $1 \le p \le \infty$

$B_p^n = \{x \in \mathbb{R}^n \quad \|x\|_p \le 1\}$

Αν $p = \infty$ $\quad B_\infty^n = [-1\ \ 1]^n$ άρα $\mathrm{vol}(B_\infty^n) = 2^n$

Για $1 \le p < \infty$ θέτουμε $I_p = \int_{\mathbb{R}^n} e^{-\|x\|_p^p}\, dx =$

$$= \int_{\mathbb{R}^n} e^{-|x_1|^p - |x_2|^p - \dots - |x_n|^p}\, dx =$$

$$= \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-|x_1|^p} \underbrace{e^{-|x_2|^p} \cdots e^{-|x_n|^p}}\, dx_1}\, dx_2 \cdots dx_n$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-|x_2|^p} \cdots e^{-|x_n|^p} \underbrace{\left(\int_{\mathbb{R}} e^{-|x_1|^p} dx_1\right)}\, dx_2 \cdots dx_n$$

$$= \left(\int_{\mathbb{R}} e^{-|x_1|^p} dx_1\right) \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \underbrace{e^{-|x_2|^p} \cdots e^{-|x_n|^p}}\, \underline{dx_2 \cdots dx_n}$$

$$= \left(\int_{\mathbb{R}} e^{-|x_1|^p} dx_1\right) \left(\int_{\mathbb{R}} e^{-|x_2|^p} dx_2\right) \cdots \left(\int_{\mathbb{R}} e^{-|x_n|^p} dx_n\right)$$

$$= \left(\int_{-\infty}^{\infty} e^{-|t|^p} dt\right)^n = \left(2 \int_0^\infty e^{-t^p}\, dt\right)^n.$$

$$I_p = \int_{\mathbb{R}^n} \underline{e^{-\|x\|_p^p}}\, dx = \int_{\mathbb{R}^n} \underline{\left(\int_{\|x\|_p}^\infty \frac{d}{dt}\left(-e^{-t^p}\right) dt\right)}\, dx$$

$$= \int_{\mathbb{R}^n} \left(\int_0^\infty \chi_{[\|x\|_p, \infty)}(t)\ p\, t^{p-1} e^{-t^p}\, dt\right) dx$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \chi_{[\|x\|_p\ \infty)}(t)\ \underline{p\, t^{p-1} e^{-t^p}}\, dx \quad dt$$

$$= \int_0^\infty p\, t^{p-1} e^{-t^p} \cdot \left(\int_{\mathbb{R}^n} \chi_{[\|x\|_p\ \infty)}(t)\ dx\right) dt$$

$\chi_{[\|x\|_p\ \infty)}(t) = 1 \iff t \ge \|x\|_p \iff \|x\|_p \le t \iff$

$\iff x \in t B_p^n \iff \chi_{t B_p^n}(x) = 1$

$$= \int_0^\infty p\, t^{p-1} e^{-t^p} \left(\int_{\mathbb{R}^n} \chi_{tB_p^n}(x)\, dx \right) dt$$

$$= \int_0^\infty p\, t^{p-1} e^{-t^p}\, \mathrm{vol}_n(tB_p^n)\, dt =$$

$$= \mathrm{vol}_n(B_p^n) \int_0^\infty p\, t^{n+p-1} e^{-t^p}\, dt$$

Άρα $\mathrm{vol}_n(B_p^n) \int_0^\infty p\, t^{n+p-1} e^{-t^p}\, dt = \left(2 \int_0^\infty e^{-t^p}\, dt \right)^n$

$$\boxed{\mathrm{vol}_n(B_p^n) = \frac{\left(2 \int_0^\infty e^{-t^p}\, dt \right)^n}{\int_0^\infty p\, t^{n+p-1} e^{-t^p}\, dt}}$$

Θέτουμε $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad x > 0$

με ολοκλήρωση κατά παράγοντες $\Gamma(x+1) = x\Gamma(x)$

$$\Big(\Gamma(x+1) = \int_0^\infty t^{(x+1)-1} e^{-t}\, dt = \int_0^\infty t^x e^{-t}\, dt =$$

$$= \int_0^\infty t^x (-e^{-t})'\, dt = t^x(-e^{-t})\Big|_{t=0}^{\infty} - \int_0^\infty (t^x)'(-e^{-t})\, dt$$

$$= \int_0^\infty x\, t^{x-1} e^{-t}\, dt = x\Gamma(x) \Big)$$

$$\Gamma(n+1) = n\Gamma(n) = n\Gamma((n-1)+1) = n(n-1)\Gamma(n-1) = \dots$$

$$\dots = n!$$

$$\int_0^\infty e^{-t^p}\, dt \xrightarrow[\substack{s^{\frac{1}{p}} = t \\ \frac{1}{p} s^{\frac{1}{p}-1} ds = dt}]{s = t^p} \int_0^\infty e^{-s} \frac{1}{p} s^{\frac{1}{p}-1}\, ds = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)$$

$$= \Gamma\left(1 + \frac{1}{p}\right)$$

$$\int_0^\infty p\, t^{n+p-1} e^{-t^p}\, dt = \int_0^\infty \cancel{p}\, (s^{\frac{1}{p}})^{n+p-1} e^{-s} \frac{1}{\cancel{p}} s^{\frac{1}{p}-1}\, ds$$

$$= \int_0^\infty s^{\frac{n}{p} + 1 - \cancel{\frac{1}{p}} + \cancel{\frac{1}{p}} - 1} e^{-s}\, ds = \Gamma\left(1 + \frac{n}{p}\right)$$

$$\boxed{\mathrm{vol}_n(B_p^n) = \frac{\left(2\,\Gamma\left(1+\frac{1}{p}\right)\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}}$$

Τύπος Stirling $\displaystyle\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} = 1$

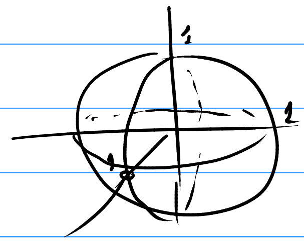
$$\sim \mathrm{vol}_n(B_p^n)^{1/n} = \frac{2\,\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{n}{p}\right)^{1/n}} \approx \frac{2\,\Gamma\left(1+\frac{1}{p}\right)}{\sqrt{2\pi\frac{n}{p}}^{\frac{1}{n}}\left(\left(\frac{n/p}{e}\right)^{\cancel{n}/p}\right)^{\cancel{1/n}}}$$

$$\approx \frac{2\,\Gamma\left(1+\frac{1}{p}\right)}{\sqrt{2\pi\frac{n}{p}}^{1/n}\; n^{1/p}}(ep)^{1/p} \approx \frac{c(p)}{n^{1/p}} \quad \text{δηλ } \exists\, c_1(p),\ c_2(p)>0$$

ώστε $$\boxed{\frac{c_1(p)}{n^{1/p}} \le \mathrm{vol}_n(B_p^n)^{1/n} \le \frac{c_2(p)}{n^{1/p}}}$$

$p=2$ $$\mathrm{vol}_n(B_2^n)^{1/n} \approx \frac{c}{\sqrt{n}}$$

$$\mathrm{vol}_n(B_2^n) \approx \left(\frac{c}{\sqrt{n}}\right)^n$$



Άσκηση 4.2.1

$$\mathrm{vol}_n(B_2^n) = \begin{cases} \dfrac{\sqrt{\pi}^n}{(n/2)!} & \text{αν } n \text{ άρτιος} \\[2ex] 2^n\sqrt{\pi}^{\,n-1}\dfrac{\left(\frac{n-1}{2}\right)!}{n!} & \text{αν } n \text{ περιττός} \end{cases}$$

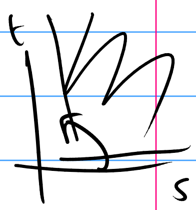
$$\mathrm{vol}_n(B_2^n) = \frac{\left(2\,\Gamma\left(1+\frac{1}{2}\right)\right)^n}{\Gamma\left(1+\frac{n}{2}\right)}$$

$\Gamma(x+1) = x\Gamma(x)$

$$\Gamma\left(1+\tfrac{1}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} e^{-t}\,dt =$$

$$= \tfrac{1}{2}\int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\,dt \qquad \frac{\sqrt{t} = s}{t = s^2 \quad dt = 2s\,ds}$$

$$= \frac{1}{\cancel{2}}\int_0^\infty \frac{1}{\cancel{s}} e^{-s^2}\,\cancel{2}\cancel{s}\,ds = \underbrace{\int_0^\infty e^{-s^2}\,ds}_{I} = \frac{\sqrt{\pi}}{2}$$



$$I^2 = \int_0^\infty e^{-s^2}\,ds \int_0^\infty e^{-t^2}\,dt = \int_0^\infty\int_0^\infty e^{-(s^2+t^2)}\,ds\,dt$$

$$s = r\cos\theta \qquad t = r\sin\theta \qquad ds\,dt = r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \underbrace{\int_0^\infty e^{-r^2}\, r\,dr}\, d\theta =$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\right)'\Big|_{r=0}^{\infty} = \frac{\pi}{2}\left(0 - \left(-\frac{1}{2}e^0\right)\right)$$

$$= \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

$$2\Gamma\left(1+\tfrac{1}{2}\right) = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\boxed{\mathrm{vol}_n(B_2^n) = \frac{\sqrt{\pi}^{\,n}}{\Gamma\left(1+\frac{n}{2}\right)}}$$

n άρτιος $\quad n = 2k \quad \Gamma\left(1+\frac{n}{2}\right) = \Gamma(1+k) = k! = \left(\frac{n}{2}\right)!$

$$\mathrm{vol}_n(B_2^n) = \frac{\sqrt{\pi}^{\,n}}{\left(\frac{n}{2}\right)!}$$

$\Gamma(x+1) = x\Gamma(x)$

n περιττός $\quad n = 2k-1 \quad \Gamma\left(1+\frac{n}{2}\right) = \Gamma\left(1+k-\frac{1}{2}\right) =$

$$= \Gamma\left(k-\tfrac{1}{2}+1\right) = \left(k-\tfrac{1}{2}\right)\Gamma\left(k-\tfrac{1}{2}\right) = \frac{2k-1}{2}\Gamma\left(k-\tfrac{3}{2}+1\right)$$

$$= \frac{2k-1}{2}\left(k-\tfrac{3}{2}\right)\Gamma\left(k-\tfrac{3}{2}\right) = \frac{2k-1}{2}\,\frac{2k-3}{2}\,\Gamma\left(k-\tfrac{5}{2}+1\right)$$

$$\boxed{k=7 \quad \frac{13}{2}\quad \frac{11}{2}\quad \frac{9}{2}\quad \frac{7}{2}\quad \frac{5}{2}\quad \frac{3}{2}\quad \frac{1}{2}}$$

$$= \cdots = \frac{2k-1}{2}\,\frac{2k-3}{2}\,\frac{2k-5}{2}\cdots\frac{1}{2}\,\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{}\overset{}{=}\;\; \left(\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}\right) =$$

$$= \frac{(2k-1)(2k-3)\cdots 1}{2^k}\sqrt{\pi} =$$

$$= \frac{(2k-1)(2k-2)(2k-3)(2k-4)\cdots 1}{2^k\,(2k-2)(2k-4)(2k-6)\cdots 2}\sqrt{\pi}$$

$$= \frac{(2k-1)!}{2^k\,2^{k-1}\,(k-1)(k-2)\cdots 1}\sqrt{\pi} =$$

$$= \frac{n!}{2^{2k-1}\,(k-1)!}\sqrt{\pi} = \frac{n!}{2^n\left(\frac{n-1}{2}\right)!}\sqrt{\pi}$$

$n=2k-1$

$\frac{n+1}{2}=k$

$\frac{n+1}{2}-1=k-1$

$\quad \| $

$\frac{n-1}{2}$

Αρα για n περιττο

$$\mathrm{vol}_n\left(B_2^n\right) = \frac{\sqrt{\pi}^{\,n}}{\dfrac{n!}{2^n\left(\frac{n-1}{2}\right)!}\sqrt{\pi}} = \frac{2^n\sqrt{\pi}^{\,n-1}\left(\frac{n-1}{2}\right)!}{n!}$$

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