On the maximal convex chains among random points in a triangle

Gergely Ambrus[1]   Imre Bárány[2]

[1]University College London &
University of Szeged

[2]MTA Rényi Institute &
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Related problem - Longest increasing subsequences

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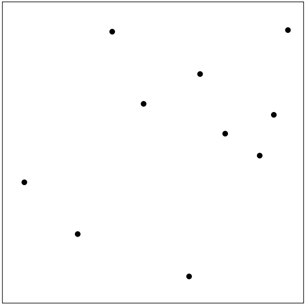
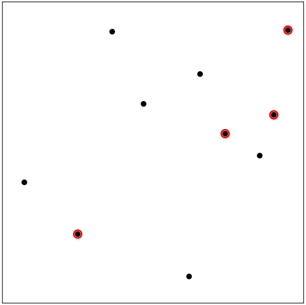
Alternative formulation: take n random points in the unit square. Maximal number of them in increasing position ?

Random points in the square



3 2 10 7 1 8 5 4 6 9

Points in increasing position



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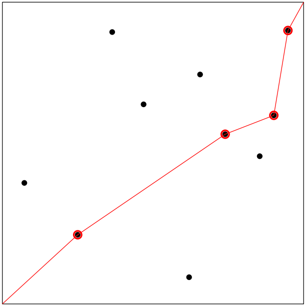
Points form an increasing chain



Points form an increasing chain from $(0, 0)$ to $(1, 1)$

Maximal convex chains

- $T = \operatorname{conv}((0,0),(1,0),(1,1))$

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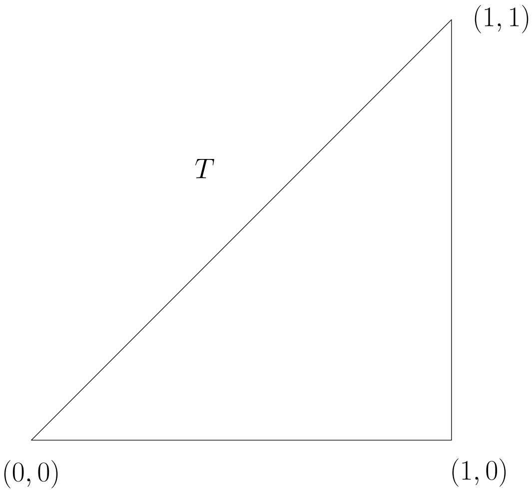
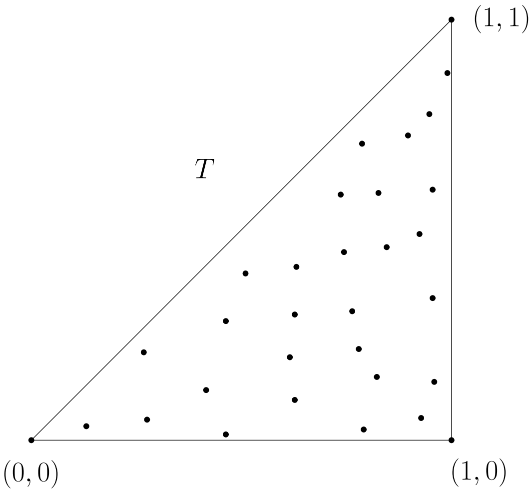
- Choose n independent random points from T with uniform distribution
  $\Rightarrow$ Their set is X_n
- Take the convex chains connecting $(0,0)$ and $(1,1)$ with vertices among the chosen points
  Convex chain: the vertices are in convex position
- Look for such chains with maximal number of vertices (referred as maximal length)

Convex chains can be useful...

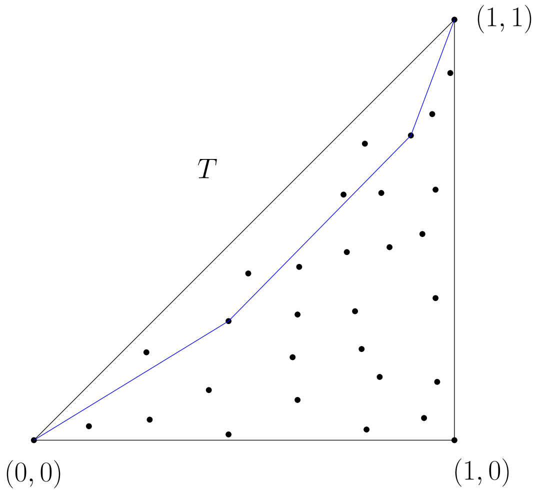
The triangle T



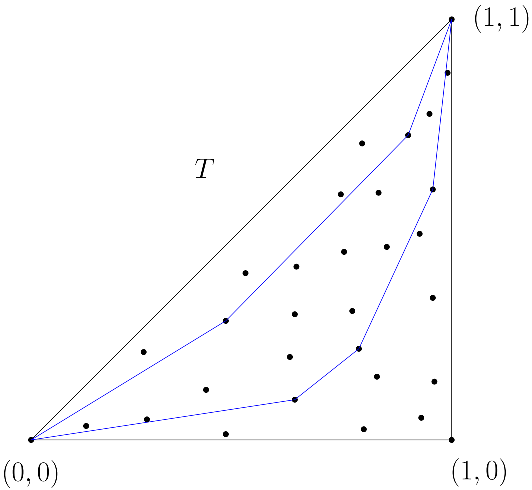
Random points in T



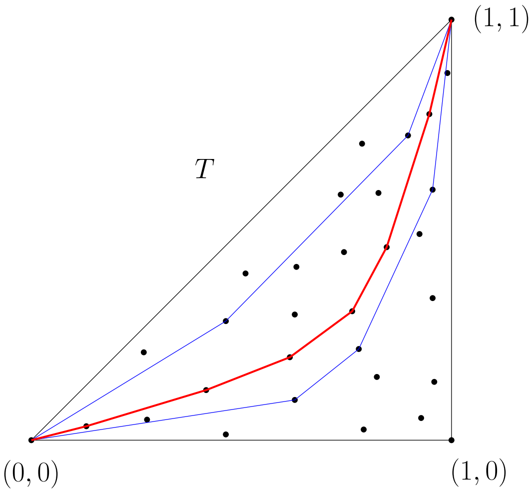
A convex chain



Another convex chain



A maximal convex chain



Maximal convex chains

Definition

L_n- the maximal number of points in X_n which are in convex position with $(0,0)$ and $(1,1)$

M_n- the convex chains of maximal length between $(0,0)$ and $(1,1)$ via X_n

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All of these are affine invariant, just depending on n.

Asymptotic behavior of the expected length

Theorem

There exists a positive constant c *for which*

$$\lim_{n\to\infty} \frac{\mathbb{E}L_n}{\sqrt[3]{n}} = c\,.$$

Proof - Upper bound

$$\mathbb{P}(k \text{ random points form a convex chain}) = \frac{2^k}{k!\,(k+1)!}$$

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If X is a positive random variable, then

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x)\, dx$$

Proof - Upper bound

For any $\gamma > 0$

$$\begin{aligned}
\mathbb{E}L_n &= \sum_{k=0}^{n} \mathbb{P}(L_n > k) \\
&\le \gamma\sqrt[3]{n} + \sum_{k > \gamma\sqrt[3]{n}} \mathbb{P}(L_n > k) \\
&\le \gamma\sqrt[3]{n} + \sum_{k > \gamma\sqrt[3]{n}} \binom{n}{k} \frac{2^k}{k!\,(k+1)!} \\
&\le \gamma\sqrt[3]{n} + n^{-1/2} C(\gamma),
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if $\gamma > \sqrt[3]{2e}$.

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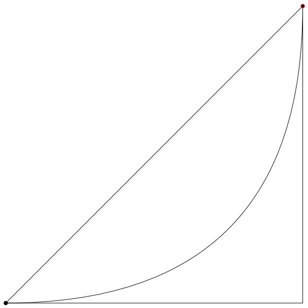
$$\limsup_{n\to\infty} \frac{\mathbb{E}L_n}{\sqrt[3]{n}} \leq \sqrt[3]{2e} = 3.9581\ldots$$

Proof - Lower bound

Parabolic arc Γ:

$$y = (1 - \sqrt{1-x})^2, \; 0 \leq x \leq 1.$$



Proof - Lower bound

Number of points above Γ: $\sim B(n, 2n/3)$.

Binomial r.v.'s are strongly concentrated: $k' \sim B(n,p),\ np = k$:

$$\mathbb{P}(k' \leq k - c\sqrt{k \log k}) \leq k^{-c^2/2}$$

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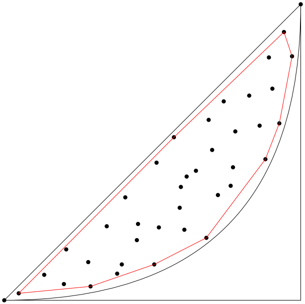
Take the convex hull of the vertices above Γ:



Proof - Lower bound

Rényi-Sulanke '63. n uniform random points in K, $A(K)$ - area of K, $AP(K)$ - affine perimeter of K

Expected number of vertices of the convex hull of the points is

$$\Gamma\left(\frac{5}{3}\right)\sqrt[3]{\frac{2}{3}}\,(A(D))^{-1/3}AP(K)\sqrt[3]{n}$$

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Most vertices are close to Γ, forming a convex chain ⇒

$$\liminf_{n\to\infty}\frac{\mathbb{E}L_n}{\sqrt[3]{n}} \geq 1.5772\ldots$$

Proof - Existence of the limit

Suppose on the contrary that

$$\alpha = \liminf_{n\to\infty} \frac{\mathbb{E}L_n}{\sqrt[3]{n}} < \limsup_{n\to\infty} \frac{\mathbb{E}L_n}{\sqrt[3]{n}} = \beta$$

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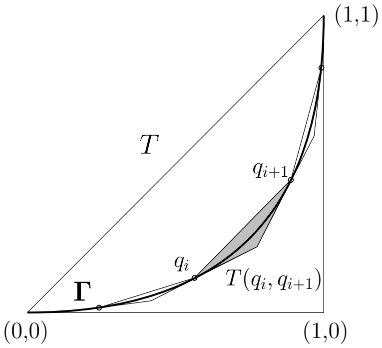
Choose n_2 uniform, independent random points in T (expected number of points in a triangle of area t is N_1)

Proof - Existence of the limit

Definition

For $q_1, q_2 \in \Gamma$, let $T(q_1, q_2)$ be the triangle determined by q_1, q_2 and the intersection point of the tangents to Γ at these points



Proof - Existence of the limit

Take as many points $(0,0) = q_0, q_1, \ldots, q_k = (1,1)$ on Γ as possible, ordered by the x coordinate, such that

$$A(T_i) = t, \; i < k,$$

$$A(T_k) \leq t,$$

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Special property of Γ:

$$\sum_{i=1}^{k} \sqrt[3]{\mathrm{A}(T(q_{i-1}, q_i))} = \sqrt[3]{1/2}$$

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$$k \geq \sqrt[3]{n_2/N_1}$$

Proof - Existence of the limit

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$$
\begin{aligned}
\alpha\sqrt[3]{n_2} \approx \mathbb{E}L_{n_2} &\geq \sum_{i \leq k} \mathbb{E}L'_i \\
&\geq \sum_{i \leq k} \mathbb{P}(k_i > n_1)\mathbb{E}L_{n_1} \\
&\geq \sum_{i \leq k-1} (1 - N_1^{-1/2})(1-\epsilon)\beta\sqrt[3]{n_1} \\
&\geq (\sqrt[3]{n_2/N_1} - 1)(1 - N_1^{-1/2})(1-\epsilon)\beta\sqrt[3]{n_1} \\
&= \beta\sqrt[3]{n_2}(1-\xi) \qquad \square
\end{aligned}
$$

Talagrand inequality

- Y is a real-valued random variable on a product probability space $\Omega^{\otimes n}$
- $|Y(x) - Y(y)| \leq d$ whenever x and y differ at d coordinates
- For any x and b with $Y(x) \geq b$ there exists an index set $\mathfrak{I}$ of at most b elements, such that $Y(y) \geq b$ holds for any y agreeing with x on $\mathfrak{I}$

If m is the median of Y, for any $\gamma > 0$ we have

$$\mathbb{P}(Y \leq m - \gamma) \leq 2 \exp\left(\frac{-\gamma^2}{4r^2 f(m)}\right)$$

$$\mathbb{P}(Y \geq m + \gamma) \leq 2 \exp\left(\frac{-\gamma^2}{4r^2 f(m+\gamma)}\right)$$

Strong concentration for $\mathbb{E}L_n$

Theorem

For every $\gamma > 0$ there exist a constant N, such that for every $n > N$

$$\mathbb{P}(|L_n - \mathbb{E}L_n| > \gamma\sqrt{\log n}\, n^{1/6}) < n^{-\gamma^2/25}.$$

Proof

- Talagrand for L_n:

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- Distance between mean and median:

$$\lim_{n\to\infty} \frac{|\mathbb{E}L_n - m|}{\sqrt{m\ logm}} = 0 \ \Rightarrow \ m < 4\sqrt[3]{n}$$

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$$\begin{aligned}
& \mathbb{P}(|L_n - \mathbb{E}L_n| \geq \gamma\sqrt{\log n}\, n^{1/6}) \\
\leq \quad & \mathbb{P}(|L_n - m| > \gamma\frac{\sqrt{3}}{2}\sqrt{m(\log m - \log 64)}) \\
\leq \quad & m^{-3\gamma^2/21} \\
\leq \quad & n^{-\gamma^2/25}
\end{aligned}$$

□

Concentration in a small triangle

The same result holds when the number of points taken is a binomial random variable:

Theorem

Let $0 \leq t \leq 1$*, and consider the triangle* T' *with vertices* $(0,0)$*,* $(\sqrt{t}, 0)$*,* $(\sqrt{t}, \sqrt{t})$*. Choose* n *independent random points uniformly in* T*, and denote* $L_{t,n}$ *the maximal number of points in* T' *which form a convex chain from* $(0,0)$ *to* $(\sqrt{t}, \sqrt{t})$*. Then for every* $\gamma > 0$ *there exists an* N*, such that for every* $n > N$*,*

$$\mathbb{P}(|L_{t,n} - \mathbb{E}L_{t,n}| > \gamma\sqrt{\log nt}\,(nt)^{1/6}) < (nt)^{-\gamma^2/25}.$$

Strong concentration for the location

The maximal chains are close to Γ with high probability.

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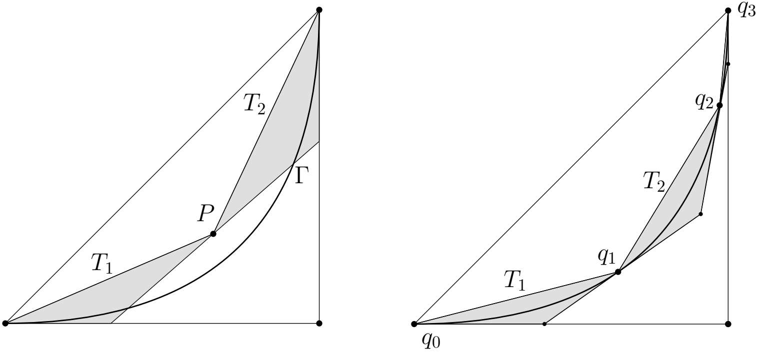
Reason:

For any point P in T,

$$\sqrt[3]{A(T_1)} + \sqrt[3]{A(T_2)} \leq \sqrt[3]{A(T)}$$



Reason - Quantitatively

For a point P far from Γ, $\sqrt[3]{A(T_1)} + \sqrt[3]{A(T_2)}$ is small.

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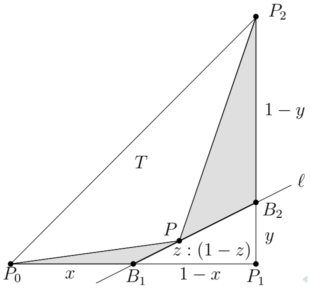
Lemma

Denote the vertices of T by $P_0 = (0,0)$, $P_1 = (1,0)$ and $P_2 = (1,1)$. Suppose that a line ℓ intersects the side P_0P_1 at $B_1 = (x,0)$ and the side P_1P_2 at $B_2 = (1,y)$. Then for any point P of $\ell \cap T$ we have

$$\sqrt[3]{1/2} - (\sqrt[3]{\mathrm{A}(P_0B_1P)} + \sqrt[3]{\mathrm{A}(PB_2P_2)}) > \frac{1}{3}(x-y)^2.$$



How far is Γ?

Definition

The random variable Q is the farthest point of the union of the chains in M_n from Γ, in notation:

$$Q = q \in T \mid \left(\mathrm{dist}(q, \Gamma) = \max_{p \in \cup M_n} \mathrm{dist}(p, \Gamma) \right),$$

Definition

For every point $q \in T$ let q' denote the closest point of Γ to q, and let φ denote the angle of the tangent of Γ at q'. Then Γ_t is the following domain containing Γ:

$$\Gamma_t = \left\{ q \in T \mid \mathrm{dist}(q, q') \leq \sqrt{3}t \frac{\cos\varphi \sin\varphi}{\cos\varphi + \sin\varphi} \right\}.$$

Strong concentration theorem for the location of the maximal chains

Theorem

Let $\gamma > 0$ *and define* $t = \gamma^{1/2} n^{-1/12} (\log n)^{1/4}$. *Then there exists* $N > 0$, *depending on* γ, *such that for any* $n > N$,

$$\mathbb{P}(Q \in \Gamma_t) > 1 - 2n^{-\gamma^2/25}.$$

Proof - location

Consider the following random variable defined on $T^{\otimes n}$:

$$X = \begin{cases} 1 & \text{if } L_n \geq \mathbb{E}L_n - \gamma\sqrt{\log n}\, n^{1/6} \\ 0 & \text{otherwise;} \end{cases}$$

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The conditional expectation $\mathbb{E}(X|Q)$ of X with respect to Q exists:

$$\int_{\{Q\in S\}} X d\mathbb{P} = \int_S \mathbb{E}(X|Q=q)\mu_Q(dq),$$

where $S \in \mathcal{B}(T)$, and μ_Q is the distribution of Q,

$$\mu_Q(S) = \mathbb{P}(Q \in S).$$

Proof - location

Lemma

If n is large enough, then for any $q \in T \setminus \Gamma_t$,

$$\mathbb{E}(X|Q = q) < 1/2.$$

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If n is large enough, then for any $q \in T \setminus \Gamma_t$,

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Let S denote the event $\{Q \in \Gamma_t\}$. Let $p = \mathbb{P}(S)$. Then

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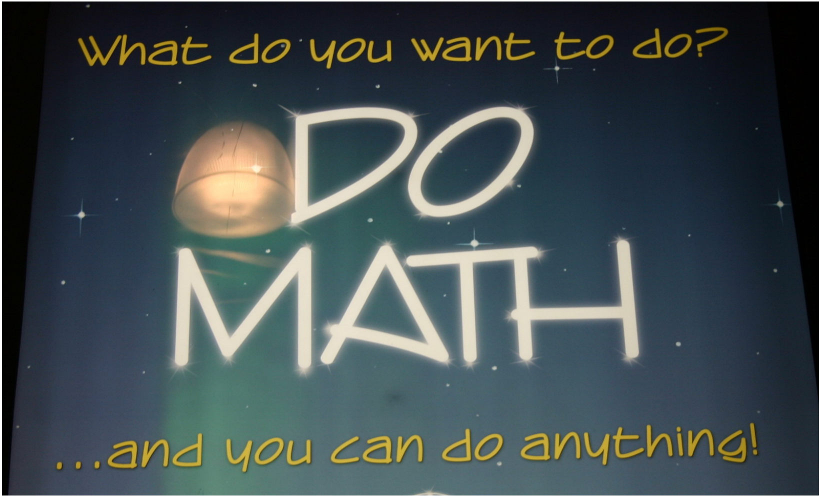
- and so on . . .

Thank you!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