Determination of a set from its covariance: complete confirmation of Matheron's conjecture.

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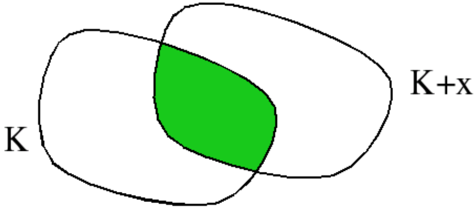
Definition of covariogram

$K \subset \mathbb{R}^n$ compact set, with $K = \overline{K^\circ}$
covariogram (or covariance) of K = function $g_K : \mathbb{R}^n \to \mathbb{R}$ defined as

$$g_K(x) := \mathrm{vol}\,(K \cap (K + x))$$

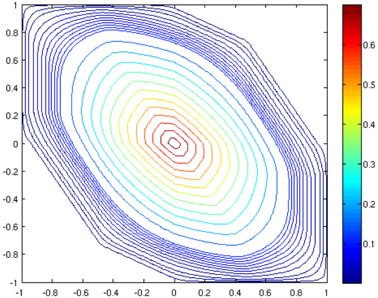


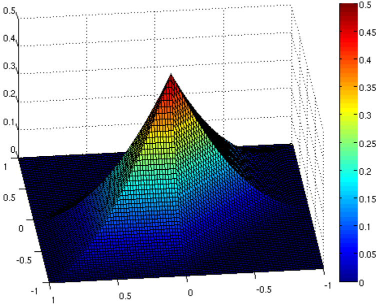
- the covariogram is the autocorrelation of 1_K,

$$g_K = 1_K * 1_{(-K)}$$

Properties

- support of $g_K = K + (-K) = \{x - y \ : \ x, y \in K\}$





- when K is convex:
  - each level set is convex (convolution bodies);
  - $(g_K)^{1/n}$ is concave;
- invariant with respect to translations and reflections (w.r.t. a point) of K.

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- Does the covariogram of a set identify the set (up to translations and reflections)?
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- Result conjectured by G. Matheron in '86, and independently asked by R. Adler and R. Pyke in '91.
- A convex body is determined in a much larger class: the class of compact sets K, with at most two connected components and $K = \overline{K^\circ}$ (G. D'Ercole,'07).

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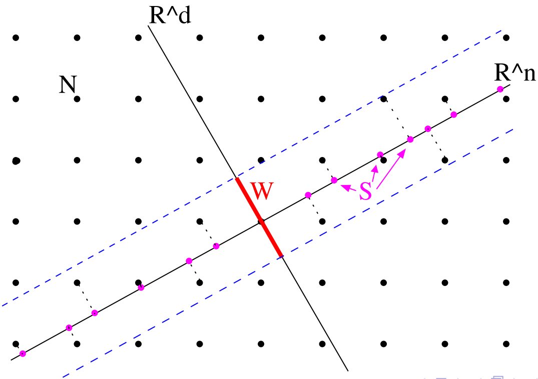
Therefore each of these data identifies K (in the planar convex case)

Theorem

- *The diffraction image of a quasicrystal S determines uniquely the atomic structure of S,*
- *if S fits into the "cut and project scheme" and the "window" associated to S is a planar convex body.*

M. Baake and U. Grimm, Zeitschrift fur Kristallographie, to appear.



Literature on Matheron's problem

- W. Nagel, J. Appl. Probability (1993).
- M. Schmitt, Mathematical Morphology in Image Processing, Dekker, 1993.
- G. Bianchi, F. Segala and A. Volčič, J. Differential Geom. (2002).
- G. Bianchi, J. London Math. Soc. (2005).
  (subclasses of planar convex bodies are determined)
- P. Goodey, R. Schneider and W. Weil, Bull. London Math. Soc. (1997).
  (most convex bodies in $\mathbb{R}^n$ are determined)
- G. Bianchi, 2006 (preprint).
  (convex polytopes in $\mathbb{R}^3$ are determined,
  false for convex polytopes in $\mathbb{R}^n$, $\forall n \geq 4$)

Proof: completing the missing part

Settings

- H and K planar, C^1 and strictly convex bodies with equal covar. g.

Goal

- It suffices to prove that an arc of ∂H is a translate of an arc of ∂K.

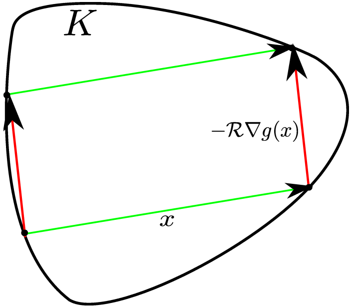
Prerequisites

G. Bianchi, J. London Math. Soc. (2005).

1. If H or K are not strictly convex, or are not C^1, then $H = \pm K + y$.
2. If an arc of ∂H, or of $-\partial H$, is a translate of an arc of ∂K then $H = \pm K + y$.

Step 1: gradient of g and inscribed parallelograms

- $\forall\ x$ there is a parallelogram inscribed in K with edges equal to x and to $-\mathcal{R}\nabla g(x)$. ($\mathcal{R}$=counterclockwise rotation by 90^0)
- A translate of this parallelogram is also inscribed in H. A priori the translation may depend on x.
- $-\mathcal{R}\nabla g(-\mathcal{R}\nabla g(x)) = -x$.



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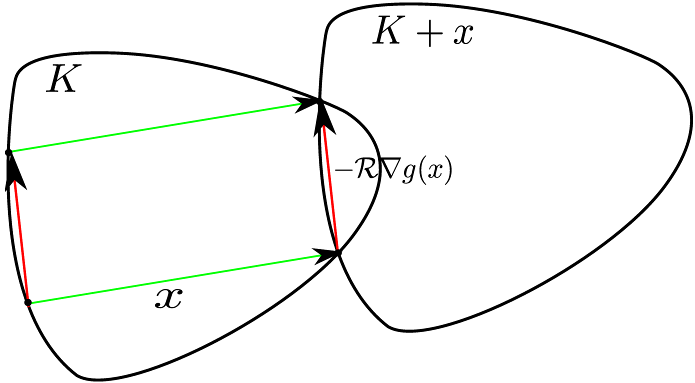
- the vector joining the two points of $\partial K \cap (\partial K + x)$ equals $-\mathcal{R}\nabla g(x)$. ($\mathcal{R}$=counterclockwise rotation by $\pi/2$).



- parallelogram = convenient representation of x and $\nabla g(x)$

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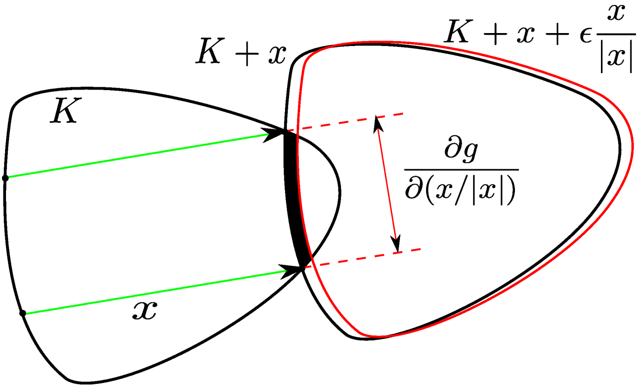
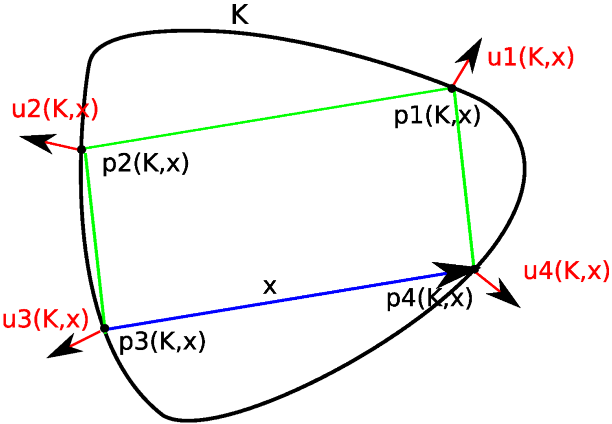


settings



Step 2: second derivatives of g

- The Hessian matrix of g is

$$D^2 g = -\frac{u_2 \otimes u_1}{\det(u_1, u_2)} - \frac{u_3 \otimes u_4}{\det(u_3, u_4)}.$$

- Moreover

$$\det D^2 g = -\frac{\det(u_2, u_3)\det(u_4, u_1)}{\det(u_1, u_2)\det(u_3, u_4)} < 0,$$

$$\det D^2 g + 1 = \frac{\det(u_2, u_4)\det(u_1, u_3)}{\det(u_1, u_2)\det(u_3, u_4)} \lesseqqgtr 0,$$

- the matrix $D^2 g(x)$ depends continuously on x, and

$$^t u_1 (D^2 g)^{-1} u_3 = 0,$$

Step 3: central symmetry and $\det D^2 g$

K centrally symmetric $\iff$ g solves the Monge-Ampere PDE

$$\det D^2 g(x) = -1 \quad \forall x \in \operatorname{int} \operatorname{supp} g \setminus \{o\}.$$

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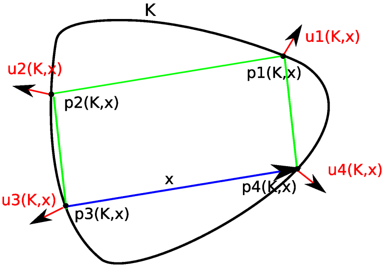

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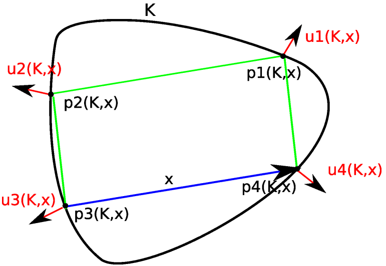
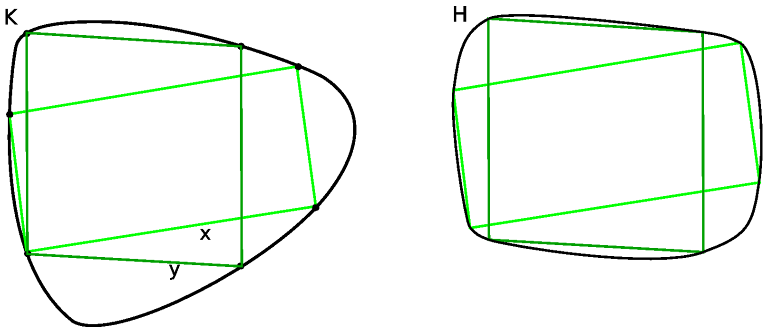


- a diagonal of each inscribed parallelogram is an affine diameter
- This settles the centrally symmetric case: K symmetric iff H symmetric. In this case $K = 1/2 \operatorname{supp} g = H$.

Step 4: controlling the relative position of parallelogr.



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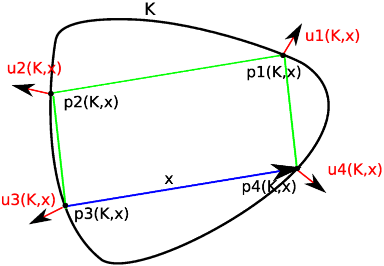
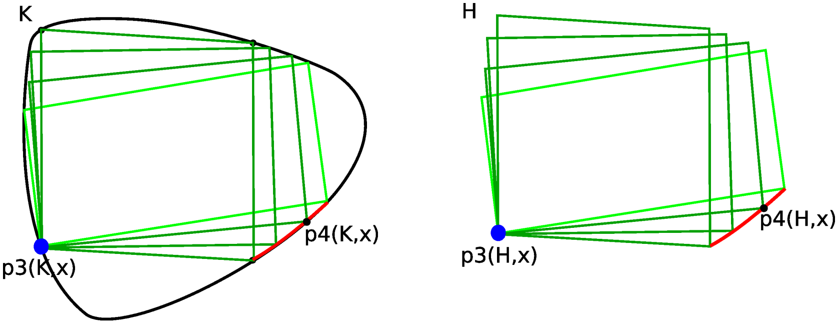


- Assume K and H not centrally symmetric.
- There exists an open set A such that, for each $x \in A$, we have (up to a reflection of H)

$$u_3(K,x) = u_3(H,x) \quad \text{and} \quad u_1(K,x) = u_1(H,x).$$

Step 5: an arc of $\pm\partial H$ is a translate of an arc of ∂K

- Assume step 4.
- Choose all possible $x \in A$ such that $p_3(K,x) =$ given (blue) point.
- Then
$$\bigcup_x p_4(K,x) = \text{red curve on } \partial K.$$
- A translate of this curve is also on ∂H.



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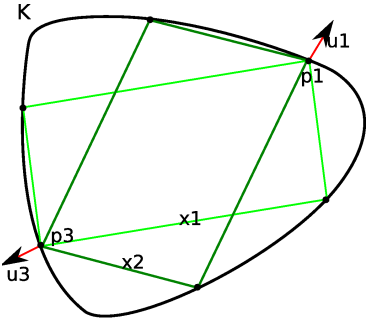

- look for two different translations x_1, x_2 such that the corresponding parallelograms (in K) have the diagonal $[p_1, p_3]$ in common.

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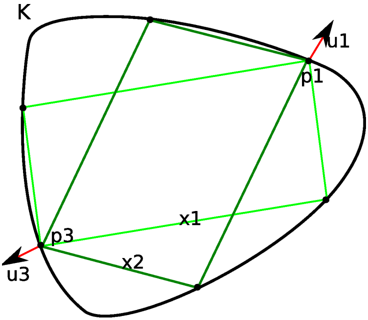
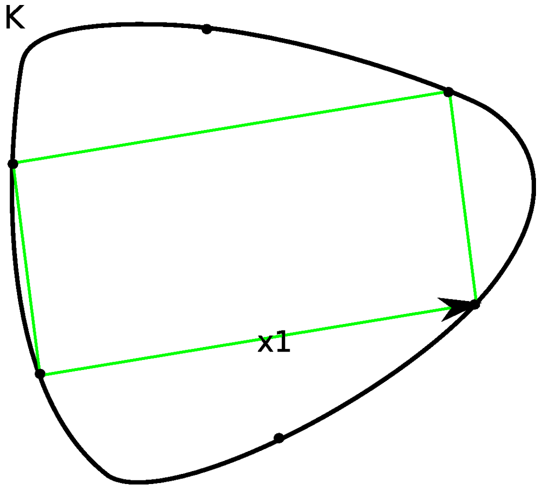


- look for two different translations x_1, x_2 such that the corresponding parallelograms (in K) have the diagonal $[p_1, p_3]$ in common.
- Why? You obtain a system for the normals in p_1 and p_3

$$\begin{cases} {}^t u_1 (D^2 g(x_1))^{-1} u_3 = 0 \\ {}^t u_1 (D^2 g(x_2))^{-1} u_3 = 0 \end{cases}$$

Step 4: controlling the relative position of parallelogr.

- Start from any x_1 such that $\det D^2 g(x_1) \neq -1$



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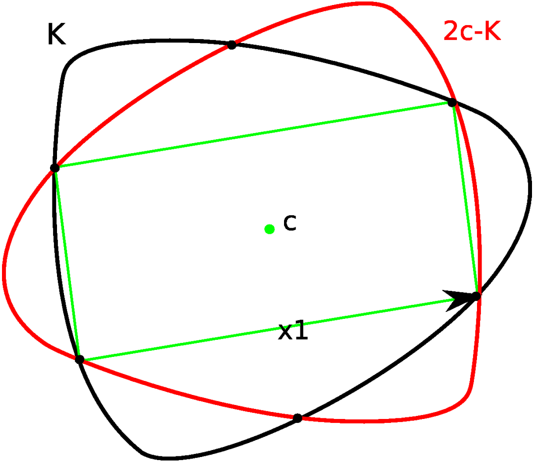

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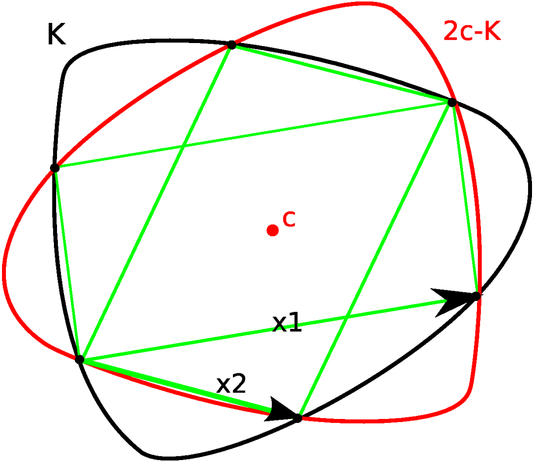

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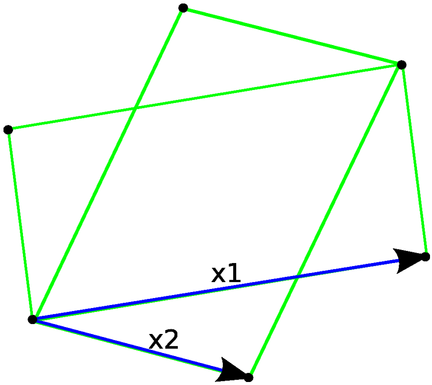
- The inscribability of the configuration below in K depends only on the covariogram.



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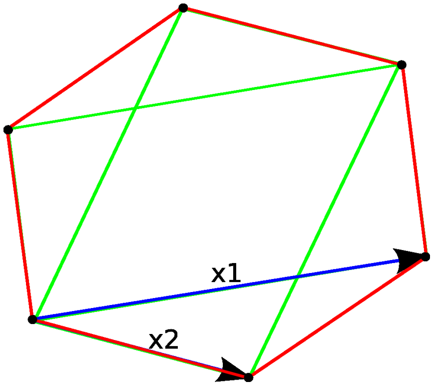

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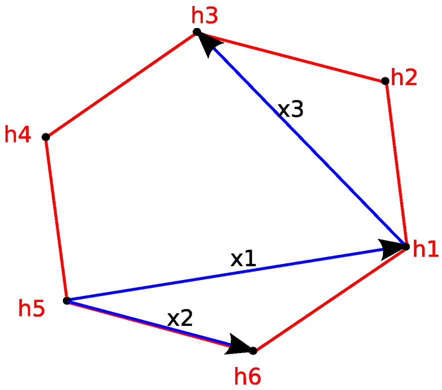

Lemma

A translate of the hexagon on the left can be inscribed in K iff

$$-\mathcal{R}\nabla g(x_1) = h_2 - h_1,$$
$$-\mathcal{R}\nabla g(x_2) = h_2 - h_6,$$
$$-\mathcal{R}\nabla g(x_3) = h_4 - h_3,$$
$$\prod_{i=1,2,3} (\det D^2 g(x_i) + 1) > 0.$$