Estimating intrinsic volumes from finitely many projections

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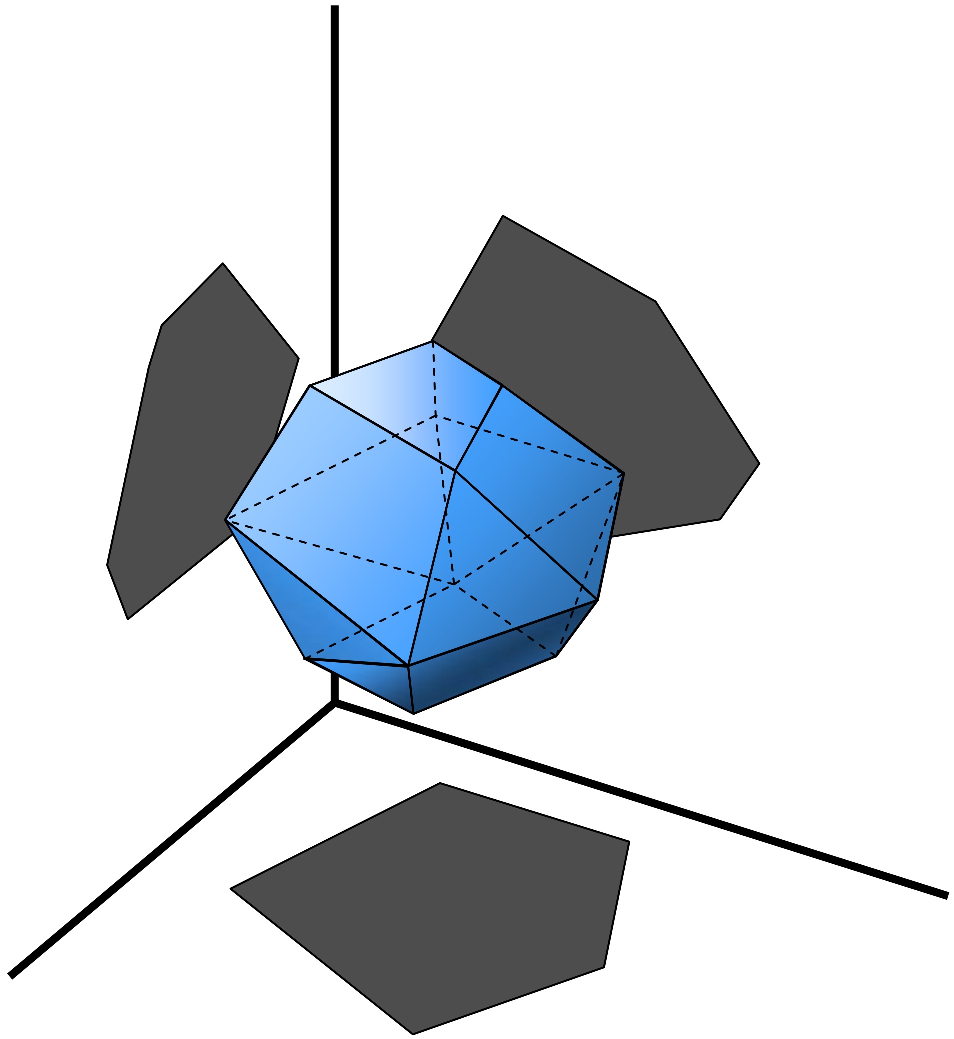
“Geometric tomography … deals with the retrieval of information about a geometric object from data about its sections, or projections, or both.”

(R.Gardner, *Geometric Tomography,* 1995)

A prototype of problem in Geometric tomography is estimating the volume of an object from the areas of its finitely many orthogonal projections.



The Loomis and Whitney inequality is a classic geometric inequality which provides an estimate of that type.

The Loomis and Whitney inequality (1949):

If E is a Borel set in $\mathbb{R}^n$, and $e_1, e_2, \ldots, e_n$ denote the unit vectors in the coordinate directions, then

$$V_n(E)^{n-1} \leq \prod_{i=1}^{n} V_{n-1}\left(E \mid e_i^{\perp}\right),$$

where equality holds for coordinate boxes.

Among convex sets equality holds only for coordinate boxes.

The quantity $V_{n-1}\left(E \mid e_i^{\perp}\right)$ is called the *brightness* of E in the direction e_i .

Generalizations and/or variants in the literature.

Hadwiger (1957), Burago-Zalgaller (1980):
If E is a Borel set in R^n, and $A_1, A_2, \ldots, A_\lambda$, $\lambda = \binom{n}{m}$, denote the coordinate subspaces of dimension m, then

$$V_n(E) \le \prod_{i=1}^{\lambda} V_m(E \mid A_i)^{\frac{n}{\lambda m}} .$$

Betke-McMullen (1982):
For every convex body K in R^n

$$Area(\partial K) \le 2 \sum_{i=1}^{n} V_{n-1}\left(K \mid e_i^{\perp}\right)$$

and equality holds if and only if K is a coordinate box.

Ball (1991):

For a convex body K in R^n

$$V_n(K)^{n-1} \le \prod_{i=1}^{m} V_{n-1}\left(K \mid u_i^{\perp}\right)^{c_i} ,$$

where directions u_i and numbers $c_i > 0$ satisfy

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n .$$

Lutwak-Yang-Zhang (work in progress):

extensions of Ball's results to different intrinsic volumes.

Intrinsic volumes V_i and quermassintegrals W_j.

Steiner's formula:

$$V_n(K+\varepsilon B) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\varepsilon^i = \sum_{i=0}^{n} \kappa_{n-i} V_i(K)\varepsilon^{n-i} ,$$

where κ_j is the volume of the unit ball in R^j.

In particular:

$$V_0(K) = \frac{1}{\kappa_n} W_n(K) = 1, \text{ the Euler characteristic;}$$

$$\frac{2\kappa_{n-1}}{n\kappa_n} V_1(K) = \frac{2}{\kappa_n} W_{n-1}(K), \text{ the mean width;}$$

$$2V_{n-1}(K) = nW_1(K), \text{ the surface area;}$$

$$V_n(K) = W_0(K), \text{ the volume.}$$

Definition. Given a convex body K in R^n, a set of directions U spanning R^n and an integer number $1 \leq r \leq n$-1, let $H_r(K, U)$ the class of all convex bodies L such that

$$V_r\left(L \mid u_i^{\perp}\right) = V_r\left(K \mid u_i^{\perp}\right)$$

for every $i = 1, 2, ..., m$.

Definition. Given a set of directions U in R^n, a *node* for U is a direction orthogonal to at least (n-1) elements from U.

C.-Colesanti-Gronchi (1995):
For every convex body K in R^n, and set of directions U spanning R^n, the only element of maximal volume in $H_{n-1}(K, U)$ is a polytope, having each facet orthogonal to some node.

Moreover, if there exists a zonotope

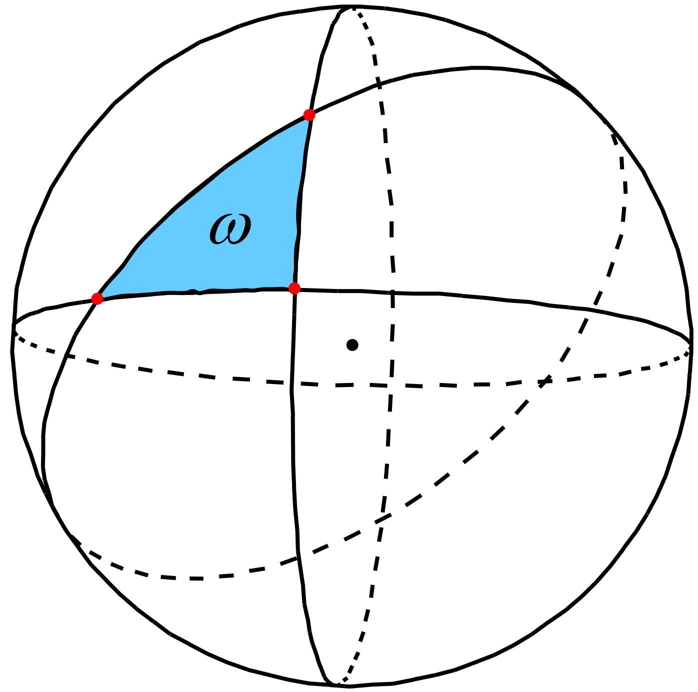
$$Z = \sum_{u \in U} \alpha(u)\, \bar{u}$$

in $H_{n-1}(K, U)$, then it has maximal volume.

Main ingredient: rearrangement of the area measure of K.

Keeping $\int_{\omega} z \, dS_{n-1}(K; z)$

fixed does not change the brightness in the given directions.



Distributing the measure on the nodes increases the volume.

Henceforth, we consider only the case of **convex bodies.**

Class of problems: Finding upper bounds of the r-th intrinsic volume of a convex body K from the s-th intrinsic volume of the projections of K

i) on the $(n-1)$-dimensional coordinate hyperplanes;

ii) on finitely many $(n-1)$-dimensional hyperplanes.

1. $r < s$.

Without any extra assumption, in these cases we cannot expect an upper bound of $V_r(K)$.

For example, for $n = 3$, $s = 2$ and $r = 1$, it is easy to find an unbounded sequence of bodies with fixed brightness along finitely many directions.

2. $r = s = n - 1$

Theorem 1. For every convex body K, there exists a polytope P from $H_{n-1}(K, U)$ of maximal surface area, having each facet orthogonal to some node.

The proof of Theorem 1 uses the same arguments as in CCG for volume.

Notice that, in general, the uniqueness of maximizer is not guaranteed.

Special case: If $U = \{e_1, e_2, ..., e_n\}$ then Theorem 1 gives the LW-inequality proved by Betke and McMullen.

3. $r = 2$, $s = 1$.

Theorem 2. If there exists a zonotope

$$Z = \sum_{u \in U} \alpha(u)\,\overline{u}$$

in $H_1(K, U)$, then Z is the only element of maximal V_2.

The proof is based on the first Minkowski inequality.

In the special case $U = \{e_1, e_2, ..., e_n\}$ the existence of a box in $H_1(K, U)$ can be proved.

Theorem 3. For every convex body K there exists a coordinate box Z such that $V_1\left(Z \mid e_i^{\perp}\right) = V_1\left(K \mid e_i^{\perp}\right)$ for every $i = 1, 2, ..., n$.

Hence, Theorem 2 gives the following corollary

Corollary 4. For every convex body K

$$2V_2(K) \le \frac{1}{n-1}\left(\sum_{i=1}^{n} V_1\left(K \mid e_i^{\perp}\right)\right)^2 - \sum_{i=1}^{n} V_1\left(K \mid e_i^{\perp}\right)^2$$

and equality holds if and only if K is a coordinate box.

4. $r = 1$, $s = 1$.

Theorem 3 can be used also for proving

Theorem 5. For every convex body K

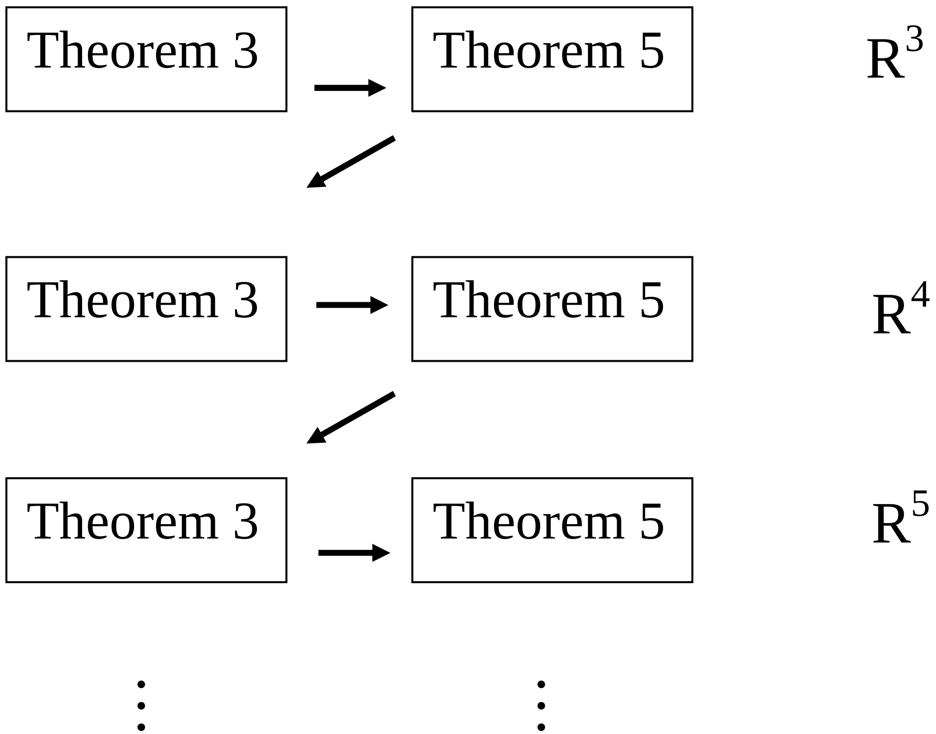
$$V_1(K) \leq \frac{1}{n-1} \sum_{i=1}^{n} V_1\left(K \mid e_i{}^{\perp}\right)$$

and equality holds if and only if K is a coordinate box.

Theorem 3 and Theorem 5 are proved by induction.

Thm 5. $V_1(K) \le \frac{1}{n-1}\sum_{i=1}^{n} V_1\left(K \mid e_i^{\perp}\right)$, equality only for coordinate boxes.

Thm 3. There exists a coordinate box Z such that $V_1\left(Z \middle| e_i^{\perp}\right) = V_1\left(K \middle| e_i^{\perp}\right)$.



To prove that Theorem 3 in R^n implies Theorem 5 in R^n, one can assume that K is symmetric with respect to all the coordinate hyperplanes.

The mean width is an integral of the support function. Thus, the inequality

$$h_K(z) \le \frac{1}{n-1} \sum_{i=1}^{n} h_K\left(z \mid e_i^{\perp}\right),$$

which is an identity for coordinate boxes, implies that

$$V_1(K) \le \frac{1}{n-1} \sum_{i=1}^{n} V_1\left(K \mid e_i^{\perp}\right) = \frac{1}{n-1} \sum_{i=1}^{n} V_1\left(Z \mid e_i^{\perp}\right) = V_1(Z).$$

To prove that Theorem 5 in $\mathrm{R}^{n\text{-}1}$ implies Theorem 3 in R^{n}, one can use the fact that Theorem 3 is equivalent to the following inequality:

$$V_1\left(K \mid e_j^{\perp}\right) \leq \frac{1}{n-1} \sum_{i=1}^{n} V_1\left(K \mid e_i^{\perp}\right).$$

The second step consists in showing that K can be thought of as contained in $e_j^{\perp}$.

Applying Theorem 5 to K in $\mathrm{R}^{n\text{-}1}$ leads to the above inequality.

Final remarks.

- Inequalities of LW type for intrinsic volumes of different order.

- Finitely many directions:
  Rearrangement of the area measure $S_1(K;\cdot)$.
  Counterexample in $\mathbb{R}^3$: Fix $u_1, u_2, \ldots, u_m$ in $S^2 \cap v^\perp$. If it were possible to distribute the measure $S_1(K;\cdot)$ on the arcs of great circles orthogonal to v or u_i, then the resulting body should be a prism.
  For suitable K, the non existence of a cylinder C such that

$$V_1\left(C \mid v^\perp\right) = V_1\left(K \mid v^\perp\right) \text{ and}$$

$$V_1\left(C \mid u^\perp\right) = V_1\left(K \mid u^\perp\right), \text{ for every } u \in v^\perp,$$

contradicts that possibility.