Equivalence of certain norms on finite-dimensional vector spaces

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Let $N >> 1$ and consider the cube $[0, N]^n$ in $\mathbb{R}^n$. This is made up of N^n unit cubes Q_α indexed by $\alpha \in \{1, 2, \ldots, N\}^n := \Omega = \Omega_N^n$.

We consider sequences (x_α) indexed by $\alpha \in \Omega$, and, in addition to the usual norms $\|x\|_q := \{\sum_{\alpha \in \Omega} |x_\alpha|^q\}^{1/q}$, we wish to consider another norm measuring certain geometric properties.

Thus define

$$\|\!|x|\!\| := \sup_T \sum_{\alpha : Q_\alpha \cap T \neq \emptyset} |x_\alpha|.$$

Here, T ranges over the family of all **tubes** in $\mathbb{R}^n$, i.e. doubly infinite cylindrical tubes with say "circular" or "square" cross section and with cross-sectional diameter say 1.

Clearly $\|\!|\cdot|\!\|$ defines a norm on the N^n -dimensional space of sequences indexed by Ω.

So there exist constants k_q and K_q (depending on N and n) so that

$$k_q |||x||| \leq \|x\|_q \leq K_q |||x|||.$$

It is the behaviour as $N \to \infty$ of these constants which interests us.

Now k_q is uninteresting and $k_q \sim C_{n,q} N^{-1/q'}$.

More interesting is K_q. We trivially get $K_q \leq C_{n,q} N^{(n-1)/q}$ and the question is whether this is the sharp estimate. The obvious place to start is to examine the proof of the upper bound and see if we can force essential equality.

Now

$$
\begin{aligned}
\|x\|_q^q &= \sum_{\alpha\in\Omega} |x_\alpha|^q \\
&= \sum_{T \text{ vertical}} \ \sum_{\alpha: Q_\alpha \cap T \neq \emptyset} |x_\alpha|^q \\
&\leq \sum_{T \text{ vertical}} \left(\sum_{\alpha: Q_\alpha \cap T \neq \emptyset} |x_\alpha| \right)^q \\
&\leq N^{(n-1)} \left(\sup_{T \text{ vertical}} \ \sum_{\alpha: Q_\alpha \cap T \neq \emptyset} |x_\alpha| \right)^q \\
&\leq N^{(n-1)} \|\!|x|\!\|^q.
\end{aligned}
$$

Think of the x_α as all being 0 or 1. (Equivalent up to factors of $\log N$.)

The first inequality is essentially sharp if and only if for each vertical tube T the sequence of (x_α) "meeting" T is supported at a point (or at $O(1)$ points).

The second inequality

$$\sum_{T \text{ vertical}} \left(\sum_{\alpha : Q_\alpha \cap T \neq \emptyset} |x_\alpha| \right)^q$$
$$\leq N^{(n-1)} \left(\sup_{T \text{ vertical}} \sum_{\alpha : Q_\alpha \cap T \neq \emptyset} |x_\alpha| \right)^q$$

is essentially sharp if and only if the occupancy across vertical tubes by (x_α) is approximately constant.

The third inequality

$$N^{(n-1)} \left(\sup_{T \text{ vertical}} \sum_{\alpha : Q_\alpha \cap T \neq \emptyset} |x_\alpha| \right)^q$$
$$\leq N^{(n-1)} \|\!| x |\!\|^q$$

is sharp if and only if the maximal tube occupancy over vertical tubes is essentially maximal tube occupancy over *all* tubes. (Or we could have just replaced "vertical" above by "direction of maximal tube occupancy" at the expense of a few constants depending on n.)

Thus, if we had an arrangement of points in Ω with approximately an absolute constant number of points in each parallel translate of the fixed tube which achieves the sup in the definition of $\|x\|$, then we would have a lower bound $K_q \geq C_{n,q} N^{(n-1)/q}$.

This means that if we could find an arrangement of at least ϵN^{n-1} points in Ω such that no tube meets more than $O(1)$ of them, we would have

$$K_q \sim N^{(n-1)/q}$$

Proposition 1 *We have*

$$c_{n,q} \frac{N^{(n-1)/q}}{(\log N)^{1/q'}} \leq K_q \leq C_{n,q} N^{(n-1)/q}.$$

More generally, given $1 \leq k \leq N$, what is the maximal number of points $\mathcal{A}_n(N,k)$ one can pick from Ω such that for all tubes T, at most k of these points lie in T ?

Or:

How much mass can you put into space so that not too much lies in any one tube?

Proposition 2 $\mathcal{A}_n(N, \log N) \geq C_n(\log N)N^{n-1}$.

The first proposition follows directly from this one.

Why are we interested in this question?

(i) X-ray tomography

Suppose we have an object G in space through which we are able to shoot X-rays and for which we are able to measure the amount of mass of G through which each X-ray has passed. Suppose this process yields an upper bound for the amount of mass in the path of each ray. Can we give a good upper bound for the total mass of the object?

(ii) Geometric measure theory

A set $E \subseteq \mathbb{R}^n$ is **tube-null** if, for every $\epsilon > 0$, there exists a cover of E by *tubes* T (of temporarily arbitrary cross section) such that $\sum |T| < \epsilon$. Thus a tube-null set is "small". It's quite easy to show that a set of Hausdorff dimension less than $(n-1)$ – or more generally a set of positive σ-finite $(n-1)$-dimensional measure must be tube-null. And it's easy to construct non-tube-null sets of any dimension greater than $n - 1/2$.

Proposition 3 *For each $s > n-1$ there exists a non-tube-null set E of positive finite s-dimensional Hausdorff measure.*

This follows upon bulding a suitable Cantor set upon the (weaker version of) Proposition 2 which says for each $0 < t < 1$,

$$\mathcal{A}(N, N^t) \geq C_{n,t} N^t N^{n-1}.$$

(iii) Harmonic analysis

An important operator in harmonic analysis is the *extension operator for the Fourier transform.* It's not too crucial for us exactly what this is, but (in case you're interested) it is the operator

$$g \mapsto \widehat{gd\sigma}(x) = \int_{\Sigma(\mathcal{U})} g(\omega) e^{-2\pi i x\cdot\omega} d\sigma(\omega)$$

where $\widehat{\cdot}$ denotes the Fourier transform, and σ is the measure whose action on test functions is given by

$$\langle \phi, \sigma \rangle = \int_{\mathcal{U} \subseteq \mathbb{R}^{n-1}} \phi(\Sigma(t)) dt$$

where $\Sigma : \mathcal{U} \to \mathbb{R}^n$ is a smooth parametrisation of a compact hypersurface in $\mathbb{R}^n$ with nonvanishing gaussian curvature such as the sphere or the base of a paraboloid.

The main point is that there is a conjecture of Mizohata and Takeuchi coming from PDEs to the effect that

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w(x)dx \leq C \sup_T w(T) \int_{\mathcal{U}} |g|^2 dt.$$

On the other hand, there is a theorem of Stein and Tomas (or, in PDE language, a Strichartz estimate) which says:

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w(x)dx \leq C\|w\|_{(n+1)/2} \int_{\mathcal{U}} |g|^2 dt.$$

So in order to test the M-T conjecture on interesting examples it is necessary to find examples of weights w so that

$$\sup_T w(T) << \|w\|_{(n+1)/2},$$

or, such that the maximal amount of mass of w in any one tube is small compared with its total mass.

Back to the main story....

So what do we know about $\mathcal{A}(N,k)$? Recall this is the maximum number of points one can pick from Ω_N so that no more than k of them meet any 1-tube.

Some easy facts:

- $\mathcal{A}(N,k) \leq C_n k N^{n-1}$. (Consider vertical tubes).

- $\mathcal{A}(N,2) \geq C_n N^{(n-1)/2}$. (Use curvature of the sphere.)

- For $N^{1/2} \leq k \leq N$, $\mathcal{A}(N,k) \geq C_n k N^{n-1}$. (Aggregate of rigid "curvature" examples over disjoint "spheres".)

In the last example it did not matter which orientations were chosen for each of the spheres. Can we choose the orientations of the rigid curvature examples intelligently in order to lessen the maximal tube occupancy?

If we choose the orientations at random, then the probability that a *given* tube contains *many* points will be very small, and there is only a limited number $N^{2(n-1)}$ of distinct tubes. So with high probability, no tube contains too many points.

More formally, doing Bernoulli trials/large deviation analysis yields Proposition 2. Thus for $\log N \le k \le N$ we have $\mathcal{A}(N,k) \sim C_n k N^{n-1}$.

It also shows that for $1 \le k \le \log N$, $\mathcal{A}(N,k) \ge C_n k N^{n-1} N^{-\Delta_n/k}$ where Δ_n is at least as big as $2(n-1)$. For example when $k = 2$ we do **not** recover the easy result $\mathcal{A}(N,2) \ge C_n N^{(n-1)/2}$.

Proposition 4 *There is an absolute constant* C_n *such that for* $1 \leq k \leq \log N$*,*

$$\mathcal{A}(N,k) \geq C_n N^{n-1} k N^{-(n-1)/k}.$$

So, when $k = 2$ we recover the "trivial" result $\mathcal{A}(N,2) \geq C_n N^{(n-1)/2}$.

More interesting are larger values of k.

When $k = 3$, we get

$$\mathcal{A}(N,3) \geq C_n N^{2(n-1)/3},$$

when $k = 4$, we get

$$\mathcal{A}(N,4) \geq C_n N^{3(n-1)/4},$$

etc., and when $k \geq \log N$ we recover the large deviation estimate

$$\mathcal{A}(N,k) \sim C_n k N^{n-1}.$$

The estimate $\mathcal{A}(N,3) \geq C_n N^{2(n-1)/3}$, at least when $n = 2$, follows from a result of W. M. Schmidt.

Although I am not aware of a proof of the higher k case in the literature, it seems likely that either it is known, or at least that the method of Schmidt will apply, so I claim no serious originality for Proposition 4.

The argument for Proposition 4 is pretty. It is adapted from an argument of Komlós, Pintz and Szemerédi concerning the Heilbronn problem – with which our problem is closely related.

The *original* result of KPS in particular implies that $\mathcal{A}_2(N, 2) \geq N^{1/2}(\log N)^{1/2}$ – a logarithmic improvement on the "trivial" result – and still the best result for $\mathcal{A}_2(N, 2)$ and for lower bounds in the Heilbronn problem as far as I know. Komlós, Pintz and Szemerédi also have the best result for upper bounds in the Heilbronn problem.

Proof of Proposition 4

First choose $k \geq 3$ points in Ω independently and uniformly at random. Then

$\mathbb{P}\{$*each* such point is in a *given* $T\} \sim N^{-k(n-1)}$.

Since there are about $N^{2(n-1)}$ different T's, then

$\mathbb{P}\{$*each* such point is in a *some* $T\} \leq N^{(2-k)(n-1)}$.

Now let $M \geq k$ and pick a set of M points in Ω independently and uniformly at random. So for each k-element subset $\{p_1, \ldots, p_k\}$ of this set,

$$\mathbb{E}(\chi_{\{p_1,\ldots,p_k \text{ all lie in some } T\}}) \leq N^{(2-k)(n-1)}.$$

There are $\binom{M}{k}$ ways σ of choosing k points $i_1, \ldots, i_k$ from $\{1, 2, \ldots, M\}$. So

$$\sum_{\sigma} \mathbb{E}(\chi_{\{p_{i_1},\ldots,p_{i_k} \text{ all lie in some } T\}}) \leq \binom{M}{k} N^{(2-k)(n-1)}$$

that is,

$$\mathbb{E}(\sum_\sigma \chi_{\{p_{i_1},\ldots,p_{i_k} \text{ all lie in some } T\}})$$

$= \mathbb{E}(\#k$-element subsets *all* of whose members lie in some T)

$$\leq \binom{M}{k} N^{(2-k)(n-1)}.$$

Therefore there exists a set S, $\#S = M$, $S \subseteq \Omega$ such that the number of k-element subsets of S, all of whose members lie in some T is

$$\leq \binom{M}{k} N^{(2-k)(n-1)}.$$

Call a k-element subset of S **bad** if all its members lie in some tube. Then the number of bad k-element subsets of S is

$$\leq \binom{M}{k} N^{(2-k)(n-1)}.$$

For each bad subset of S remove one point of S, resulting in a subset $S' \subseteq S$ with

$$\#S' \geq \#S - \binom{M}{k} N^{(2-k)(n-1)} = M - \binom{M}{k} N^{(2-k)(n-1)}$$

such that *no* k-element subset of S' lies in any tube, i.e. so that no tube contains more than $(k-1)$ members of S'.

Given k and N we want to maximise

$$M - \binom{M}{k} N^{(2-k)(n-1)}$$

over $M \geq k$. Using Stirling's formula we can make this as big as $M/2$ provided

$$M \leq CkN^{(n-1)(k-2)/(k-1)}.$$

Choosing M to be about this value, we see that S' is a set of cardinality

$$CkN^{(n-1)(k-2)/(k-1)}$$

and no tube contains more than $k-1$ points of S'.

Many problems in this area remain open and are interesting for other areas of mathematics. For example, the Heilbronn problem:

Given n, consider possible arrangements of n points in the unit cube $Q \subseteq \mathbb{R}^2$. The aim is to find, amongst all such arrangements, the maximum of the minimal areas of the $\binom{n}{3}$ triangles with vertices in the arrangement.

Komlós, Pintz and Szemerédi showed that there is an arrangement for which all triangles have area $\geq C \log n/n^2$ (and also hold the record for the best upper bound $O(n^{-8/7+\epsilon})$, for which the "trivial" estimate is $O(1/n)$.)

Previous work was done by Roth and Schmidt.

Narrowing the gap here is an interesting problem.

So no triangle with vertices in the set can be contained in a tube of width $(\log n)/n^2$, which means that, taking $N = n^2/\log n$, there exists an arrangement of $n \sim N^{1/2}$ points in Ω_N with no more than two in any 1-tube, i.e. that $\mathcal{A}(N, 2) \geq CN^{1/2}(\log N)^{1/2}$.