Isoperimetric inequalities for uniformly log-concave measures and uniformly convex bodies

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Introduction

Isoperimetry

- $(\mathbb{R}^n, \|\cdot\|, \mu)$; μ - Borel probability measure on $(\mathbb{R}^n, \|\cdot\|)$, absolutely continuous.
- Minkowski's boundary measure of Borel set A:

$$\mu^+_{\|\cdot\|}(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon,\|\cdot\|}) - \mu(A)}{\varepsilon},$$

$$A_{\varepsilon,\|\cdot\|} = \{x \in \mathbb{R}^n \mid \exists y \in A , \|x - y\| < \varepsilon\}$$

- $(\mathbb{R}^n, \|\cdot\|, \mu)$ satisfies an *isoperimetric inequality*:

$$\mu^+_{\|\cdot\|}(A) \geq I\left(\widetilde{\mu(A)}\right) \quad , \quad I : [0, 1/2] \to \mathbb{R}_+$$

$$\widetilde{\mu(A)} := \min(\mu(A), 1 - \mu(A))$$

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$$\widetilde{\mu(A)} := \min(\mu(A), 1 - \mu(A)) \quad \left(\mu^+_{\|\cdot\|}(A) = \mu^+_{\|\cdot\|}(A^C)\right)$$

Concentration

- By integrating an isoperimetric inequality, we obtain an equivalent global version. More generally:
- $(\mathbb{R}^n, \|\cdot\|, \mu)$ satisfies a *concentration inequality* if $\exists \alpha \geq 0 \; \exists \beta, c_1, c_2 > 0$:

$$\forall A \quad \mu(A) = \frac{1}{2} \implies \mu(A_{\varepsilon,\|\cdot\|}) \geq 1 - c_1 \exp(-c_2 n^{\alpha} \varepsilon^{\beta})$$

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- $$\text{Isoperimetry} \quad \not\Leftarrow \quad \text{Concentration} \quad (\text{e.g. } c_1 > 1/2)$$

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- Isoperimetry $\Rightarrow$ Functional Inequalities $\Rightarrow$ Concentration

Example

$(\mathbb{R}^n, |\cdot|, \gamma_n)$; γ_n - standard Gaussian density on $(\mathbb{R}^n, |\cdot|)$.

Thm (Sudakov-Tsirel'son, Borell 1974)

$(\mathbb{R}^n, |\cdot|, \gamma_n)$ satisfies the isoperimetric inequality:

$$\gamma^+_{n,|\cdot|}(A) \geq \varphi \circ \Phi^{-1}(\widetilde{\gamma_n(A)})$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \Phi(y) = \int_{-\infty}^{x} \varphi(x) dx$$

$$\varphi \circ \Phi^{-1}(x) \simeq x \log^{\frac{1}{2}}(1/x) \quad \forall x \in [0, 1/2]$$

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Corollary

$(\mathbb{R}^n, |\cdot|, \gamma_n)$ satisfies the concentration inequality:

$$\forall A \quad \gamma_n(A) = \frac{1}{2} \implies \gamma_n(A_{\varepsilon,|\cdot|}) \geq 1 - 1/2 \exp(-\varepsilon^2/2)$$

Main Results

M. Gromov–V. Milman Concentration for Uniformly Convex Spaces

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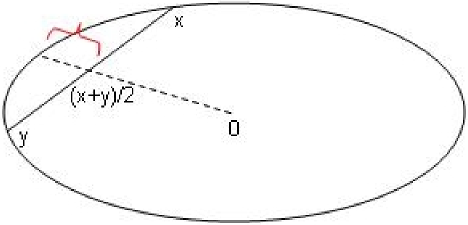
<u>Def:</u> *modulus of convexity* of $(X, \|\cdot\|)$: $\delta_X, \delta_{\|\cdot\|} : [0,2] \to [0,1]$

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| ; \|x\|, \|y\| \leq 1 \quad \|x-y\| \geq \varepsilon \right\}$$



$K_X, K_{\|\cdot\|}, K = \{\|x\| \leq 1\}$

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- $(X, \|\cdot\|)$ or K_X are *uniformly convex* if $\delta_X(\varepsilon) > 0 \;\; \forall \varepsilon > 0$.
- $(X, \|\cdot\|)$ or K_X are *p-convex* (α) if $\delta_X(\varepsilon) \geq \alpha \varepsilon^p \;\; \forall \varepsilon > 0$.
- Example - ℓ_q spaces $(1 < q < \infty)$:

$$\begin{cases} 2 \leq q < \infty & \text{q-convex } (\alpha = q/2^q) \\ 1 < q \leq 2 & \text{2-convex } (\alpha = q-1) \end{cases}$$

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Gromov-Milman Concentration Vs. Isoperimetry

Thm (Gromov–Milman 1987, Arias-de-Reyna–Ball–Villa 1998)

$(\mathbb{R}^n, \|\cdot\|, \lambda_K)$; λ_K = uniform probability measure on $K = K_{\|\cdot\|}$;
Satisfies a concentration inequality:

$$\forall A \quad \lambda_K(A) = \frac{1}{2} \implies \lambda_K(A_{\varepsilon,\|\cdot\|}) \geq 1 - 2\exp(-2n\delta_{\|\cdot\|}(\varepsilon))$$

Thm 1 (M.–Sodin 2007)

Essentially an isoperimetric version for $(\mathbb{R}^n, \|\cdot\|, \lambda_K)$:

- If K is p-convex (α):
$$\lambda^+_{K,\|\cdot\|}(A) \geq c\alpha^{1/p} n^{1/p} \widetilde{\lambda_K(A)} \log^{1-1/p} \frac{1}{\widetilde{\lambda_K(A)}}.$$
- Analogous version for general modulus δ_K.

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Corollaries of Thm 1

- Essentially recovers the Gromov–Milman concentration.

If K is p-convex (α):

$$\lambda_K(A) = \frac{1}{2} \implies \lambda_K(A_{\varepsilon,\|\cdot\|}) \geq 1 - \exp\left(-\left(\log^{\frac{1}{p}} 2 + \frac{c\alpha^{\frac{1}{p}} n^{\frac{1}{p}} \varepsilon}{p}\right)^p\right)$$

$$\text{for large } \varepsilon \quad = 1 - \exp(-(c'/p)^p n\, \alpha\varepsilon^p)$$

$$\text{(GM)} \quad \geq 1 - c_1 \exp(-c_2 \quad n\, \alpha\varepsilon^p)$$

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$$\text{(BZ)} \quad \lambda^+_{K,\|\cdot\|}(A) \geq c\alpha^{1/p} n^{1/p} (\widetilde{\lambda_K(A)})^{\frac{n-1}{n}},$$

$$\text{(MS)} \quad \lambda^+_{K,\|\cdot\|}(A) \geq c\alpha^{1/p} n^{1/p} \widetilde{\lambda_K(A)} \log^{1-1/p} \frac{1}{\widetilde{\lambda_K(A)}}.$$

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- Recovers (up to constants) a log-Sobolev-type functional inequality for p-convex bodies of Bobkov–Ledoux (2000).

Log-Concave Measures

- $d\mu = f(x)dx$ is *log-concave* if $f(x) = \exp(-g(x))$, $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex.
- Examples: $1_K(x)dx$; c $\exp(-\|x\|^p)$ dx, $p \geq 1$.
- Known analogy between convex bodies and log-concave measures.
- $(\mathbb{R}^n, \|\cdot\|, \mu)$; *modulus of log-concavity* of μ w.r.t. $\|\cdot\|$: $\delta_{\mu,\|\cdot\|} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$:

$$\delta_{\mu,\|\cdot\|}(\varepsilon) := \inf \left\{ \frac{g(x)+g(y)}{2} - g\left(\frac{x+y}{2}\right) \middle| \begin{array}{c} g(x), g(y) < \infty \\ \|x-y\| \geq \varepsilon \end{array} \right\}$$

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Uniformly Log-Concave Measures

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- μ is p-log-concave ($\alpha = \alpha_{\|\cdot\|}$) if $\delta_{\mu,\|\cdot\|}(\varepsilon) \geq \alpha\varepsilon^p \quad \forall \varepsilon > 0$.

Examples:

- γ_n ; $g(x) = \frac{|x|^2}{2} + c \Rightarrow$ (Parall. iden.) $\delta_{\gamma_n,|\cdot|}(\varepsilon) = \varepsilon^2/8$.
- $(\mathbb{R}^n, \|\cdot\|)$ is p-convex $\Leftrightarrow$ (using Figiel–Pisier) $d\mu = c\exp(-\|x\|^p)$ is p-log-concave.
- $1_K(x)dx$ is never uniformly log-concave.

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Observation: $\delta_{\mu\phi,\|\cdot\|} \geq \delta_{\mu,\|\cdot\|}$ for any log-concave ϕ, norm $\|\cdot\|$.

Thm 2

Thm 2 (M.–Sodin 2007)

$(\mathbb{R}^n, \|\cdot\|, \mu)$, μ is uniformly log-concave, $\delta = \delta_{\mu,\|\cdot\|}$:

$$\mu^+_{\|\cdot\|}(A) \geq C_\delta \widetilde{\mu(A)} \gamma\left(\log \frac{1}{\widetilde{\mu(A)}}\right) \quad \gamma(t) = \frac{t}{\delta^{-1}(t)}$$

(explicit expression for C_δ; for $\delta(\varepsilon) \geq \alpha\varepsilon^p$, $C_\delta \geq c > 0$).

Corollaries for $\delta(\varepsilon) \geq c\varepsilon^2 \Rightarrow \gamma(t) = \sqrt{t}$

- $(\mathbb{R}^n, |\cdot|, \gamma_n)$ - recovers (up to constants) Gaussian isoperimetry (Sudakov–Tsirel'son, Borell 1974).
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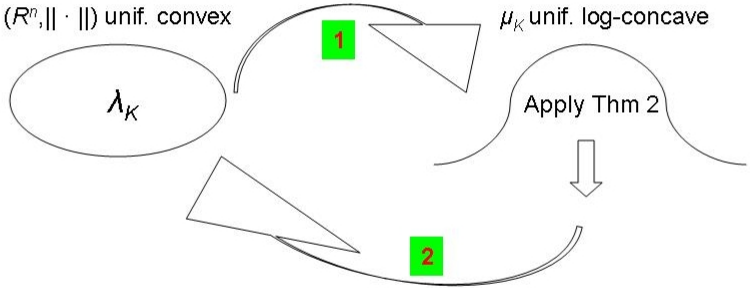
Ideas of Proofs

Thm 2 ⇒ Thm 1 (ideas of Bobkov–Ledoux)



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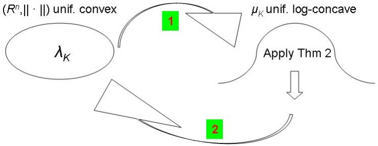

Step 1:

- for p-convex: $d\mu = c \exp(-\|x\|^p) dx$ is p-log-concave.
- for general $\delta_{\|\cdot\|}$: $d\mu = c \exp(-n \|x\|^2) 1_K(x) dx$.
  Can show: $\delta_{\mu, \|\cdot\|}(\varepsilon) \geq c n \delta_{\|\cdot\|}(\varepsilon/4)$.

  Using Figiel–Pisier, can show that $\forall \|x\|, \|y\| \leq 1$:

$$\frac{\|x\|^2 + \|y\|^2}{2} - \left\| \frac{x+y}{2} \right\|^2 \geq c \, \delta_{\|\cdot\|} \left(\frac{\|x-y\|}{4} \right).$$

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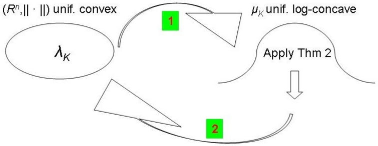

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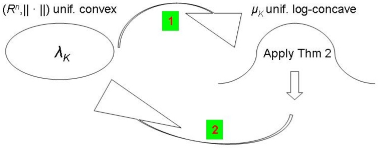

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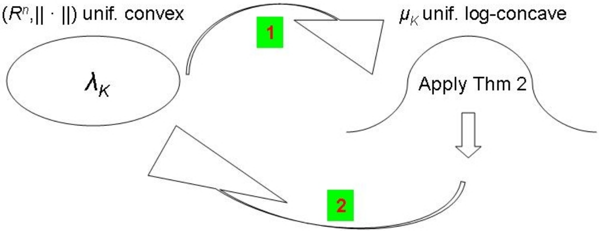

Step 2 (Idea of Bobkov–Ledoux):
Construct a Lipschitz map (w.r.t. $\|\cdot\|$) which pushes forward μ_K onto λ_K (transfers isoperimetric inequalities).

- For p-convex: $\mu_K = c\exp(-\|x\|^p)dx \to \lambda_K$ done by Bobkov–Ledoux.
- For arbitrary $\delta_{\|\cdot\|}$: $\mu_K = f(x)dx \to \lambda_K$, we generalize their result.

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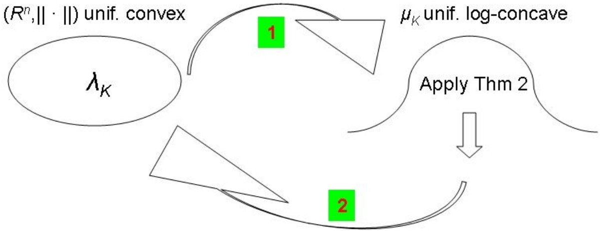

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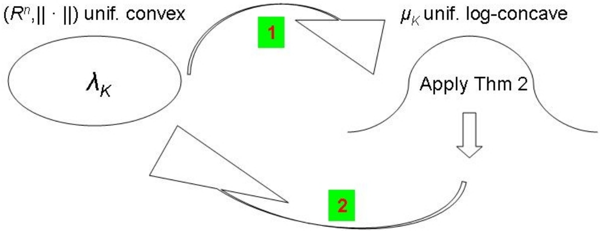

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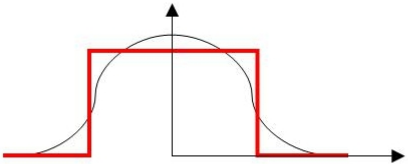
Lipschitz Radial Map



Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be integrable.
Following Bobkov–Ledoux, use *radial-map* $T_f : \mathbb{R}_+ x \to \mathbb{R}_+ x$, $\forall x \in \mathbb{R}^n$, which pushes-forward $f(x)dx$ onto $1_{K_f}(x)dx$, so $T_f : f(xr)r^{n-1}dr \to 1_{[0,a]}(xr)r^{n-1}dr \;\; \forall x \in \mathbb{R}^n$.

$$K_f = \left\{ x \in \mathbb{R}^n ; n \int_0^\infty f(xr) r^{n-1} dr \geq 1 \right\}$$

Ball: f is (even) log-concave ⇒ K_f is a (symmetric) convex body.

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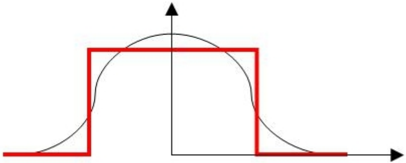

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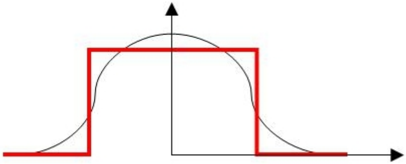

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Lipschitz Radial Map

Thm (M.–Sodin 2007)

For any even log-concave f,
as a map $T_f : (\mathbb{R}^n, \|\cdot\|_{K_f}) \to (\mathbb{R}^n, \|\cdot\|_{K_f})$, $\|T_f\|_{Lip} \leq Cf(0)^{1/n}$.

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Proof of Thm 2 (isop. inequality for $(\mathbb{R}^n, \|\cdot\|, \mu)$)

- “Localization Principle": used by Gromov–Milman, developed by Kannan–Lovász–Simonovits, advocated and used by Bobkov.
- Assume $f = \frac{d\mu}{dx}$ semi-continuous, so $\mu|_E = f|_E dx_E$. In its local form, the Localization Lemma reduces the study of isoperimetric inequalities on $(\mathbb{R}^n, \|\cdot\|, \mu)$ to $(L, \|\cdot\|, \mu|_L \phi)$, $L \subset \mathbb{R}^n$ affine line and ϕ log-concave on L.
- Observation:
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- Reduces Thm 2 to proving the isoperimetric inequality for $(\mathbb{R}, |\cdot|, \sigma)$, $\delta_{\sigma, |\cdot|} \geq \delta_{\mu, \|\cdot\|}$, i.e. 1-D uniformly log-concave measures on $\mathbb{R}$.

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Thank you!