An extension of a Bourgain–Lindenstrauss–Milman inequality

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1) Definitions

Let $(E, \|\cdot\|)$ be a normed space, and let $v_1, \cdots, v_n \in E \setminus \{0\}$.

Define a norm $|||\cdot|||$ on $\mathbb{R}^n$:

$$|||x||| = \mathbb{E}\,\|\sum_{i=1}^{n} \varepsilon_i x_i v_i\| \;, \tag{1}$$

where the expectation is over the choice of n independent random signs $\varepsilon_1, \cdots, \varepsilon_n$.

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where the expectation is over the choice of n independent random signs $\varepsilon_1, \cdots, \varepsilon_n$.

Remark

This is an *unconditional* norm; that is,

$$|||(x_1, x_2, \cdots, x_n)||| = |||(|x_1|, |x_2|, \cdots, |x_n|)||| \ .$$

2) Motivation

We ask whether it is possible to average $O(n)$ of the terms, rather than the 2^n terms in (1) ?

In order to obtain a norm that is isomorphic to $|||\cdot|||$ and is in particular (isomorphically) unconditional.

3) Theorem (Sodin & F.)

Let $N = (1+\delta)n$, $\delta > 0$, and let

$$\{\varepsilon_{ij} \,\big|\, 1 \le i \le n, 1 \le j \le N\}$$

be a collection of independent random signs. Then

$$\mathbb{P}\left\{\forall x \in \mathbb{R}^n \;\; c\delta^2 \,|||x||| \le \frac{1}{N}\sum_{j=1}^{N} \| \sum_{i=1}^{n} \varepsilon_{ij} x_i v_i \| \le C|||x|||\right\} \ge 1 - e^{-c'\delta n} ,$$

where $c', c, C > 0$ are universal constants.

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- Their argument yields the upper bound for the full range of δ, so the innovation is in the lower bound for small $\delta > 0$.
- With the stated dependence on δ, the corresponding result for the scalar case, i.e. $\dim E = 1$, was proved by Rudelson, improving previous bounds on $c(\delta)$ by Kashin, Johnson - Schechtman, Litvak - Pajor - Rudelson - Tomczak-Jaegermann - Vershynin and Artstein-Avidan - Friedland - Milman - Sodin.

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- Their argument yields the upper bound for the full range of δ, so the innovation is in the lower bound for small $\delta > 0$.
- With the stated dependence on δ, the corresponding result for the scalar case, i.e. $\dim E = 1$, was proved by Rudelson, improving previous bounds on $c(\delta)$ by Kashin, Johnson - Schechtman, Litvak - Pajor - Rudelson - Tomczak-Jaegermann - Vershynin and Artstein-Avidan - Friedland - Milman - Sodin.
- This scalar case is one of the two main ingredients of our proof, the second one being Talagrand's concentration inequality.

5) Theorem (Talagrand)

Let $w_1, \cdots, w_n \in E$ be vectors in a normed space $(E, \|\cdot\|)$, and let $\varepsilon_1, \cdots, \varepsilon_n$ be independent random signs. Then for any $t > 0$

$$\mathbb{P}\left\{\left|\|\sum_{i=1}^{n}\varepsilon_i w_i\| - \mathbb{E}\|\sum_{i=1}^{n}\varepsilon_i w_i\|\right| \geq t\right\} \leq C_1 e^{-c_1 t^2/\sigma^2},$$

where $c_1, C_1 > 0$ are universal constants, and

$$\sigma^2 = \sigma^2(w_1, \cdots, w_n) = \sup\left\{\sum_{i=1}^{n}\varphi(w_i)^2 \,\middle|\, \varphi \in E^*, \ \|\varphi\|^* \leq 1\right\} .$$

6) Proof

Let us denote $|||x|||_N = \frac{1}{N}\sum_{j=1}^{N} \|\sum_{i=1}^{n} \varepsilon_{ij} x_i v_i\|$.

This is a random norm depending on the choice of ε_{ij}.

Let $S^{n-1}_{|||\cdot|||} = \{x \in \mathbb{R}^n : |||x||| = 1\}$ be the unit sphere of $(\mathbb{R}^n, |||\cdot|||)$;

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Let $S^{n-1}_{|||\cdot|||} = \{x \in \mathbb{R}^n : |||x||| = 1\}$ be the unit sphere of $(\mathbb{R}^n, |||\cdot|||)$;

We estimate the following probability

$$\mathbb{P}\left\{\forall x \in S^{n-1}_{|||\cdot|||} \; c\delta^2 \leq |||x|||_N \leq C\right\} \quad \geq$$

7) Proof

$$\mathbb{P}\left\{\forall x \in S^{n-1}_{|||\cdot|||}\ c\delta^2 \leq |||x|||_N \leq C\right\}$$
$$\geq 1 - \mathbb{P}\left\{\exists x \in S^{n-1}_{|||\cdot|||},\ |||x|||_N > C\right\}$$
$$- \mathbb{P}\left\{\left(\forall y \in S^{n-1}_{|||\cdot|||}, |||y|||_N \leq C\right) \wedge \left(\exists x \in S^{n-1}_{|||\cdot|||}, |||x|||_N < c\delta^2\right)\right\} \ .$$

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\end{aligned}
$$

Remark

As we mentioned, the needed estimate for the upper bound follows from the argument in BLM.

8) Lower Bound

Denote $\sigma^2(x) = \sigma^2(x_1 v_1, \cdots, x_n v_n)$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

and we recall that

$$\sigma^2(x_1 v_1, \cdots, x_n v_n) = \sup \left\{ \sum_{i=1}^{n} \varphi(x_i v_i)^2 \,\middle|\, \varphi \in E^*, \ \|\varphi\|^* \leq 1 \right\} .$$

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We estimate the last term

$$\mathbb{P}\left\{ \left(\forall y \in S^{n-1}_{|||\cdot|||}, |||y|||_N \leq C \right) \wedge \left(\exists x \in S^{n-1}_{|||\cdot|||}, |||x|||_N < c\delta^2 \right) \right\} .$$

9) Lower Bound

We decompose the sphere $S^{n-1}_{|||\cdot|||} = U \cup V$,

$$U = \left\{ x \in S^{n-1}_{|||\cdot|||} \,\middle|\, \sigma(x) \geq \sigma_0 \right\} \ ,$$

$$V = \left\{ x \in S^{n-1}_{|||\cdot|||} \,\middle|\, \sigma(x) < \sigma_0 \right\} \ ,$$

where σ_0 is a universal constant that we choose later.

10) $x \in U$

Recall Rudelson's estimate for the scalar case, i.e. $\dim E = 1$:

Let $N = (1+\delta)n$, $0 < \delta < 1$, and let

$$\{\varepsilon_{ij} \,\big|\, 1 \le i \le n, 1 \le j \le N\}$$

be a collection of independent random signs. Then, with high probability, for any $y \in \mathbb{R}^n$

$$\frac{1}{N}\sum_{j=1}^{N} \Big|\sum_{i=1}^{n} \varepsilon_{ij} y_i\Big| \ge c_4 \delta^2 |y| \;,$$

where $c_4 > 0$ a universal constant, and $|\cdot|$ is the standard Euclidean norm.

11) $x \in U$

Therefore, for any $x \in \mathbb{R}^n$ and for any $\varphi \in E^*$ with $\|\varphi\|^* \leq 1$, we have

$$\begin{aligned} |||x|||_N &\geq \frac{1}{N}\sum_{j=1}^{N}\left|\varphi(\sum_{i=1}^{n}\varepsilon_{ij}x_i v_i)\right| = \frac{1}{N}\sum_{j=1}^{N}\left|\sum_{i=1}^{n}\varepsilon_{ij}\varphi(x_i v_i)\right| \\ &\geq c_4\delta^2\sqrt{\sum_{i=1}^{n}\varphi(x_i v_i)^2}. \end{aligned} \tag{2}$$

12) $x \in U$

Recall that $U = \left\{ x \in S^{n-1}_{|||\cdot|||} \,\middle|\, \sigma(x) \geq \sigma_0 \right\}$.

Inequality (2) holds for every $\varphi \in E^*$ with $\|\varphi\|^* \leq 1$, and hence we get

$$\begin{aligned} |||x|||_N &\geq c_4 \delta^2 \sqrt{\sup \left\{ \sum_{i=1}^{n} \varphi(x_i v_i)^2 \,\middle|\, \varphi \in E^*, \; \|\varphi\|^* \leq 1 \right\}} \\ &= c_4 \delta^2 \sigma(x) \\ &\geq c_4 \delta^2 \sigma_0 \; . \end{aligned}$$

13) $x \in V$

Now, let us consider the other case where

$$x \in V = \left\{ x \in S^{n-1}_{|||\cdot|||} \,\middle|\, \sigma(x) < \sigma_0 \right\}.$$

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Let $\mathcal{N}_\theta \subset V$ be a θ-net for V, it is known that $|\mathcal{N}_\theta| \leq (3/\theta)^n$.

Note that for any $y \in \mathcal{N}_\theta$ we have $\sigma(y) < \sigma_0$.

Therefore by Talagrand's inequality

$$\mathbb{P}\left\{ \| \sum_{i=1}^{n} \varepsilon_i y_i v_i \| < 1/2 \right\} \leq C_1 \exp(-\frac{c_1}{4\sigma_0^2}) .$$

14) $x \in V$

And hence we get that

$$\mathbb{P}\left\{\frac{1}{N}\sum_{j=1}^{N}\|\sum_{i=1}^{n}\varepsilon_{ij}y_i v_i\| < \frac{1}{4}\right\}$$

$$= \mathbb{P}\left\{\sum_{j=1}^{N}\|\sum_{i=1}^{n}\varepsilon_{ij}y_i v_i\| < \frac{1}{2}\cdot\frac{N}{2}\right\} \leq 2^N \left\{C_1 \exp\left(-\frac{c_1}{4\sigma_0^2}\right)\right\}^{N/2} .$$

15) $x \in V$

Now, we choose $0 < \theta < \frac{1}{8C}$ and $\sigma_0 > 0$ such that the following inequality holds (where $C > 0$ comes from the upper bound):

$$2^N \left\{ C_1 \exp\left(-\frac{c_1}{4\sigma_0^2} \right) \right\}^{N/2} \cdot |\mathcal{N}_\theta| \leq 2^{-N}$$

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Then with high probability we have for any $y \in \mathcal{N}_\theta$ that $|||y|||_N \geq 1/4$.

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Then with high probability we have for any $y \in \mathcal{N}_\theta$ that $|||y|||_N \geq 1/4$.

Using the upper bound, we infer:

$$\begin{aligned} |||x|||_N &\geq |||y|||_N - |||x - y|||_N \\ &\geq 1/4 - C/8C = 1/4 - 1/8 = 1/8 \;, \quad x \in V \;, \end{aligned}$$

which concludes the proof. □

16) Definitions

Let $(E, \|\cdot\|)$ be a normed space, and let $v_1, \cdots, v_n \in E \setminus \{0\}$.

Define a norm $|||\cdot|||$ on $\mathbb{R}^n$:

$$|||x||| = \int_{\mathbb{R}^n} \|\sum_{i=1}^{n} a_i x_i v_i\| d\mu(a) \ ,$$

where a_i is the i^{th} coordinate of a vector a and μ is a log-concave probability measure μ on $\mathbb{R}^n$.

17) Theorem

Let $N = (1+\delta)n$, $\delta > 0$, and let

$$\{a(1), \cdots, a(N) \in R^n\}$$

be a set of N independent random vectors, distributed with respect to a log-concave probability measure μ on $\mathbb{R}^n$.

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be a set of N independent random vectors, distributed with respect to a log-concave probability measure μ on $\mathbb{R}^n$.

Then with high probability for any $x \in \mathbb{R}^n$

$$c(\delta)|||x||| \leq \frac{1}{N}\sum_{j=1}^{N} \| \sum_{i=1}^{n} a(j)_i x_i v_i \| \leq C|||x|||,$$

where $c(\delta) > 0$ is a constant depending only on δ.