Roots of Ehrhart polynomials

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joint work with
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Samos
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- Let $\mathcal{P}^n$ be the set of lattice polytopes $P \subset \mathbb{R}^n$ with $\operatorname{int} P \neq \emptyset$.

The Ehrhart Polynomial

- Ehrhart, 1967. *Let* $P \in \mathcal{P}^n$ *and* $k \in \mathbb{N}$. *Then*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i,$$

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- $\mathrm{G}_0(P) = 1$
- $\mathrm{G}_n(P) = \mathrm{vol}\,(P)$
- $\mathrm{G}_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet}} \frac{\mathrm{vol}_{\,n-1}(F)}{\det(\mathrm{aff}\,F \cap \mathbb{Z}^n)}$

- There are formulas/identities for the coefficients via generating functions or via the Todd differential operator...
  Barvinok, 1992,1993, 1994; Brion, 1988, 1992; Brion and Vergne 1997; Cappell and Shaneson 1994; Chen, 2002; Diaz and Robins, 1999; Kantor and Khovanskii 1993; Morelli 1993; Pommersheim 1993; 1996; 1997; Pukhlikov and Khovanskii 1992;...

- Stanley, 1980; Betke, Gritzmann, 1986. *Let Z be a lattice* zonotope. *Then*

$$\mathrm{G}_i(Z) = \sum_{F \text{ } i\text{-face}} \frac{\mathrm{vol}_{\,i}(F)}{\det(\mathrm{aff}\, F \cap \mathbb{Z}^n)}\, \gamma(Z, F).$$

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- Liu, 2005. *Let $C(n, m)$ be a cyclic n-polytope with m integral vertices on the moment curve $t \to (t, t^2, \ldots, t^n)$. Then*

$$\mathrm{G}_i(C(n, m)) = \mathrm{vol}_i(C(n, m)|\mathbb{R}^i).$$

- In general the G_i's, $i \in \{1, \ldots, n-2\}$, might be negative:

  Reeve simplices. For $n = 3$ and $l \geq 1$, let

$$R(l) = \mathrm{conv}\,\{0, e_1, e_2, (1, 1, l)^{\intercal}\}.$$

  Then $\mathrm{G}(R(l)) = 4$, $\mathrm{G}_3(R(l)) = l/6$, $\mathrm{G}_2(R(l)) = 1$, $\mathrm{G}_0(R(l)) = 1$ and

$$\mathrm{G}_1(R(l)) = \frac{12 - l}{6}.$$

- Ehrhart's Reciprocity Law, 1967.

$$\mathrm{G}(\mathrm{int}\,(k\,P)) = (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(P)\,(-k)^i.$$

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- Betke, Kneser, 1985. *Every additive and unimodular invariant functional on $\mathcal{P}^n$ is a linear combination of $\mathrm{G}_i()$, $0 \leq i \leq n$.*

- For $P \in \mathcal{P}^n$ and $s \in \mathbb{C}$, let

$$\mathrm{G}(s, P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\, s^i = \prod_{i=1}^{n} \left(1 + \frac{s}{\gamma_i(P)}\right).$$

$-\gamma_i(P)$, $1 \leq i \leq n$, are the roots of the Ehrhart polynomial $\mathrm{G}(s, P)$.

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- $$\mathrm{G}_n(P) = \prod_{i=1}^{n} \frac{1}{\gamma_i}, \quad \mathrm{G}_{n-1}(P) = \sum_{j=1}^{n} \prod_{i \neq j} \frac{1}{\gamma_i}, \ldots$$

- Let $T_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$ be the standard simplex. Then

$$\mathrm{G}(s, T_n) = \binom{s+n}{n}$$

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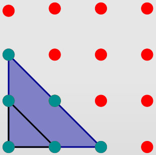
- Or, since $\mathrm{int}\,(k\, T_n) \cap \mathbb{Z}^n = \emptyset$ for $k = 1, \dots, n$

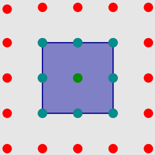


it is

$$0 = \mathrm{G}(\mathrm{int}\,(k\, T_n)) = (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(T_n)\,(-k)^i$$
$$= (-1)^n\,\mathrm{G}(-k, T_n).$$

Thus $-\gamma_i(T_n) = -i$, $1 \le i \le n$.

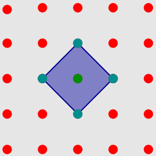
- Let $C_n = \{x \in \mathbb{R}^n : |x_i| \leq 1,\ 1 \leq i \leq n\}$ be the cube.



Then $\mathrm{G}(s, C_n) = (2\,s+1)^n$, and so

$$-\gamma_i(C_n) = -\frac{1}{2},\ 1 \leq i \leq n.$$

- Let $C_n^\star = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$ be the regular crosspolytope.



Then

$$\mathrm{G}(s, C_n^\star) = \sum_{i=0}^{n} \binom{n}{i} \binom{s+n-i}{n}$$

and

$$-\gamma_i(C_n^\star) = ?.$$

- Beck, De Loera, Develin, Pfeifle, Stanley, 2001. First systematic study of the roots of Ehrhart polynomials.

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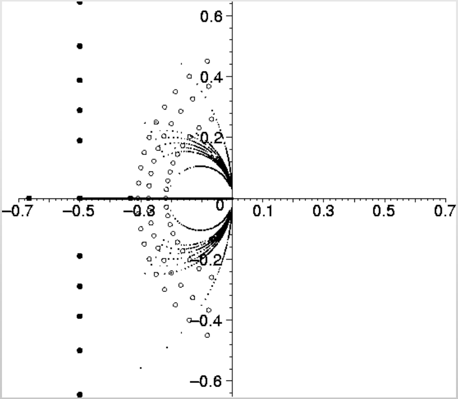
- Roots of 2-dimensional lattice polygons with real part $\geq -2/3$.



- **Theorem**. *The roots of the Ehrhart polynomials of 3-dimensional lattice polytopes are contained in*

$$[-3,-1] \cup \left\{ a + \mathrm{i}\, b : -1 \leq a < 1,\ a^2 + b^2 \leq 3 \right\}$$

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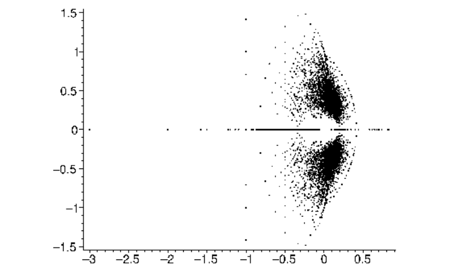
- Roots of 100000 random lattice simplices in $\mathbb{R}^3$ (taken from the paper of Beck, De Loera, Develin, Pfeifle, Stanley, 2001).



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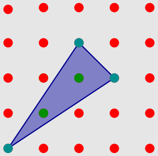
- Braun, 2006

$$\left| s + \frac{1}{2} \right| \leq n \left(n - \frac{1}{2} \right).$$

- For $l \in \mathbb{N}$, let $S_n(l) = \mathrm{conv}\left\{e_1, \ldots, e_n, -l \sum_{i=1}^{n} e_i\right\}$.



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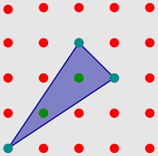

- $\mathrm{G}(\mathrm{int}\, S_n(l)) = l$ and $\mathrm{vol}\,(S_n(l)) = (n\, l + 1)/n!$.

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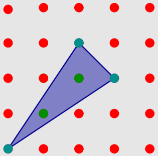

- $\mathrm{G}(\mathrm{int}\, S_n(l)) = l$ and $\mathrm{vol}\,(S_n(l)) = (n\, l + 1)/n!$.

- **Theorem**. *Let $P \in \mathcal{P}^n$. Then*

$$\mathrm{vol}\,(P) \geq \frac{n\,\mathrm{G}(\mathrm{int}\ P) + 1}{n!},$$

and the bound is best possible.

- Proposition. *For* $\mathrm{G}(\mathrm{int}\, P) > 1$ *lattice polytopes of minimal volume are not necessarily unimodular equivalent, but they have the same Ehrhart polynomial.*

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- **Theorem.** *All roots of the polynomial* $\mathrm{G}(s, S_n(1))$ *have real part* $-1/2$. *If* $-\gamma_n$ *is a root of* $\mathrm{G}(s, S_n(1))$ *with maximal norm, then*

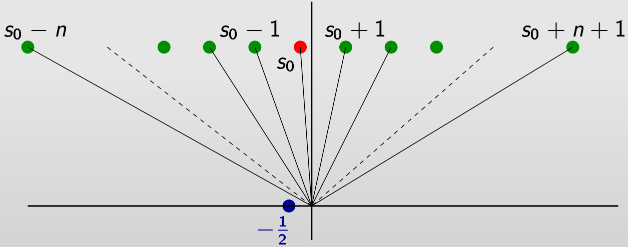
$$\left|-\gamma_n + \frac{1}{2}\right| = \frac{n(n+2)}{2\,\pi} + O(1),$$

as n *tends to infinity.*

$$\mathrm{G}(s, S_n(1)) = \sum_{i=0}^{n} \binom{s+n-i}{n} = \binom{s+n+1}{n+1} - \binom{s}{n+1}.$$

Hence, if s_0 is a root then

$$s_0(s_0-1)\cdot\ldots\cdot(s_0-n) = (s_0+n+1)(s_0+n)\cdot\ldots\cdot(s_0+1).$$



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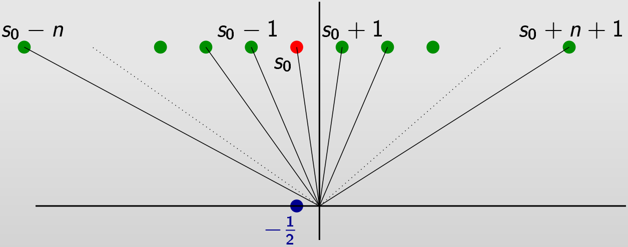

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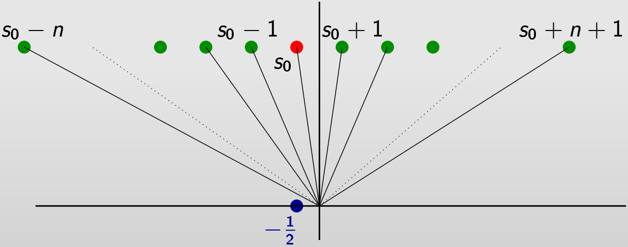
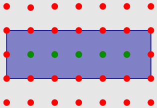


- **Remark.** *For $n = 2, 3$ the polynomial $\mathrm{G}(s, S_n(1))$ has roots of maximal norm among all Ehrhart polynomials of polytopes with interior points.*

- Blichfeldt, 1921; van der Corput, 1935. *Let $P \in \mathcal{P}^n$ be 0-symmetric. Then*

$$\mathrm{vol}\,(P) \leq 2^{n-1}\,\left(\mathrm{G}(\mathrm{int}\,P) + 1\right).$$



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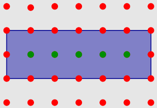


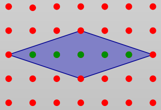
- **Problem**. *Let* $P \in \mathcal{P}$ *be* 0*-symmetric. Then*

$$\mathrm{vol}\,(P) \geq \frac{2^{n-1}}{n!}\,\left(\mathrm{G}(\mathrm{int}\,P) + 1\right).$$

Equality, for instance for

$$C_n^\star(2\,l - 1) = \mathrm{conv}\,\{\pm l\,e_1, \pm\,e_2, \ldots, \pm\,e_n\}, \quad l \geq 1,$$

with $2\,l - 1$ interior points.



- Kirschenhofer, Pethoe, Tichy, 1999; Bump, Choi, Kurlberg, Vaaler, 2000; Rodriguez-Villegas, 2002.

$$\Re(-\gamma_i(C_n^\star)) = -\frac{1}{2}, \quad 1 \le i \le n.$$

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- **Proposition.** *Let $P \in \mathcal{P}^n$. If all roots of $\mathrm{G}(s, P)$ have real part $-1/2$ then, up to an unimodular translation, P is a reflexive polytope of volume $\le 2^n$.*

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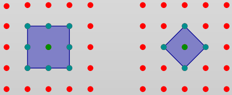
  - $P \in \mathcal{P}^n$ $(0 \in \operatorname{int} P)$ is called reflexive if
  $$P^\star = \{y \in \mathbb{R}^n : x\,y \le 1, \text{ for all } x \in P\} \in \mathcal{P}^n,$$
  i.e., if the polar polytope $P^\star$ is again a lattice polytope.



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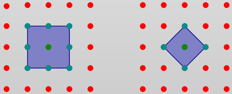
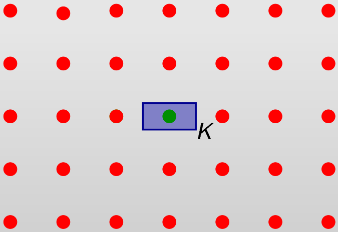
    

  - $\Leftrightarrow \mathrm{vol}\,(P) = \frac{2}{n}\mathrm{G}_{n-1}(P)$.

- Let $K \subset \mathbb{R}^n$ be 0-symmetric convex body.

$$\lambda_i(K) = \min \{\lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i\}$$

is called the i-th successive minimum, $1 \leq i \leq n$.



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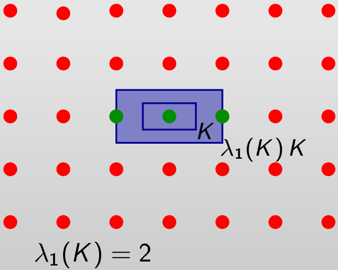

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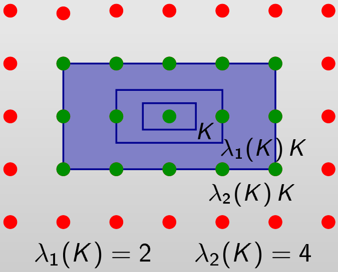

- Minkowski's 1st Theorem, 1896.

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$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \le \mathrm{vol}\,(K) \le \prod_{i=1}^{n}\frac{2}{\lambda_i(K)}.$$

  - Thus
$$\left(n!\prod_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)^{1/n} \ge \left(\prod_{i=1}^{n} \gamma_i(P)\right)^{1/n} \ge \left(\prod_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)^{1/n}$$
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- **Theorem.**

$$\frac{1}{n}\left(\sum_{i=1}^{n}\gamma_i(P)\right) \leq \frac{1}{n}\left(\sum_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)$$

and the bound is best possible, e.g., for the cube C_n and for the crosspolytope $C_n^\star$.

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- There is no lower bound.

- **Theorem.**

$$\frac{1}{n}\left(\sum_{i=1}^{n}\gamma_i(P)\right) \leq \frac{1}{n}\left(\sum_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)$$
$$\left(\Leftrightarrow \frac{\mathrm{G}_{n-1}(P)}{\mathrm{vol}\,(P)} \leq \sum_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)$$

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- Lemma. *Let P be a 0-symmetric polytope with facets $F_i = P \cap \{a_i x = b_i\}$, $|a_i| = 1$. Let L be an k-dimensional linear subspace and let $I_L = \{i : a_i \in L\}$. Then*

$$\mathrm{vol}\,(P) \geq \frac{1}{k} \sum_{i \in I_L} \mathrm{vol}\,_{n-1}(F_i)\, b_i.$$

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- For $s \in \mathbb{C}$ let

$$\mathrm{L}(K, s) = \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} s + 1 \right) = \sum_{i=0}^{n} \mathrm{L}_i(K)\, s^i.$$

- So we want to show for 0-symmetric lattice polytopes P

$$\mathrm{G}(P,1) = \sum_{i=0}^{n} \mathrm{G}_i(P) \leq \mathrm{L}(P,1) = \sum_{i=0}^{n} \mathrm{L}_i(P).$$

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- Corollary. The inequality for the arithmetic mean implies

$$\mathrm{G}_{n-1}(P) \leq \mathrm{L}_{n-1}(P).$$