Contracting Clusters of Critical Percolation

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- Changing any finite set of edges cannot destroy or create an infinite cluster. Therefore $\psi(p)$ is either 0 or 1.
- Therefore there exists some p_c, depending on G, such that $\psi(p) = 0$ for $p < p_c$ and $\psi(p) = 1$ for $p > p_c$.

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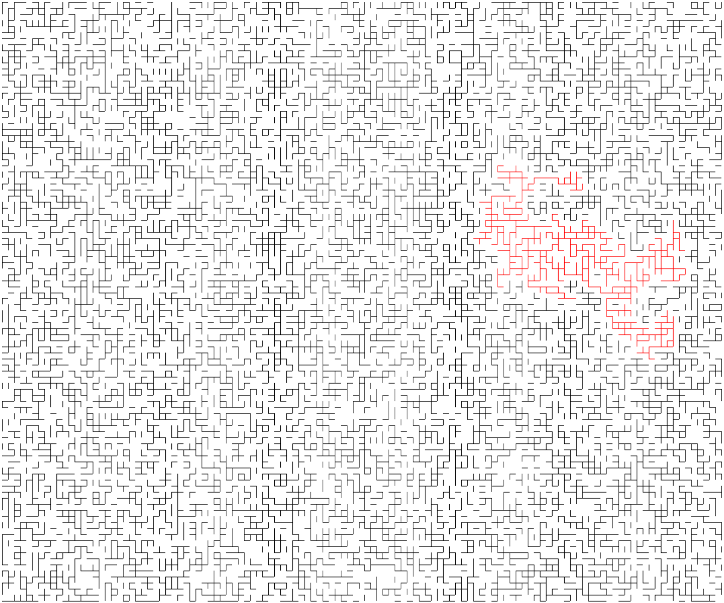

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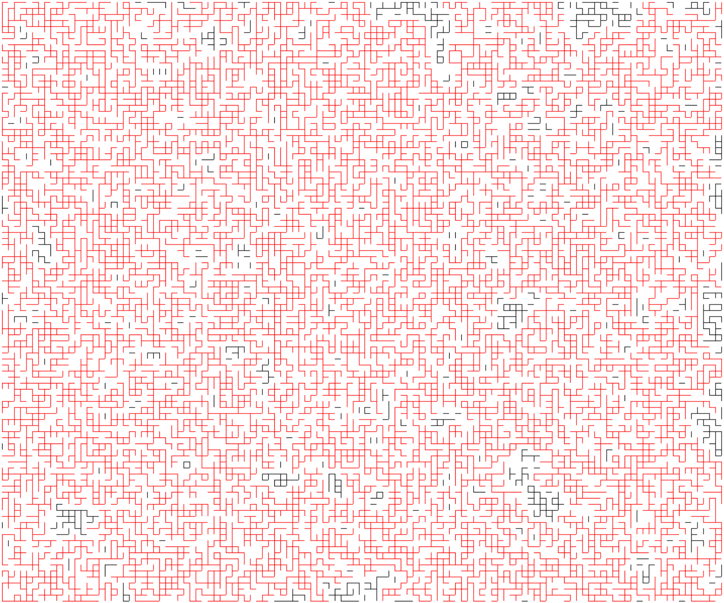

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- for G a d-regular tree, $p_c = \frac{1}{d-1}$ and $\psi(p_c) = 0$. This is equivalent to a Galton-Watson branching process.
- The complete graph on n vertices exhibits similar behavior (even though it is finite) with "$p_c = \frac{1}{n}$" and "$\psi(p_c) = 0$", Erdős & Rényi (1959).

$p \neq p_c$

- In the subcritical case, component sizes decay exponentially in the volume, i.e. for every $p < p_c$ there exist some $\lambda > 0$ such that

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- In most senses, the supercritical cluster "looks like a stretched-out grid".

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(d). $\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2$. *In $d = 6$ there are logarithmic corrections. The conjecture for the value $\frac{91}{5}$ is related to a conjecture that the distribution of large finite clusters is conformally invariant.

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- $d = 3, 4, 5, 6$: not even a.

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- Results of the type "$\mathcal{C}$ is like a grid".

Random walk on the incipient infinite cluster

- Take critical percolation, and condition on the cluster of the origin $\mathcal{C}$ to satisfy $|\mathcal{C}| > n$. Take $n \to \infty$. It turns out that the distributions of $\mathcal{C}$ converge in the appropriate sense to a limit, Kesten (1986), van der Hofstadt & Járai (2004). This limit is called the incipient infinite cluster.

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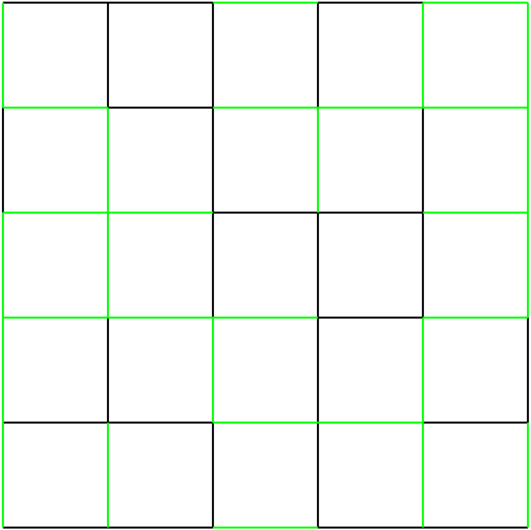
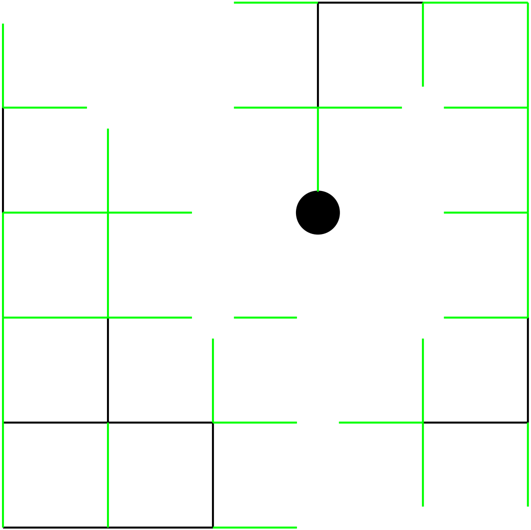
- In $d > 6$, the exact exponent is $\frac{1}{3}$.

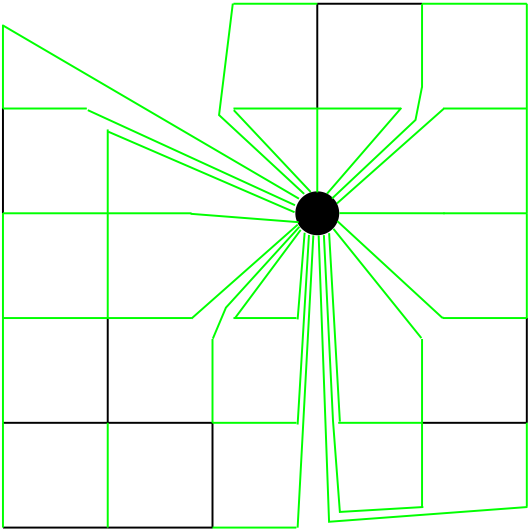
No edges are removed, edges are only colored in two colors.



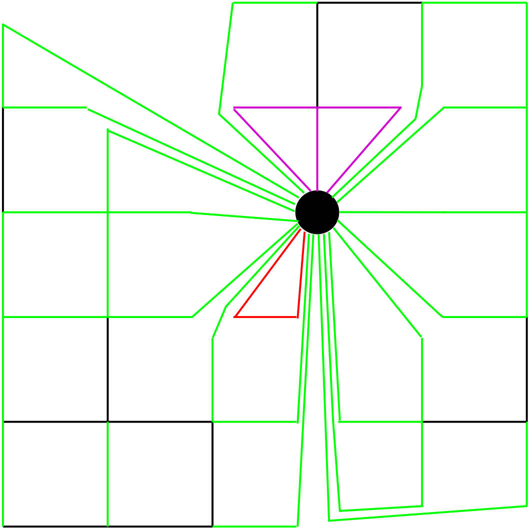
Take a black cluster and replace it with a single vertex.



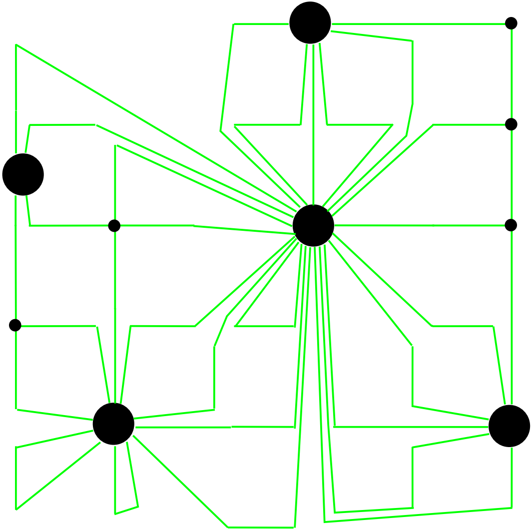
Connect it to all edges which used to connect to the cluster.



Note that this can create loops and multiple edges.



Repeat for all clusters.



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- Hence we will focus on $p = p_c$, in which case we will call the graph CCCP.

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$$d(x,y) \approx \log\log|x-y|.$$

For comparison, on a supercritical cluster, $d(x,y) \approx |x-y|$, while on the IIC $d(x,y) \approx |x-y|^2$.

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  For comparison, on a supercritical cluster, $d(x, y) \approx |x - y|$, while on the IIC $d(x, y) \approx |x - y|^2$.
- The graph satisfies the same *isoperimetric inequality* as $\mathbb{Z}^d$, i.e. for any finite $A \subset G$

$$|\partial A| \geq c|A|^{(d-1)/d}$$

  where $|\partial A|$ is the number of edges going out of A, and $|A| = \sum_{v \in A} \deg v$ i.e. the total number of edges going out of vertices of A. $\frac{d-1}{d}$ is sharp.

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- The incipient infinite cluster behaves like a critical branching tree, embedded into $\mathbb{Z}^d$ randomly (this is known as “integrated superbrownian excursion”).
- CCCP behaves in strange and unexpected ways. We don't have in mind any simple model that would reproduce all data above.

We are now going to sketch some of the ideas that went into the proof of $d(x, y) \approx \log \log |x - y|$.

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- For $d > 6$, $\mathbb{P}(x \leftrightarrow y) \approx |x - y|^{2-d}$. Hara, van der Hofstad & Slade (2003).

First moment

- Let $x, y \in \mathbb{Z}^d$, and assume for simplicity that $d = 8$. What is the probability that one can jump from x to y by no more than 2 moves in CCCP (in general, $\left\lfloor \frac{1}{2} d \right\rfloor - 2$ moves)?

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- We can estimate $\mathbb{E}L$ using the FKG inequality:

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\mathbb{E}L &= \sum_{(v,w)} \mathbb{P}(x \leftrightarrow v, w \leftrightarrow y) \\
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Second moment

- Let now x_1, x_2, y_1 and $y_2 \in \mathbb{Z}^d$, and assume they are all $\approx r$ apart. Let L_i be, as before, the number of edges (v_i, w_i) such that $x_i \leftrightarrow v_i$ and $w_i \leftrightarrow y_i$. We want to upper-bound $\mathbb{E}L_1L_2 - \mathbb{E}L_1\mathbb{E}L_2$.

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$$\begin{aligned}\mathbb{E}L_1L_2 &= \mathbb{E}L_1L_2\mathbf{1}_{L_1 \text{ and } L_2 \text{ occur disjointly}} + \\ &\quad + \mathbb{E}L_1L_2\mathbf{1}_{L_1 \text{ and } L_2 \text{ do not occur disjointly}} \leq \\ &\leq \mathbb{E}L_1\mathbb{E}L_2 + \mathbb{E}L_1L_2\mathbf{1}_{L_1 \text{ and } L_2 \text{ do not occur disjointly}}.\end{aligned}$$

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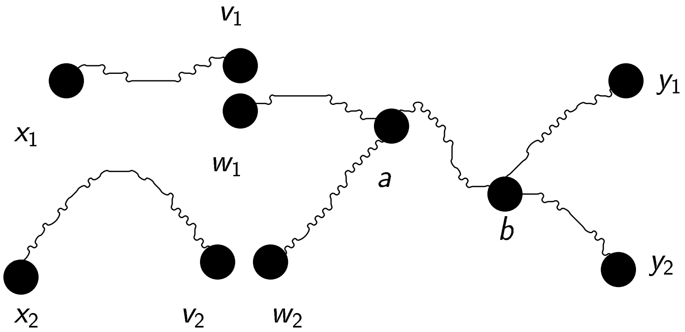
- To estimate the last summand, assume for simplicity that it is the connections $w_1 \leftrightarrow y_1$ and $w_2 \leftrightarrow y_2$ that occur non-disjointly.

Second moment II

Examine the event that $w_1 \leftrightarrow y_1$ and $w_2 \leftrightarrow y_2$ occur non-disjointly. This must be as in the following picture:



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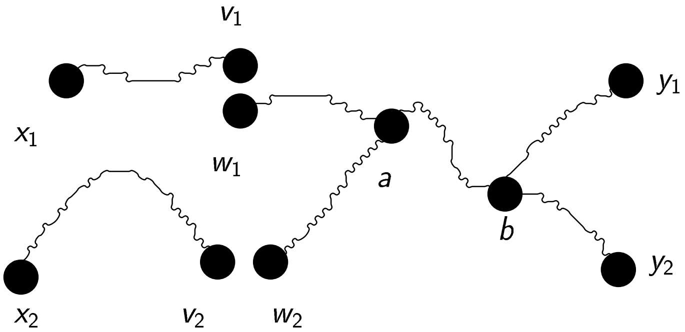

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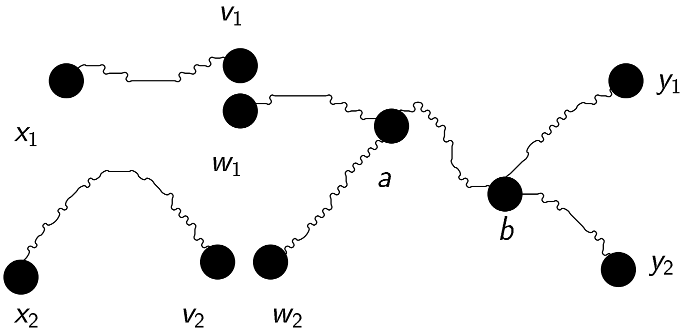

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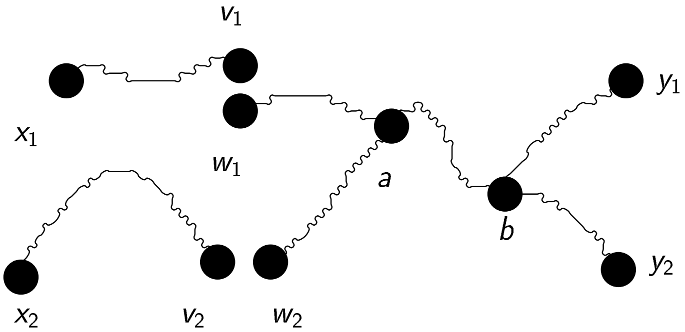

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- Therefore if we have two large clusters at scale r, each has size $\approx r^4$ and therefore the expected number of connections is $\approx \left(r^4\right)^2 \cdot r^{-4} = r^4$ while the variance is only $\approx \left(r^4\right)^4 \cdot r^{-10} = r^6$. We get that they are connected with probability $> 1 - cr^{-2}$.

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- Similar diagrammatic bounds show the whole log log result. Roughly we show that at scale r there are clusters which go as far as r^2, so you can move between scales with a bounded number of jumps. We omit all further details.

Thank you