From the Mahler conjecture to Gauss linking forms

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The Mahler conjecture

Let $K = -K \subseteq \mathbb{R}^n$ be a symmetric convex body. Then

$$K^\circ = \{\vec{y} \mid \forall \vec{x} \in K,\ \vec{x} \cdot \vec{y} \leq 1\}.$$

is its *polar body*. The *Mahler volume*

$$v(K) = (\text{Vol } K)(\text{Vol } K^\circ)$$

is affinely invariant.

Conjecture 1 (Mahler) *$v(K)$ is maximized by the ℓ^2-ball B_n. It is minimized by the cube C_n.*

Actually, K need only be pointed ($\vec{0} \in K$). Then the conjectured minimum is the simplex Δ_n.

Prior results

Theorem 2 (Blaschke, Santaló, Saint-Raymond) *For all K, there exists $\vec{0} \in K$ such that*

$$v(K) \leq v(B_n),$$

with equality if and only if K is an ellipsoid E.

Theorem 3 (Bourgain-Milman) *There exists $c > 0$ such that*

$$v(K) \geq c^n v(E).$$

The Bourgain-Milman theorem is part of a great family of results due to V. Milman and many others.

The Mahler conjecture implies $c = \frac{2}{\pi}$ if $K = -K$, and $c = \frac{e}{2\pi}$ in general.

The new result

Theorem 4 (K.) *If $K = -K$, then*

$$v(K) \geq \frac{2^n}{\binom{2n}{n}} v(E).$$

So our $c = \frac{1}{2}$, and the Mahler conjecture holds up to $\left(\frac{\pi}{4}\right)^n$. This bound is false when $K \neq -K$, because $\pi > e$.

Corollary 5 *Even if $K \neq -K$, then*

$$v(K) \geq \frac{4^n}{\binom{2n}{n}^2} v(E).$$

Here $c = \frac{1}{4}$; the asymmetric Mahler conjecture holds up to $\left(\frac{\pi}{2e}\right)^n$.

The bottleneck conjecture

What I really prove is a theorem in indefinite geometry.

Theorem 6 *Let* $H^{\pm}$ *be the unit pseudospheres of indefinite geometry* $\mathbb{R}^{(a,b)}$*. Let* $N^{\pm}$ *be necks (spacelike and timelike cores), and let*

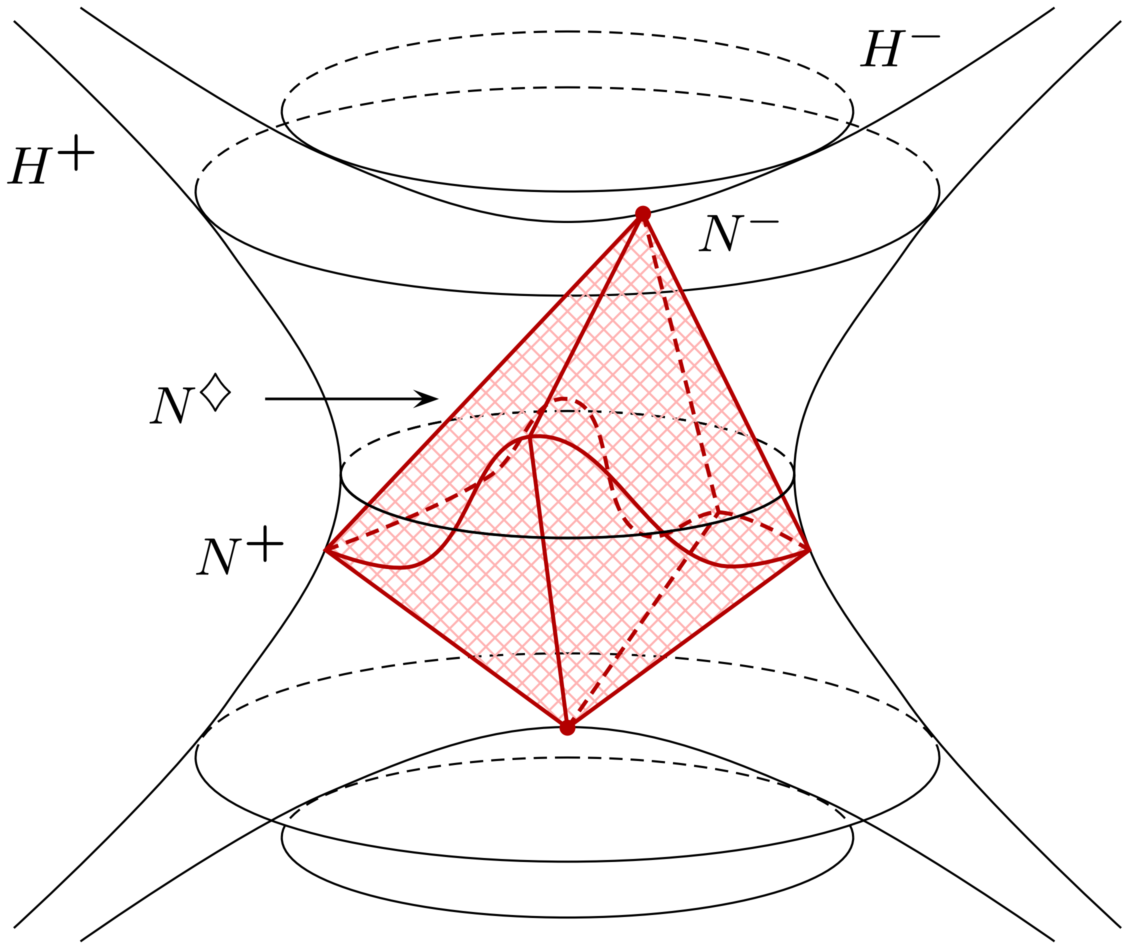
$$N^{\diamondsuit} = \overline{N^{+} * N^{-}}$$

be their filled join. Then Vol $N^{\diamondsuit}$ *is minimized when* $N^{+} \perp N^{-}$*.*

From 1987 to 2006 this was the "bottleneck conjecture" (name due to W. Kuperberg).

N^{+} is *spacelike* means that $\vec{v} \in T_{\vec{x}} N^{+}$ is a positive (or spacelike) vector; likewise N^{-}.

A picture of the bottleneck problem



From Mahler to necks

Let

$$K^{\pm} = \{(\vec{x}, \vec{y}) \in K \times K^{\circ} \mid \vec{x} \cdot \vec{y} = \pm 1\}.$$

They are subsets of pseudospheres of $\mathbb{R}^n \times \mathbb{R}^n$,

$$H^{\pm} = \{(\vec{x}, \vec{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \vec{x} \cdot \vec{y} = \pm 1\},$$

with respect to a signature (n, n) inner product.

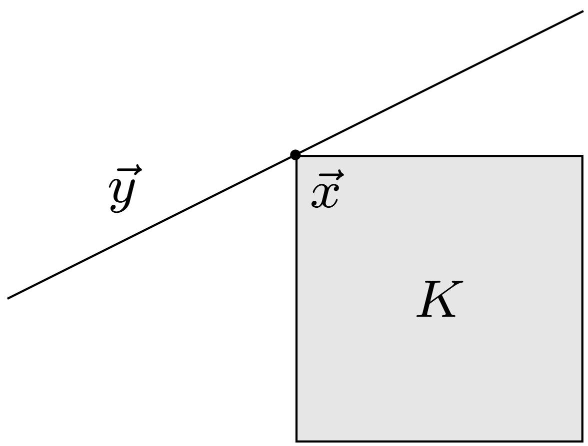
$$(\vec{x}_1, \vec{y}_1) \cdot (\vec{x}_2, \vec{y}_2) = \frac{\vec{x}_1 \cdot \vec{y}_2 + \vec{x}_2 \cdot \vec{y}_1}{2}.$$

Note that

$$H^{\pm} \cong S^{n-1} \times \mathbb{R}^n \qquad K^{\pm} \cong S^{n-1}.$$

Because, K^+ is pairs $\vec{x} \in \partial K$ and $\vec{y} \in \partial K^{\circ}$ such that $\vec{y}$ (as a dual vector) supports K at $\vec{x}$.

From Mahler to necks



For example, if $K = C_2$, then K^+ is a non-planar octagon.

In addition, if K and K° are positively curved:

- K^+ is spacelike and K^- is timelike.
- $K^\pm$ is a topological core of $H^\pm$.
- The geometric join $K^+ * K^-$ is boundary-starlike.
- Thus

$$K^\diamondsuit = \overline{K^+ * K^-} \subseteq K \times K^\circ \qquad \text{Vol } K^\diamondsuit \leq v(K).$$

From Mahler to necks

The volume

$$v(K) = \text{Vol } K \times K^\circ$$

is *maximized* when K is an ellipsoid, which is when Vol $K^\diamondsuit$ is *minimized*. If $f(K)$ is maximized by ellipsoids, we obtain a lower bound by proving that $g(K) \leq f(K)$ is minimized by ellipsoids.

How good is the bound? If $K = B_n$, then

$$B_n^\diamondsuit = \sqrt{2}B_n * \sqrt{2}B_n \qquad \text{Vol } B_n^\diamondsuit = \frac{2^n}{\binom{2n}{n}}(\text{Vol } B_n)^2.$$

At the other end, if $K = C_n = [-1,1]^n$, then

$$K^\diamondsuit = K \times K^\circ.$$

From necks to linking forms

Again, the real result concerns $N^{\pm} \subset H^{\pm} \subset \mathbb{R}^{(a,b)}$. Once again $N^{+} * N^{-}$ is boundary-starlike and $N^{\diamondsuit} = \overline{N^{+} * N^{-}}$. Then

$$\text{Vol } N^{\diamondsuit} = \frac{\int_{N^{+} \times N^{-}} \vec{x} \wedge \vec{y} \wedge d\vec{x}^{\wedge a-1} \wedge d\vec{y}^{\wedge b-1}}{ab\binom{a+b}{a}}.$$

(That is, $(\vec{x}, \vec{y}) \in N^{+} \times N^{-}$. The integrand is a "double wedge" in the algebra $\Lambda^*(\mathbb{R}^{(a,b)}) \otimes \Omega^*(\mathbb{R}^{(a,b)})$.)

The idea is just to divide $N^{\diamondsuit}$ into slices subtended by $\vec{0}$ and infinitesimal simplices at $\vec{x}$ and $\vec{y}$. The slices are thin simplices; the integrand is a determinant.

From necks to linking forms

The integral resembles the Gauss linking integral in $\mathbb{R}^3$:

$$\mathrm{lk}(K_1, K_2) = \int_{K_1 \times K_2} \frac{(\vec{x} - \vec{y}) \wedge d\vec{x} \wedge d\vec{y}}{4\pi |\vec{x} - \vec{y}|^3}.$$

It even more resembles the SO(4)-invariant linking integral in S^3 (DeTurck-Gluck, K.):

$$\mathrm{lk}(K_1, K_2) = \int_{K_1 \times K_2} \phi(\vec{x} \cdot \vec{y}) \vec{x} \wedge \vec{y} \wedge d\vec{x} \wedge d\vec{y},$$

where

$$\phi(\cos\alpha) = \frac{(\pi - \alpha)(\cos\alpha) + (\sin\alpha)}{4\pi^2 (\sin\alpha^3)}.$$

If we compactify $H^{\pm}$ to make $S^{2n-1} = \overline{H^+ \cup H^-}$, then $\mathrm{lk}(N^+, N^-) = 1$ always.

From necks to linking forms

There is an SO(a,b)-invariant linking form on $H^+ \cup H^-$. Let

$$\omega = \phi(\vec{x} \cdot \vec{y})\vec{x} \wedge \vec{y} \wedge d\vec{x}^{\wedge a-1} \wedge d\vec{y}^{\wedge b-1}.$$

The main necessary condition is that $\omega(\vec{x}, \vec{y})$ is a "weak cycle":

$$d_x d_y \omega = -d_y d_x \omega = 0.$$

It is also sufficient if ω is SO(a,b)-invariant. The weak cycle condition yields the ODE

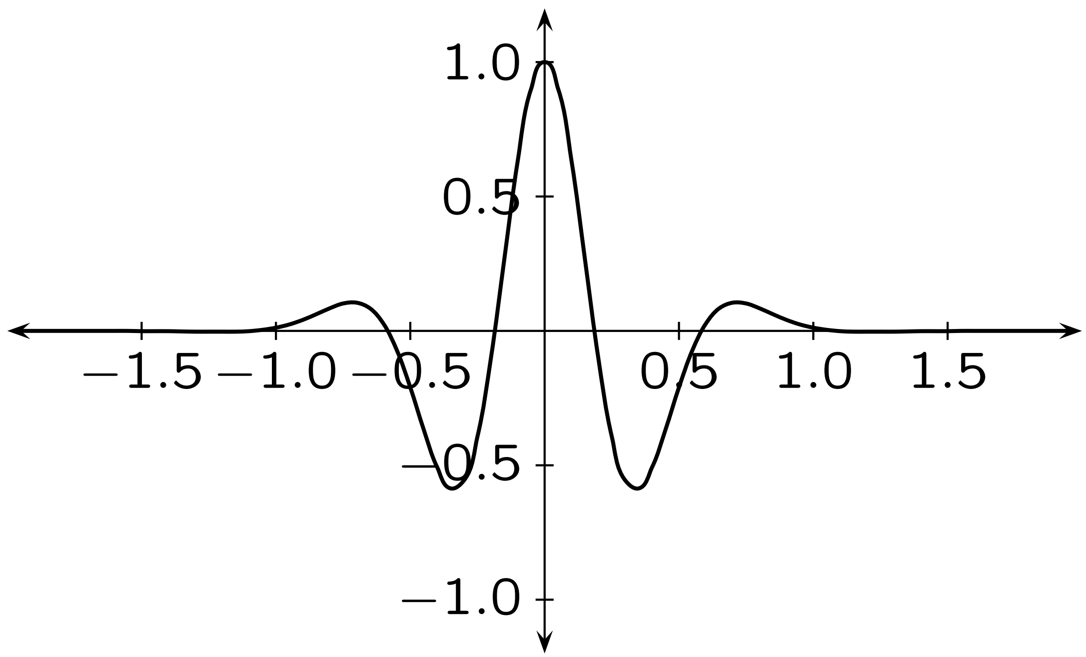
$$f'' + (a+b)(\tanh \alpha) f' + abf = 0,$$

where

$$f(\alpha) = \phi(\sinh \alpha).$$

From necks to linking forms

The ODE for $f(\alpha)$ is a damped harmonic oscillator, whose even solutions look like this:



End of the proof

So $\phi(t) \leq 1$. Moreover, $\vec{x} \wedge \vec{y} \wedge d\vec{x}^{\wedge a-1} \wedge d\vec{y}^{\wedge b-1} > 0$, because $N^{\pm}$ have spacelike and timelike points and tangencies.

Finally $l(N^+, N^-) \leq w(N^+, N^-)$, where

$$l(N^+, N^-) = \int_{N^+ \times N^-} \phi(\vec{x} \cdot \vec{y}) \vec{x} \wedge \vec{y} \wedge d\vec{x}^{\wedge a-1} \wedge d\vec{y}^{\wedge b-1}$$
$$w(N^+, N^-) = \int_{N^+ \times N^-} \vec{x} \wedge \vec{y} \wedge d\vec{x}^{\wedge a-1} \wedge d\vec{y}^{\wedge b-1},$$

with equality when $\vec{x} \cdot \vec{y} = 0$ on all of $N^+ \times N^-$. But

$$l(N^+, N^-) \propto \mathrm{lk}(N^+, N^-) = 1$$

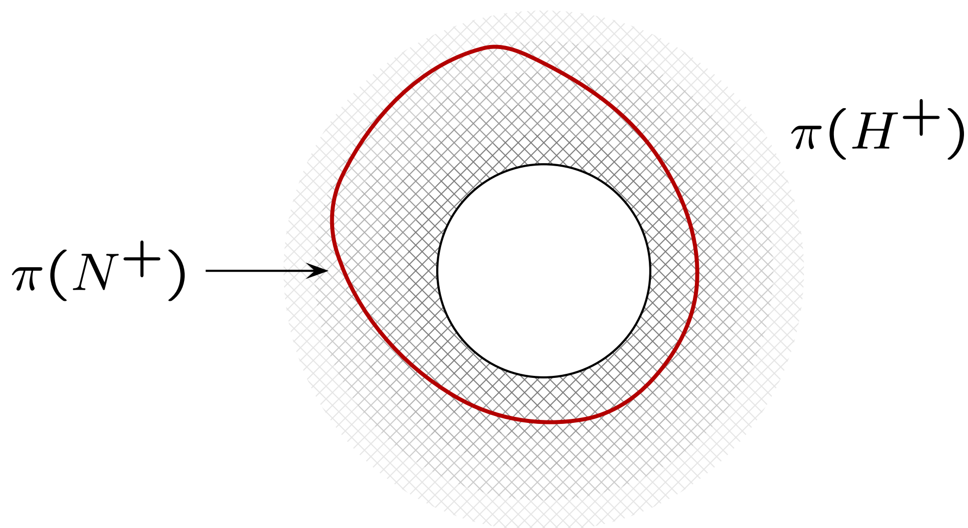
is constant, while

$$\mathrm{Vol}\ N^{\diamondsuit} \propto w(N^+, N^-).$$

Thus $N^{\diamondsuit}$ is minimized when $N^+ \perp N^-$.

Where the proof came from

The proof came from the following picture, when $b = 1$. Let π be the projection perpendicular to N^- (which is just two points).



Vol $N^\diamondsuit = ($Vol $\overline{N^-})($Vol $\overline{\pi(N^+)})/a$. Since $\pi(N^+)$ encircles the hole of $\pi(H^+)$, the result follows. This encirclement is a baby linking number.

More table-turning?

There are some interesting loose ends, like whether $K^\diamondsuit$ is always convex. ($N^\diamondsuit$ need not be.)

The most ambitious question is another example of table-turning.

Conjecture 7 *The probabilistic expectation*

$$E_{K\times K^\circ}\left[(\vec{x}\cdot\vec{y})^2\right]$$

is maximized by ellipsoids.

This conjecture is well-known to imply the famous isotropic constant conjecture, that L_K is universally bounded. My point is that if we view Conjecture 7 as an *exact* maximization problem, using the symmetry of ellipsoids, it may be within reach. (If it is true!)