On the infimum convolution inequality

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Samos, June 25 2007

Talagrand's two level concentration

Let ν be a symmetric exponential measure on $\mathbb{R}$, i.e. the probability measure with the density $\frac{1}{2}e^{-|x|}$ and for $1 \leq p \leq \infty$,

$$B_p^n = \Big\{x \in \mathbb{R}^n \colon \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \leq 1\Big\}$$

Theorem (Talagrand 1991)

There exists a constant C such that for any n and Borel set A in $\mathbb{R}^n$

$$\nu^n(A + \sqrt{t}B_2^n + tB_1^n) \geq 1 - \frac{1}{\nu^n(A)}e^{-t/C}.$$

Maurey's property (τ)

In 1991 B. Maurey proposed the following definition.

Definition

Let μ be a probability measure on $\mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \to [0,\infty)$. We say that a pair (μ,φ) has the property (τ) if for any bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int e^{-f}\,d\mu \int e^{f\Box\varphi}\,d\mu \le 1,$$

where

$$f\Box\varphi(x) := \inf_y (f(y) + \varphi(x-y))$$

is the infimum convolution of f and φ.

Basic properties

Proposition (Tensorization)

If pairs (μ_i, φ_i), $i = 1, \ldots, k$ *have property* (τ) *and* $\varphi(x_1, \ldots, x_k) = \varphi_1(x_1) + \ldots + \varphi_k(x_k)$, *then the couple* $(\otimes_{i=1}^{k} \mu_i, \varphi)$ *also has property* (τ).

Proposition (Transport of measure)

Suppose that (μ, φ) *has property* (τ) *and* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is such that*

$$\psi(Tx - Ty) \leq \varphi(x - y) \text{ for all } x, y \in \mathbb{R}^n.$$

Then the pair $(\mu \circ T^{-1}, \psi)$ *has property* (τ).

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Main Maurey's results

Proposition (Concentration of measure)

If the pair (μ, φ) *has the property* (τ) *then for all* $t > 0$,

$$\mu(A + B_\varphi(t)) \geq 1 - \frac{1}{\mu(A)} e^{-t},$$

where

$$B_\varphi(t) := \{x \in \mathbb{R}^n \colon \varphi(x) \leq t\}.$$

Theorem (Maurey 1991)

Let $w(x) = \frac{1}{36}x^2$ *for* $|x| \leq 4$ *and* $w(x) = \frac{2}{9}(|x| - 2)$ *otherwise. Then the pair* $(\nu^n, \sum_{i=1}^n w(x_i))$ *has property* (τ). *In particular*

$$\forall_{t \geq 0}\ \nu^n(A + 6\sqrt{t}B_2^n + 9tB_1^n) \geq 1 - \frac{1}{\nu^n(A)} e^{-t}.$$

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In search of a cost function

Let μ be a "nice" probability measure on $\mathbb{R}^n$. What function φ can we choose so (μ, φ) would have property (τ)?

If $f(x) = \langle t, x \rangle$ for some $t \in \mathbb{R}^n$, then

$$f \Box \varphi(x) = \inf_y (\varphi(y) + \langle t, x - y \rangle) = \langle t, x \rangle - \sup_y (\langle t, y \rangle - \varphi(y))$$
$$= \langle t, x \rangle - \mathcal{L}\varphi(t).$$

Therefore

$$\int e^{f \Box \varphi} d\mu \int e^{-f} d\mu = e^{\Lambda_\mu(t) + \Lambda_\mu(-t) - \mathcal{L}\varphi(t)},$$

where

$$\Lambda_\mu(t) := \ln \int e^{\langle t, x \rangle} d\mu(x).$$

Hence property (τ) for (μ, φ) implies $\mathcal{L}\varphi(t) \geq \Lambda_\mu(t) + \Lambda_\mu(-t)$.

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Infimum Convolution Inequality

If μ is a symmetric measure on $\mathbb{R}^n$ and the cost function φ is convex then property (τ) for (μ, φ) implies $\mathcal{L}\varphi(t) \geq 2\Lambda_\mu(t)$, i.e.

$$\varphi(x) \leq 2\Lambda_\mu^*(\frac{x}{2}),$$

where

$$\Lambda_\mu^*(x) = \mathcal{L}\Lambda_\mu(x) = \sup_t \Big(\langle t, x\rangle - \ln \int e^{\langle t,y\rangle} d\mu(y)\Big).$$

This motivates the following

Definition

We say that a symmetric probability measure μ on $\mathbb{R}^n$ satisfies the infimum convolution inequality with a constant C ($\mathrm{IC}(C)$ in short) if the pair $(\mu, \Lambda_\mu^*(C^{-1}x))$ has property (τ), i.e. for all bounded measurable functions f,

$$\int e^{-f} d\mu \int e^{f \Box \Lambda_\mu^*(C^{-1}\cdot)} d\mu \leq 1.$$

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Rephrased Maurey's results

Proposition (Tensorization)

If μ_i satisfy $\mathrm{IC}(C_i)$ for $i = 1, \ldots, k$, then $\otimes_{i=1}^{k} \mu_i$ satisfies $\mathrm{IC}(C)$ with $C = \max_i C_i$.

Theorem

The measure ν^n satisfies $\mathrm{IC}(C)$ with universal C.

Proof. Standard calculation shows that $\Lambda_\nu(t) = \frac{1}{1-t^2}$ for $|t| < 1$ and $\Lambda_\nu^*(x) \sim \min(|x|, x^2)$. The assertion follows by Maurey's theorem. □

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Questions

- Which other measures on $\mathbb{R}$ satisfy IC?
- Which nice (nonproduct) measures on $\mathbb{R}^n$ satisfy IC (with constant not dependending on dimension)?
- What kind of concentration is implied by IC?

Level sets

How look the sets

$$B_\mu(u) := \{x \in \mathbb{R}^n \colon \Lambda_\mu^*(x) \le u\}?$$

Definition

Let μ be a symmetric probability measure on $\mathbb{R}^n$ and $p \ge 1$ be such that $\int |x_i|^p d\mu < \infty$ for all i. We set

$$\mathcal{M}_\mu(p) := \left\{ t \in \mathbb{R}^n \colon \int |\langle t, x\rangle|^p d\mu(x) \le 1 \right\}$$

and

$$\begin{aligned}\mathcal{Z}_\mu(p) &:= (\mathcal{M}_\mu(p))^\circ \\ &= \{y \in \mathbb{R}^n \colon |\langle t, y\rangle|^p \le \mathbf{E}|\langle t, x\rangle|^p d\mu(x) \text{ for all } t \in \mathbb{R}^n\}\end{aligned}$$

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Regular measures

Definition

We say that a measure μ on $\mathbb{R}^n$ is α-regular if for all $p \geq q \geq 2$ and all $t \in \mathbb{R}^n$,

$$\Big(\int |\langle t, x\rangle|^p d\mu(x)\Big)^{1/p} \leq \alpha \frac{p}{q} \Big(\int |\langle t, x\rangle|^q d\mu(x)\Big)^{1/q}.$$

Proposition

All symmetric logconcave measures are 1-regular.

Proposition

Suppose that μ is a symmetric, isotropic α-regular measure on $\mathbb{R}^n$. Then

$$B_\mu(t) \sim_\alpha \begin{cases} \sqrt{t} B_2^n & 0 \leq t \leq 2 \\ \mathcal{Z}_\mu(t) & t \geq 2. \end{cases}$$

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Concentration Inequality

Definition

We say that a propability measure μ *on* $\mathbb{R}^n$ *satisfy the concentration inequality with constant* C *(*$\mathrm{CI}(C)$ *in short) if for all* $t \geq 2$ *and all Borel sets* A*,*

$$\mu(A + C\mathcal{Z}_\mu(t)) \geq 1 - \frac{1}{\mu(A)} e^{-t}. \qquad (1)$$

Remark. The condition (1) is equivalent to the condition

$$\mu(A + C\mathcal{Z}_\mu(t)) \geq \min(e^t \mu(A), \frac{1}{2})$$

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Equivalence between IC and CI inequalities

Proposition

If μ is α-regular symmetric, then $\mathrm{IC}(C)$ implies $\mathrm{CI}(C_\alpha C)$.

Definition

We say that a measure μ satisfies the Cheeger's inequality with constant κ if

$$\mu^+(A) \geq \kappa \min(\mu(A), 1 - \mu(A)),$$

where $\mu^+(A) := \liminf_{u \to 0+} \frac{\mu(A+tB_2^n)-\mu(A)}{t}$.

Proposition

If μ is an α-regular symmetric and satisfies Cheeger's inequality with constant κ, then $\mathrm{CI}(C)$ implies $\mathrm{IC}(C(\alpha, \kappa, C))$.

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Weak and strong moments

For which measures μ on $\mathbb{R}^n$ weak and strong moments are comparable in the sense that for all $p \geq 2$,

$$\Big(\int \|x\|^p d\mu\Big)^{1/p} \leq \Big(\int \|x\|^2 d\mu\Big)^{1/2} + C \sup_{\|x^*\| \leq 1} \Big(\int |x^*(x)|^p d\mu\Big)^{1/p}. \qquad (2)$$

Definition

We will say that μ has property $\mathrm{WSM}(C)$ if (2) holds for all $p \geq 2$.

If μ is logconcave, isotropic and satisfies $\mathrm{WSM}(C)$ then we get for all $p \geq 2$,

$$\Big(\int \|x\|_2^p d\mu\Big)^{1/p} \leq \sqrt{n} + Cp.$$

(comp. Klartag CLT and Paouris Concentration of mass)

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WCM for IC measures

Proposition

For any measure μ, $\mathrm{CI}(C)$ implies WCM(KC).

The proof is based on standard integration by parts argument.

Remark. For any measure μ on $\mathbb{R}^n$ and $p \geq n$,

$$\Big(\int \|x\|^p d\mu\Big)^{1/p} \leq 10 \sup_{\|x^*\|\leq 1} \Big(\int |x^*(x)|^p\Big)^{1/p}.$$

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Product measures

Theorem

Every symmetric logconcave measure on $\mathbb{R}$ *satisfies* $\mathrm{IC}(C)$ *with universal constant* C.

Proof is based on transport of measure and Maurey's result for exponential measure.

Corollary

Every symmetric product logconcave measure on $\mathbb{R}^n$ *satisfies* $\mathrm{IC}(C_1)$, $\mathrm{CI}(C_2)$ *and* $\mathrm{WSM}(C_3)$ *with universal constants* C_i.

Uniform measures on B_p^n

Let $\mu_{p,n}$ denote the uniform distribution on $n^{1/p}B_p^n$, i.e.

$$\mu_{p,n}(A) = \frac{\mathrm{vol}(A \cap n^{1/p}B_p^n)}{\mathrm{vol}(n^{1/p}B_p^n)}.$$

Theorem

Measures $\mu_{p,n}$, $1 \leq p \leq \infty$, $n = 1, 2 \ldots$ satisfy $\mathrm{IC}(C)$ with a universal constant C not depending on p and n.

Sketch of the proof

Suppose that $1 \leq p \leq 2$, then $\mu_{p,n}$ satisfies Cheeger's inequality with some universal κ (Sodin), so it is enough to show that $\mu_{p,n}$ satisfy $\mathrm{CI}(C)$. For $t \geq n$, $\mathcal{Z}_t(\mu_{p,n}) \sim n^{1/p} B_p^n$ and for $2 \leq t \leq n$, $\mathcal{Z}_t(\mu_{p,n}) \sim \sqrt{t} B_2^n + t^{1/p} B_p^n$. We need to show that for $2 \leq t \leq n$,

$$\mu_{p,n}(A + C\sqrt{t}B_2^n + Ct^{1/p}B_p^n) \geq \min(e^t \mu(A), \frac{1}{2}). \qquad (3)$$

Let $\nu_{p,n}$ be the probability measure on $\mathbb{R}^n$ with the density $c_p^n \exp(-\|x\|_p^p)$. This measure is product logconcave, so it satisfies $\mathrm{IC}(C)$, so $\mathrm{CI}(C)$ and it is easy to check that for all $t \geq 2$, $\mathcal{Z}_t(\nu_{p,n}) \sim \sqrt{t} B_2^n + t^{1/p} B_p^n$. Hence (3) holds with $\mu_{p,n}$ replaced by $\nu_{p,n}$.

There exists $T = T_{p,n} \colon \mathbb{R}^n \to \mathbb{R}^n$ of the form $Tx = \frac{x}{\|x\|_p} f_{p,n}(\|x\|_p)$ that transports $\nu_{p,n}$ onto $\mu_{p,n}$. It is easy to check that $\|Tx - Ty\|_p \leq C\|x - y\|_p$. However T is not Lipschitz with respect to Euclidean norm.

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$$\mu_{p,n}(A + C\sqrt{t} B_2^n + Ct^{1/p} B_p^n) \geq \min(e^t \mu(A), \frac{1}{2}). \qquad (3)$$

Let $\nu_{p,n}$ be the probability measure on $\mathbb{R}^n$ with the density $c_p^n \exp(-\|x\|_p^p)$. This measure is product logconcave, so it satisfies $\mathrm{IC}(C)$, so $\mathrm{CI}(C)$ and it is easy to check that for all $t \geq 2$, $\mathcal{Z}_t(\nu_{p,n}) \sim \sqrt{t} B_2^n + t^{1/p} B_p^n$. Hence (3) holds with $\mu_{p,n}$ replaced by $\nu_{p,n}$.

There exists $T = T_{p,n} \colon \mathbb{R}^n \to \mathbb{R}^n$ of the form $Tx = \frac{x}{\|x\|_p} f_{p,n}(\|x\|_p)$ that transports $\nu_{p,n}$ onto $\mu_{p,n}$. It is easy to check that $\|Tx - Ty\|_p \leq C\|x - y\|_p$. However T is not Lipschitz with respect to Euclidean norm.

Sketch of the proof ctd

One can however show that T is Lipschitz on $C\sqrt{n}B_2^n \setminus \frac{1}{C}n^{1/p}B_p^n$. We have $\nu_{p,n}(\frac{1}{C}n^{1/p}B_p^n) \leq e^{-n}$ and we can easily deal with it. Unfortunately

$$\nu_{p,n}\Big(\mathbb{R}^n \setminus C\sqrt{n}B_2^n\Big)$$

is of order the $\exp(-n^{p/2}) >> \exp(-n)$. To treat this case we need a slightly improved Talagrand's inequality

Proposition

Suppose that $(A + 10tB_1^n) \cap 4\sqrt{n}B_2^n = \emptyset$, then

$$\nu^n(A + 10tB_1^n) \geq e^t \nu^n(A).$$

This inequality may be transported to $\nu_{p,n}$ (tB_1^n would be replaced by $t^{1/p}B_p^n$) and then to $\mu_{p,n}$.

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Convex versions

One may also consider concex IC and convex CI. We say μ has convex $\mathrm{IC}(C)$ if for all *convex* functions f

$$\int e^{-f} d\mu \int e^{f \Box \Lambda_\mu^*(C^{-1}\cdot)} d\mu \leq 1.$$

And μ has convex $\mathrm{CI}(C)$ if for all Borel convex sets A

$$\mu(A + C\mathcal{Z}_\mu(t)) \geq 1 - \frac{1}{\mu(A)} e^{-t}.$$

For example uniform distrinution on $\{-1,1\}^n$ has convex $\mathrm{CI}(C)$ (Talagrand) and convex $\mathrm{IC}(C)$ (Maurey).
Convex CI is equivalent to convex IC for uniform measures.

Open questions

- Characterization of $\mathrm{IC}(C)$ measures on $\mathbb{R}$.
- Does uniform distributions on Orlicz Balls satisfy IC?
- Uniform distribution on 1-symmetric convex sets?
- Logconcave measures?