A simple proof of the functional Santaló inequality

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The Blaschke-Santaló inequality

Let K be a convex body in $\mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that

$$|K_z|_n \, |(K_z)^\circ|_n \le |D|_n |D^\circ|_n = v_n^2,$$

where $|\cdot|_n$ stands for the volume, $K_z = K + z$, $(K_z)^\circ$ is its polar body, D the Euclidean ball and v_n its volume.

Remark

It is well known that one can choose z such that $(K_z)^\circ$ has its centroid at 0. Hence we can rewrite the inequality as follows: if K is such that K° has its center of mass at 0 then

$$|K|_n \, |K^\circ|_n \le v_n^2.$$

Notation

If g is a non-negative function such that both g and Ng are integrable

$$\mathrm{bar}(g) = \frac{\int g(x)x\,dx}{\int g(x)\,dx}$$

denotes its center of mass (or barycenter). The center of mass (or centroid) of convex body is by definition the barycenter of its indicator function.

Let us define a functional analogue of the polar body:

Definition

Let f be a non-negative function on $\mathbb{R}^n$, integrable with respect to the Lebesgue measure. We define the polar function of f by

$$f^\circ : x \longrightarrow \inf_{y \in \mathbb{R}^n} \frac{\mathrm{e}^{-x \cdot y}}{f(y)}$$

Remark

$\left(\mathrm{e}^{-|x|^2/2}\right)^\circ = \mathrm{e}^{-|x|^2/2}$.

Equivalently:

$$(\mathrm{e}^{-\phi})^\circ = \mathrm{e}^{-\mathcal{L}\phi},$$

where $\mathcal{L}\phi$ is the Legendre transform of ϕ:

$$\mathcal{L}\phi(x) = \sup_{y\in\mathbb{R}^n} \{x \cdot y - \phi(y)\}.$$

The function f° is log-concave and moreover $f = (f^\circ)^\circ$ iff f is log-concave.

The functional Santaló inequality [AKM,2004]

Let f be a non-negative integrable function on $\mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that

$$\int f_z(x)\,dx \int (f_z)^\circ(y)\,dy \;\le\; \Big(\int \mathrm{e}^{-|x|^2/2}\,dx\Big)^2 = (2\pi)^n,$$

where $f_z(x) = f(x-z)$.

Remark

Again the theorem can be formulated the following way: if f is such that $\mathrm{bar}(f^\circ) = 0$ then

$$\int f(x)\,dx \int f^\circ(y)\,dy \;\le\; (2\pi)^n.$$

This implies the usual Santaló inequality: if K is a convex body we write $f_K = \mathrm{e}^{-N_K^2/2}$, where N_K is the gauge of K. One has

$$(f_K)^\circ = f_{K^\circ}$$

Besides $\forall K$:

$$\int f_K = c_n |K|_n$$

with c_n depending only on the dimension, it can be computed by taking $K = D$ the Euclidean ball. Note also that $(f_K)_z = f_{(K_z)}$.

Theorem

Let f and g be non-negative Borel functions on $\mathbb{R}^n$ satisfying

$$\forall x, y \quad f(x)g(y) \leq \mathrm{e}^{-x\cdot y}$$

Let H be an affine hyperplane, H_+, H_- the two half-spaces separated by H and let $\lambda \in [0,1]$ be defined by $\lambda \int_{\mathbb{R}^n} f = \int_{H_+} f$. There exists $z \in H$ such that

$$\int f(x)\,dx \int g(y)\,\mathrm{e}^{y\cdot z}\,dy \;\leq\; \frac{1}{4\lambda(1-\lambda)}(2\pi)^n.$$

In particular, there exists $z \in \mathbb{R}^n$ such that

$$\int f(x)\,dx \int g(y)\mathrm{e}^{y\cdot z}\,dy \;\leq\; (2\pi)^n.$$

This theorem yields the following functional Santaló inequality:

Corollary

Let f and g satisfy

$$\forall x, y \quad f(x)g(y) \leq \mathrm{e}^{-x\cdot y}$$

If f (or g) has its barycenter at 0 then

$$\int f(x)\,dx \int g(y)\,dy \;\leq\; (2\pi)^n.$$

Remark

In Artstein, Klartag and Milman's result, one has to assume that the function that has its barycenter at 0 is log-concave.

Proof of the corollary.

Suppose for example that $\mathrm{bar}(g) = 0$. Let us define

$$h : z \in \mathbb{R}^n \to \int_{\mathbb{R}^n} g(x) \mathrm{e}^{x \cdot z} \, dx,$$

The function h attains its minimum at 0. On the other hand, by the preceding theorem, there exists z such that

$$\int f(x) \, dx \int g(x) \, \mathrm{e}^{x \cdot z} dx \leq (2\pi)^n. \qquad \square$$

We recall the Prékopa-Leindler inequality: if f, g, h are non-negative function on $\mathbb{R}^n$ satisfying

$$f(x)^\lambda g(y)^{1-\lambda} \leq h(\lambda x + (1-\lambda)y)$$

for all x, y in $\mathbb{R}^n$ and for some fixed $\lambda \in (0,1)$, then

$$\Big(\int_{\mathbb{R}^n} f(x)\, dx\Big)^\lambda \Big(\int_{\mathbb{R}^n} g(y)\, dy\Big)^{1-\lambda} \leq \int_{\mathbb{R}^n} h(z)\, dz.$$

Lemma

Let ϕ_1, ϕ_2, ρ be non-negative Borel functions on $\mathbb{R}_+$, with ρ integrable. If

$$\forall s, t > 0, \quad \phi_1(s)\phi_2(t) \le \rho(\sqrt{st})^2$$

then

$$\int_{\mathbb{R}_+} \phi_1(s)\, ds \int_{\mathbb{R}_+} \phi_2(t)\, dt \le \Big(\int_{\mathbb{R}_+} \rho(r)\, dr\Big)^2.$$

Proof.

Apply the Prékopa-Leindler inequality (with $\lambda = 1/2$) to

$$\begin{aligned} f(x) &= \phi_1(\mathrm{e}^x)\mathrm{e}^x \\ g(y) &= \phi_2(\mathrm{e}^y)\mathrm{e}^y \\ h(z) &= \rho(\mathrm{e}^z)\mathrm{e}^z. \quad \square \end{aligned}$$

Remark

When $\rho(r) = \mathrm{e}^{-r^2/2}$ this lemma becomes: if

$$\forall s, t > 0, \quad \phi_1(s)\phi_2(t) \leq \mathrm{e}^{-st}$$

then

$$\int_{\mathbb{R}_+} \phi_1(s)\, ds \int_{\mathbb{R}_+} \phi_2(t)\, dt \leq \frac{\pi}{2}.$$

Proof of the theorem (case $\lambda = 1/2$)

In dimension 1: assume $\int f = 1$, let r be a median of f. We apply the preceding inequality to $\phi_1(s) = f(s+r)$ and $\phi_2(t) = g(t)\mathrm{e}^{rt}$. We get

$$\int_r^\infty f(s)\,ds \int_0^\infty g(t)\mathrm{e}^{rt}\,dt \leq \frac{\pi}{2}$$

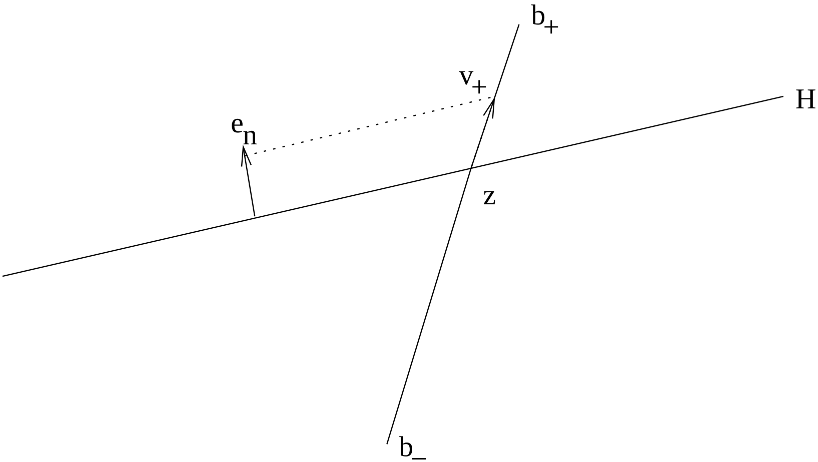
hence

$$\int_0^\infty g(t)\mathrm{e}^{rt}\,dt \leq \pi.$$

Similarly, working with $\phi_1(s) = f(-s+r)$ and $\phi_2(t) = g(-t)\mathrm{e}^{-rt}$, we obtain

$$\int_{-\infty}^0 g(t)\mathrm{e}^{rt}\,dt \leq \pi.$$

Adding the last two inequalities yields the result in dimension 1.



We define

$$b_+ = \mathrm{bar}(f_{|H_+}) \quad \text{and} \quad b_- = \mathrm{bar}(f_{|H_-}).$$

We are going to prove that

$$\int_{\mathbb{R}^n} g(x)\mathrm{e}^{z\cdot x}\,dx \le (2\pi)^n.$$

Let L be the hyperplane parallel to H passing through 0 and $e_1, \dots, e_n$ an orthonormal basis such that $\mathrm{span}(e_1 \dots e_{n-1}) = L$. We define

$$v_+ = \frac{b_+ - z}{(b_+ - z)\cdot e_n}$$

and

$$F_+ : y \in L \to \int_{\mathbb{R}_+} f(z + y + sv_+)\,ds.$$

Using the linear map A defined by

$$\begin{aligned} e_i &\longrightarrow e_i, \quad i = 1 \dots n-1 \\ e_n &\longrightarrow v_+ \end{aligned}$$

one gets

$$\int_L F_+(y)\,dy = \int_{H_+} f(x)\,dx = \frac{1}{2}.$$

and

$$\mathrm{bar}(F_+) = \pi(b_+ - z) = 0,$$

where π is the projection with image L and kernel $\mathbb{R}v_+$.

Defining $B = (A^{-1})^t$ and

$$G_+ : y' \in L \to \int_{\mathbb{R}_+} g(By' + te_n) \mathrm{e}^{z \cdot (By' + te_n)} \, dt,$$

we have

$$(y + sv_+) \cdot (By' + te_n) = y \cdot y' + st$$

for all $s, t \in \mathbb{R}$ and $y, y' \in L$. Hence

$$f(z + y + sv_+) g(By' + te_n) \mathrm{e}^{z \cdot (By' + te_n)} \leq \mathrm{e}^{-st - y \cdot y'}.$$

Applying the lemma to

$$\begin{aligned}\phi_1(s) &= f(z + y + sv_+)\\ \phi_2(t) &= g(By' + te_n)\mathrm{e}^{z\cdot(By'+te_n)+y\cdot y'}\end{aligned}$$

we get

$$F_+(y)\, G_+(y') \;\le\; \frac{\pi}{2}\,\mathrm{e}^{-y\cdot y'}$$

for every $y, y' \in L$. Then, by the induction assumption,

$$\int_L F_+(y)\, dy \;\int_L G_+(y')\, dy' \;\le\; \frac{\pi}{2}(2\pi)^{n-1}.$$

which yields

$$\int_{L_+} g(Bx)\,\mathrm{e}^{z\cdot Bx}\, dx \;\le\; \frac{1}{2}(2\pi)^n.$$

Similarly, working with

$$F_- : y \in L \;\; \rightarrow \;\; \int_{\mathbb{R}_+} f(y - sv_+)\, ds \qquad \text{and}$$

$$G_- : y' \in L \;\; \rightarrow \;\; \int_{\mathbb{R}_+} g(By' - te_n)\, \mathrm{e}^{z\cdot(By' - te_n)}\, dt$$

we would obtain

$$\int_{L_-} g(Bx)\, \mathrm{e}^{z\cdot Bx}\, dx \; \leq \; \frac{1}{2}(2\pi)^n.$$

We get

$$\int_{\mathbb{R}^n} g(Bx)\, \mathrm{e}^{z\cdot Bx}\, dx \leq (2\pi)^n$$

which is the result since B has determinant 1.

The Fradelizi-Meyer inequality

Let f be a non-negative Borel function on $\mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that for any non-negative Borel functions g, ρ defined respectively on $\mathbb{R}^n$ and $\mathbb{R}_+$ and satisfying

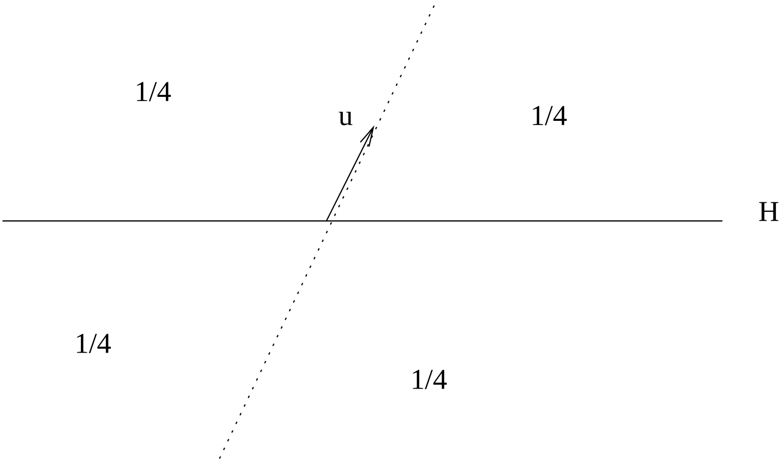
$$\forall x, y \in \mathbb{R}^n,\ \big(x \cdot y \geq 0\big) \ \Rightarrow\ f(z+x)g(y) \ \leq\ \rho\big(\sqrt{x \cdot y}\big)^2,$$

we have

$$\int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} g(y)\, dy \ \leq\ \Big(\int_{\mathbb{R}^n} \rho(|x|_2)\, dx \Big)^2.$$

Such a point z is called a Santaló point for the function f.

Clearly in dimension 1 any median for f is a Santaló point. In dimension 2 the following construction gives a Santaló point.



For dimension 3 and beyond it is more complicated. We use a construction due to Yao and Yao:

Definition

Let f be a non-negative Borel function on $\mathbb{R}^n$. We say that $c \in \mathbb{R}^n$ is a projective center if the following alternative holds

- if $n = 1$ then c is the median of f
- if $n > 1$ then the horizontal hyperplane H containing c is median for f and there exists $u \in \mathbb{S}_+^{n-1}$ such that c is a projective center for both

$$F_+ : y \in H \to \int_{\mathbb{R}_+} f(y + tu)\, dt$$

and

$$F_- : y \in H \to \int_{\mathbb{R}_+} f(y - tu)\, dt.$$

Copying the proof of the preceding theorem one can prove:

Theorem

Let f be a non-negative Borel integrable function on $\mathbb{R}^n$, any projective center for f is also a Santaló point.