“Vertex index of convex bodies and asymmetry of convex polytopes”

(based on joint works with K. Bezdek and E.D. Gluskin)

Motivation

Conjecture. *Every d-dimensional convex body can be covered by 2^d smaller positively homothetic copies of itself.*

In other words, for every convex body $\mathbf{K} \subset \mathbb{R}^d$ there exists $0 < \lambda < 1$ and points $x_i \in \mathbb{R}^d$, $i \leq 2^d$, such that

$$\mathbf{K} \subset \bigcup_{i=1}^{2^d} (x_i + \lambda \mathbf{K})$$

Remarks 1. One needs exactly 2^d translations in the case of d-dimensional cube (for every $1/2 \leq \lambda < 1$).

2. The best known result is $Cd \ln d \; 2^d$.

Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$.

Def. 1. A $p \in \mathbb{R}^d \setminus \mathbf{K}$ illuminates a boundary point q of $\mathbf{K}$ if the ray emanating from p and passing through q intersects the interior of $\mathbf{K}$ (after the point q).

Def. 2. A family of exterior points of $\mathbf{K}$, $\{p_1, p_2, \ldots, p_m\} \subset \mathbb{R}^d \setminus \mathbf{K}$, illuminates $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of p_i's.

Boltyanski-Hadwiger conjecture. *Every d-dimensional convex body can be illuminated by 2^d points.*

Remarks 1. Clearly, we need 2^d points to illuminate the d-dimensional cube.

2. Two conjectures above are equivalent.

Although computing the smallest number of points illuminating a given body is very important, it does not provide any quantitative information on points of illumination. In particular, one can take such poins to be very far from the body. To control that, in 1992 K. Bezdek introduced the *illumination parameter*, $\mathrm{ill}(\mathbf{K})$, of $\mathbf{K}$ as follows:

$$\mathrm{ill}(\mathbf{K}) = \inf \left\{ \sum_i \|p_i\|_{\mathbf{K}} \quad \middle| \quad \{p_i\}_i \text{ illuminates } \mathbf{K} \right\}.$$

Here $\|x\|_{\mathbf{K}}$ denotes the gauge (or Minkowsky functional) of $\mathbf{K}$, i.e.

$$\|x\|_{\mathbf{K}} = \inf\{\lambda > 0 \mid x \in \lambda\mathbf{K}\}.$$

This insures that far-away points of illumination are penalized.

K. Bezdek posed the problem of finding the upper bound for the illumination parameter. He also provided some estimates and conjectured that ill$(\mathbf{B}_2^d) = 2d^{3/2}$ and ill$(\mathbf{K}) \geq 2d$ for every 0-symmetric body $\mathbf{K}$.

Motivated by the notion of the illumination parameter in 2004 Swanepoel introduced the *covering parameter*, of a convex body $\mathbf{K}$,

$$\mathrm{cov}(\mathbf{K}) = \inf \Big\{ \sum_i \frac{1}{1-\lambda_i} \ \mid$$

$$\mathbf{K} \subset \bigcup_i (x_i + \lambda_i \mathbf{K}), 0 < \lambda_i < 1, x_i \in \mathbb{R}^d \Big\}.$$

In this way homothets almost as large as $\mathbf{K}$ are penalized. Swanepoel obtained the following inequality.

Theorem. *There exists an absolute constant $C > 0$ such that for every d and every 0-symmetric convex body $\mathbf{K}$ in $\mathbb{R}^d$ one has*

$$\mathrm{ill}(\mathbf{K}) \leq 2\ \mathrm{cov}(\mathbf{K}) \leq C\ 2^d d^2 \ln d.$$

Vertex Index of a convex body.

Idea. To measure the smallest possible closeness to 0 of the vertex set of a polytope containing K. In other words, we want to inscribe a (0-symmetric) convex body into a polytope with small number of vertices which are not far away from the origin.

Let $\mathbf{K}$ be a 0-symmetric convex body in $\mathbb{R}^d$. Let $\mathbf{K} \subset \mathbf{P} = \text{conv}\,\{p_i\}_{i \le m}$. We introduce the *vertex index* of $\mathbf{K}$ as follows:

$$\text{vein}(\mathbf{K}) = \inf \left\{ \sum_i \|p_i\|_{\mathbf{K}} \;\; \Big| \;\; \mathbf{K} \subset \text{conv}\,\{p_i\}_i \right\}.$$

Claim 1. *For any* 0*-symmetric convex body* $\mathbf{K}$ *in* $\mathbb{R}^d$ *and any invertible linear operator* T *one has*

$$\text{vein}(\mathbf{K}) = \text{vein}(T\mathbf{K}).$$

Claim 2. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be* 0*-symmetric convex bodies in* $\mathbb{R}^d$. *Then*

$$\text{vein}(\mathbf{K}) \leq d\,(\mathbf{K}, \mathbf{L}) \cdot \text{vein}(\mathbf{L}),$$

where $d(\cdot, \cdot)$ *denotes the Banach-Mazur distance.*

Claim 3. *For any convex body* $\mathbf{K}$ *in* $\mathbb{R}^d$ *one has*

$$\text{vein}(\mathbf{K}) \leq \text{ill}(\mathbf{K}).$$

Moreover, if $\mathbf{K}$ *is smooth then*

$$\text{vein}(\mathbf{K}) = \text{ill}(\mathbf{K}).$$

Remark. Note that $\text{ill}(\mathbf{B}^d_\infty) = 2^d$, while below we will see that $\text{vein}(\mathbf{K}) \leq Cd^{3/2} \ln d$.

Theorem. *There are absolute positive constants c and C such that for every 0-symmetric convex body $\mathbf{K}$ in $\mathbb{R}^d$ one has*

$$\frac{d^{3/2}}{\sqrt{2\pi e}\ \mathrm{ovr}(\mathbf{K})} \leq \mathrm{vein}(\mathbf{K}) \leq C\ d^{3/2}\ \ln(2d).$$

Here $\mathrm{ovr}(\mathbf{K})$ is the outer volume ratio of $\mathbf{K}$,

$$\mathrm{ovr}(\mathbf{K}) = \inf \left(\frac{\mathrm{vol}\,(\mathcal{E})}{\mathrm{vol}\,(\mathbf{K})} \right)^{1/d}$$

where the infimum is taken over all ellipsoids $\mathcal{E} \supset \mathbf{K}$ and $\mathrm{vol}\,(\cdot)$ denotes the volume.

Remark. There exists a body $\mathbf{K}$ such that

$$\mathrm{ovr}(\mathbf{K}) \geq c\ \frac{\sqrt{d}}{\ln d} \quad \text{and} \quad \mathrm{vein}(\mathbf{K}) \geq c\ \frac{d^{3/2}}{\ln(2d)}.$$

Theorem. *For every d one has*

$$\frac{d^{3/2}}{\sqrt{2\pi e}} \leq \text{vein}(\mathbf{B}_2^d) \leq 2d^{3/2}$$

and

$$\frac{d^{3/2}}{\pi e} \leq \text{vein}(\mathbf{B}_\infty^d) \leq 5d^{3/2}.$$

Moreover,

$$\text{vein}(\mathbf{B}_1^d) = 2d.$$

Conjecture.

$$\text{vein}(\mathbf{B}_2^d) = 2d^{3/2}.$$

Theorem. The conjecture is true in dimensions 2 and 3.

Asymmetry of convex polytopes.

One of natural ways to measure the asymmetry of a convex body $\mathbf{K}$ is the following parameter

$$\partial(\mathbf{K}) = \inf_{a \in \mathbf{K}} \sup_{x \in \mathbf{K}} \| - x\|_{\mathbf{K}-a}.$$

It is not difficult to see that $\partial(\mathbf{K})$ is equivalent (up to the constant 2) to the minimal possible Banach-Mazur distance between $\mathbf{K}$ and a symmetric convex body. It is also known that for every convex body $\mathbf{K}$ one has

$$\partial(\mathbf{K}) \leq d.$$

In 2001 with Gluskin we investigated the behavior of $\partial(\mathbf{K})$ when $\mathbf{K}$ is a polytope with small number of vertices. More precisely, we investigated the following function

$$f(k,d) = \inf \partial(\mathbf{K}),$$

where the infimum is taken over all non-degenerated convex polytopes in $\mathbb{R}^d$ with $d+k$ vertices $(1 \leq k \leq d)$.

Clearly, $f(k,d) = 1$ for $k = d$ (take the octahedron). On the other hand, it is known that

$$f(1,d) = d = \sup\left\{\partial(\mathbf{K}) \;\mid\; \mathbf{K} \subset \mathbb{R}^d\right\}$$

i.e. a d-dimensional simplex is the most asymmetric body. It is very natural to ask how fast $f(\cdot,d)$ decreases. We proved that

$$d/k \leq f(k,d) \leq \lceil d/k \rceil,$$

where $\lceil a \rceil$ denotes the smallest integer larger than or equal to a.

If $\mathbf{K}$ is a convex polytope with vertices x_i then

$$\partial(\mathbf{K}) = \inf_{a \in \mathbf{K}} \max_i \| - x_i \|_{\mathbf{K}-a}.$$

Thus the functional $\partial(\cdot)$ takes into account only one (the worst) vertex of $\mathbf{K}$. Here we suggest another averaging-type functional to measure asymmetry. Namely, given convex polytope $\mathbf{K}$ with m vertices x_i's we consider

$$\phi(\mathbf{K}) = \inf_{a \in \mathbf{K}} \frac{1}{m} \sum_{i=1}^{m} \| - x_i \|_{\mathbf{K}-a}$$

and the function $g(k,d) = \inf \phi(\mathbf{K})$, where the infimum is taken over all non-degenerated convex polytopes in $\mathbb{R}^d$ with $d+k$ vertices ($k \le d$).

Clearly, $g(k,d) \le f(k,d)$. We show that

$$g(k,d) \ge d/(2k) \qquad (\text{so} \quad g(k,d) \ge f(k,d)/4).$$

More precisely, we proved

Theorem. *Let $\mathbf{K} \subset \mathbb{R}^d$ be a polytope with $d+k$ verices, where $1 \le k \le d$. Then*

$$\inf_{a \in \mathbf{K}} \sum_{i=1}^{d+k} \| - x_i \|_{\mathbf{K}-a} \ge \max \left\{ 2d, \; \frac{d(d+k)}{2k} \right\}.$$

An application.

Theorem. *Let* $\mathbf{K} = -\mathbf{K} \subset \mathbb{R}^d$*. Then*

$$\mathrm{ill}(\mathbf{K}) \geq \mathrm{vein}(\mathbf{K}) \geq 2d.$$

Remark. Note that $\mathrm{ill}(\mathbf{B}_1^d) = \mathrm{vein}(\mathbf{B}_1^d) = 2d$.

Proof: Let $\mathbf{K} \subset \mathbf{L} = \mathrm{conv}\,\{p_i\}_{i \leq m}$. WLOG we can assume that $\|p_i\|_{\mathbf{K}} \geq 1$ for every i. If $m \geq 2d$ then we trivially have

$$\sum_{i=1}^{m} \|p_i\|_{\mathbf{K}} \geq m \geq 2d.$$

Assume $m < 2d$. Since $\mathbf{K} = -\mathbf{K} \subset \mathbf{L}$, we have $\|-x\|_{\mathbf{L}} \leq \|x\|_{\mathbf{K}}$ for every $x \in \mathbb{R}^d$. Therefore, applying our Theorem, we obtain

$$\sum_{i=1}^{m} \|p_i\|_{\mathbf{K}} \geq \sum_{i=1}^{m} \|-p_i\|_{\mathbf{L}} \geq 2d,$$

which completes the proof. □

Now we outline the proof of the estimate. First we need the following Proposition.

Proposition 1 *Let $T = \{t_{ij}\}$ be $m \times m$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigenvalues of T. Then*

$$\sum_{j=1}^{m} |\lambda_j| \leq \sum_{i,j=1}^{m} |t_{ij}|.$$

Proof: As usual, let s_j's denote the singular values of T. By Weil's Theorem,

$$\sum_{j=1}^{m} |\lambda_j| \leq \sum_{j=1}^{m} |s_j| = \gamma(T) \leq \sum_{i,j=1}^{m} |t_{ij}|\gamma(E_{ij}),$$

where E_{ij} is the marix with 1 in ith row jh column and 0 otherwise. Clearly, $\gamma(E_{ij}) = 1$, which completes the proof. □

Lemma. *Let* $\Lambda = \{\lambda_{ij}\}$ *be* $m \times m$ *matrix of rank* $k \geq 1$ *with non-negative entries such that* $\lambda_{ii} \geq 1$ *for every* $i \leq m$. *Then*

$$\sum_{i,j=1}^{m} \lambda_{ij} \geq 3m - 2k.$$

Moreover, if $m \geq 2k$ *then*

$$\sum_{i,j=1}^{m} \lambda_{ij} \geq \frac{m(m-1)}{2k-1} + m \geq \frac{m^2}{2k} + m.$$

Remark. Note that the estimate of the Lemma is asymptotically sharp. Indeed, consider a block-diagonal matrix with k blocks $[m/k] \times [m/k]$ or $\lceil m/k \rceil \times \lceil m/k \rceil$ of rank one, such that each block has entries 1 only. Then we have

$$\sum_{i,j=1}^{m} \lambda_{ij} \leq \frac{m^2}{k} + \frac{k}{4}.$$

First, WLOG, we assume that $\lambda_{ii} = 1$ for every i (otherwise pass to the matrix $\{\lambda_{ij}/\lambda_{ii}\}_{ij}$).

Consider $T = \Lambda - I$, where I is the identity and denote its entries by t_{ij}. Clearly, $t_{ij} \geq 0$ and $t_{ii} = 0$ for every $i, j \leq m$. By $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the eigenvalues of T.

Since Λ is of rank k, at least $m-k$ of eigenvalues of T are equal to -1 (indeed, $T = -I$ on KerΛ). Since

$$0 = \sum_{i=1}^{m} t_{ii} = \text{Trace}T = \sum_{i=1}^{m} \lambda_i,$$

we obtain

$$\sum_{i=1}^{m} |\lambda_i| \geq 2m - 2k.$$

Proposition 1 implies

$$\sum_{i,j=1}^{m} t_{ij} \geq 2m - 2k,$$

which shows

$$\sum_{i,j=1}^{m} \lambda_{ij} \geq 3m - 2k.$$

Now we assume that $m \geq 2k$. Let $\sigma \subset \{1, 2, \ldots, m\}$ be of cardinality l for some $2k \leq l \leq m$. Let

$$\bar{\Lambda} = \left\{\lambda_{ij}\right\}_{i,j\in\sigma}.$$

Clearly the rank of $\bar{\Lambda}$ does not exceed k, so, by the first part, we have

$$\sum_{i,j\in\sigma} \lambda_{ij} \geq 3l - 2k.$$

Using averaging argument, we obtain

$$\begin{aligned}
\sum_{i,j=1}^{m} \lambda_{ij} &= m + \sum_{\substack{i,j=1 \\ i\neq j}}^{m} \lambda_{ij} \\
&= m + \binom{m-2}{l-2}^{-1} \sum_{\substack{\sigma\subset\{1,2,\ldots,m\} \\ |\sigma|=l}} \sum_{\substack{i,j\in\sigma \\ i\neq j}} \lambda_{ij} \\
&\geq m + \binom{m-2}{l-2}^{-1} \binom{m}{l} (2l - 2k) \\
&= m + 2\frac{m(m-1)}{l(l-1)} (l - k).
\end{aligned}$$

The choice $l = 2k$ completes the proof. $\square$

Theorem 1 *Let $1 \le k \le d$ and $m = k + d$. Let $\mathbf{K}$ be a convex polytope in $\mathbb{R}^d$ with m vertices x_1, x_2, ..., x_m. Then*

$$\sum_{i=1}^{m} \|-x_i\|_{\mathbf{K}} \ge \frac{m^2}{2k} \ge \max\left\{2d,\ \frac{md}{2k}\right\}.$$

Proof: Consider the operator $T : \mathbb{R}^m \to \mathbb{R}^d$ defined by $Te_i = x_i$. Denote the kernel of T by L. Clearly, L is a k-dimensional subspace of $\mathbb{R}^m$. The orthogonal projection onto $L^\perp$ we denote by P.

$$A := \sum_{i=1}^{m} \|-x_i\|_{\mathbf{K}} =$$

$$\sum_{i=1}^{m} \sup\left\{\langle f, -x_i\rangle \ \mid f \in \mathbb{R}^d, \left\langle f, x_j\right\rangle \le 1\ \forall j \le m\right\}.$$

Using $\langle f, x_i\rangle = \langle f, Te_i\rangle = \langle T^* f, e_i\rangle$, we get $A =$

$$\sum_{i=1}^{m} \sup\left\{\langle h, -e_i\rangle \ \mid h \in \mathbb{R}^m \cap L^\perp, \left\langle h, e_j\right\rangle \le 1\ \forall j \le m\right\}.$$

Now denote

$$S := \{h \in \mathbb{R}^m \mid \left\langle h, e_j \right\rangle \leq 1 \ \forall j \leq m\}$$

and for every $i \leq m$ denote

$$Q_i := \{h \in \mathbb{R}^m \mid \langle h, e_i \rangle \geq -1\}.$$

Then

$$S^0 = \{h \in \mathbb{R}^m \mid 0 \leq \left\langle h, e_j \right\rangle \ \forall j \leq m, \ \sum_{j=1}^{m} \left\langle h, e_j \right\rangle \leq 1\}$$

and

$$Q_i^0 = \{h \in \mathbb{R}^m \mid -1 \leq \langle h, e_i \rangle \leq 0, \left\langle h, e_j \right\rangle = 0 \, \forall j \neq i\}.$$

Using duality and our notation, we observe

$$\begin{aligned} A &= \sum_{i=1}^{m} \sup_{h \in S \cap L^{\perp}} \langle h, -e_i \rangle = \sum_{i=1}^{m} \sup_{h \in S \cap L^{\perp}} \|h\|_{Q_i} \\ &= \sum_{i=1}^{m} \sup_{h \in Q_i^0} \|h\|_{PS^0} = \sum_{i=1}^{m} \|-e_i\|_{PS^0}. \end{aligned}$$

Note

$$\|z\|_{S^0} := \begin{cases} \sum_{j=1}^{m} \left\langle z, e_j \right\rangle & \text{if} \quad \left\langle z, e_j \right\rangle \geq 0 \quad \forall j \leq m, \\ \infty & \text{otherwise,} \end{cases}$$

which implies

$$\|z\|_{PS^0} = \inf_{y \in L} \|z + y\|_{S^0} = \inf \sum_{j=1}^{m} \left\langle z + y, e_j \right\rangle$$

where the last infimum is taken over all $y \in L$ satisfying $\left\langle y, e_j \right\rangle \geq -\left\langle z, e_j \right\rangle \quad \forall j \leq m$.

Thus

$$A = \sum_{i=1}^{m} \inf \left(\sum_{j=1}^{m} \left\langle y, e_j \right\rangle - 1 \right),$$

where the infimum (which depends on i) is taken over all $y \in L$ satisfying $\langle y, e_i \rangle \geq 1$ and $\left\langle y, e_j \right\rangle \geq 0 \quad \forall j \leq m$.

Assume it attains on $y_i \in L$, $i \leq m$. Denoting $y_{ij} := \left\langle y_i, e_j \right\rangle$, we observe that $y_{ij} \geq 0$ and $y_{ii} \geq 1$, for every $i, j \leq m$, and that the matrix $\{y_{ij}\}$ has rank at most k. Since $m = d + k \geq 2k$, applying Lemma, we obtain

$$A = \sum_{i=1}^{m} \sum_{j=1}^{m} y_{ij} - m \geq \frac{m(m-1)}{2k-1} \geq \frac{m^2}{2k}.$$

$\square$