Marchenko-Pastur distribution for random vectors with log-concave law

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Joint work with Leonid Pastur

Global regime for sample covariance matrices

We consider a sequence of real or complex $m \times n$ matrices

$$\Gamma_{n,m} \quad n = 1, 2, \ldots$$

$$\Gamma_{n,m} = \begin{pmatrix} \gamma_{11}^{(n)} & \cdots & \gamma_{1n}^{(n)} \\ \gamma_{21}^{(n)} & \cdots & \gamma_{2n}^{(n)} \\ \vdots & \ddots & \vdots \\ \gamma_{m1}^{(n)} & \cdots & \gamma_{mn}^{(n)} \end{pmatrix}$$

with $m \sim cn$ and $c > 1$. We suppose that these matrices are **isotropic**, that is:

$$\text{for all } i, j \quad \mathbb{E}\gamma_{ij}^{(n)} = 0 \quad \text{and} \quad \mathbb{E}|\gamma_{ij}^{(n)}|^2 = \frac{1}{n}.$$

in the complex case, we suppose moreover that $\mathbb{E}(\gamma_{ij}^{(n)})^2 = 0$.

Eigenvalue counting measure

Denote $\lambda_1 \leq ... \leq \lambda_n$ the eigenvalues of the real symmetric (or hermitian) $n \times n$ matrix $\Gamma^*\Gamma$ and introduce their Normalized Counting (or empirical) Measure $N_{n,m}$, setting for any interval $\Delta \subset \mathbb{R}$

$$N_{n,m}(\Delta) = \mathrm{Card}\{\ell \in [1, n] : \lambda_\ell \in \Delta\}/n.$$

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Let $c > 1$, it was shown by **Marchenko and Pastur [MP] (1967)** that if all the components of the matrices are i.i.d. random variables, then there exists a non-random probability measure N such that for any interval $\Delta \subset \mathbb{R}$ we have the convergence in probability

$$\lim_{n\to\infty,\ m\to\infty,\ m/n\to c} N_{n,m}(\Delta) = N(\Delta)$$

where N is the so-called Marchenko-Pastur law.

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N is supported on the interval $[a, b]$ with $a = \left(1 - \frac{1}{\sqrt{c}}\right)^2, b = \left(1 + \frac{1}{\sqrt{c}}\right)^2$ and with density

$$\frac{c}{2\pi x}\sqrt{(b-x)(x-a)}, \quad x \in [a, b].$$

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Note: The particular case with Gaussian components is known since the 30th in statistics as the Wishart matrix.

General setting and spherical case

Let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. random vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and $\{\tau_\alpha\}_{\alpha=1}^m$ be i.i.d. random variables with common probability law σ. Set

$$H_{n,m} = \sum_{\alpha=1}^m \tau_\alpha Y_\alpha \otimes Y_\alpha.$$

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$$\Gamma_{n,m} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}$$

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Denote $\lambda_1 \leq ... \leq \lambda_n$ the eigenvalues of the real symmetric (or hermitian) $n \times n$ matrix $H_{n,m}$ and introduce their Normalized Counting (or empirical) Measure $N_{n,m}$, setting for any interval $\Delta \subset \mathbb{R}$

$$N_{n,m}(\Delta) = \mathrm{Card}\{\ell \in [1,n] : \lambda_\ell \in \Delta\}/n.$$

It was shown by **Marchenko and Pastur [MP]**(1967) that if $\{Y_\alpha\}_{\alpha=1}^m$ are uniformly distributed over the unit sphere of $\mathbb{R}^n$ (or $\mathbb{C}^n$), then there exists a non-random probability measure N such that for any interval $\Delta \subset \mathbb{R}$ we have the convergence in probability

$$\lim_{n\to\infty,\ m\to\infty,\ m/n\to c} N_{n,m}(\Delta) = N(\Delta).$$

$$L_\alpha(X) = Y_\alpha \otimes Y_\alpha(X) = (X, Y_\alpha)Y_\alpha,\ \forall X \in \mathbb{R}^n(\mathbb{C}^n)$$

Non independent entries: the ℓ_p case

A more general but similar case as the spherical one was observed by **L. Pastur and A. P.** (2004).

Non independent entries: the ℓ_p case

Let $p \geq 1$. Let $\{Y_\alpha\}_{\alpha=1}^m$ be uniformly distributed over a ball $\{\sum_1^n |x_i|^p \leq r^p\}$ of the n-dimensional ℓ_p^n space that has been rescaled so that the matrix

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This property is very particular to the ℓ_p space (or of similar spaces) and is not true in general even for unconditional space (see **S. Bobkov** and **J. O. Wojtaszczyk**).

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The method is based on the following result of **Barthe - Guédon - Mendelson - Naor**.

Let $X_1, \ldots, X_n$ and Z be independent so that the first have a distribution with a density of the form $c_p e^{-|t|^p}$ and the last has an exponential law. Let $X = (X_1, \ldots, X_n)$, then

$$\frac{X}{\left(\sum_1^n |X_i|^p + Z\right)^{1/p}}$$

generates the Lebesgue measure on $\{\sum_1^n |x_i|^p \leq 1\}$.

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This representation which creates almost independent entries allows to transfer results on Wishart matrices to this case of random sampling by bringing back to the classical Marchenko-Pastur theorem for i.i.d. entries.

Log-concave setting

Definition of isotropic vectors. A random real vector $Y \in \mathbb{R}^n$ is called isotropic if

$$\mathbf{E}\{(Y, X)\} = 0 \quad \text{and} \quad \mathbf{E}\{|(Y, X)|^2\} = n^{-1}|X|^2, \ \forall X \in \mathbb{R}^n,$$

where $|X|$ denotes the Euclidean norm of X.

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A random **complex vector** $Y \in \mathbb{C}^n$ is called isotropic if $(\Re Y, \Im Y) \in \mathbb{R}^{2n}$ is isotropic; in others words, $\Re Y$ and $\Im Y$ are isotropic and not correlated.

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Alert on normalization!!

Recall also that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called log-concave if for any $\theta \in [0, 1]$ and any $X_1, X_2 \in \mathbb{R}^n$, then $f\big(\theta X_1 + (1 - \theta)X_2\big) \geq f(X_1)^\theta f(X_2)^{1-\theta}$.

A measure μ on $\mathbb{R}^n$ with a log-concave density will be called **log-concave.**

Universal principle for log-concave measure

Theorem (L. Pastur and A. P.) Let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. isotropic random vectors of $\mathbb{R}^n$(or $\mathbb{C}^n$) with a log-concave distribution and $\{\tau_\alpha\}_{\alpha=1}^m$ be i.i.d. random variables with a common probability law σ. Consider random matrices

$$H_{n,m} = \sum_{\alpha=1}^{m} \tau_\alpha Y_\alpha \otimes Y_\alpha.$$

Then there exist a probability measure N such that for any interval $\Delta \subset \mathbb{R}$ we have in probability

$$\lim_{n\to\infty, m\to\infty, m/n\to c\in[0,\infty)} N_{n,m}(\Delta) = N(\Delta).$$

- N is uniquely determined by its Stieltjes transform which satisfies the following equation:

$$f(z) = -\left(z - c\int_{\mathbb{R}} \frac{\tau\sigma(d\tau)}{1+\tau f(z)}\right)^{-1}. \qquad (*)$$

Stieltjes transform method: [MP] (1967)

Introduce the Stieltjes transform

$$f(z) = \int_{\mathbb{R}} \frac{N(d\lambda)}{\lambda - z}, \ \Im z \neq 0$$

of a measure N and the resolvent of a real symmetric (hermitian) matrix A

$$G_A(z) = (A - z)^{-1}, \ \Im z \neq 0.$$

The use of the Stieltjes transform is based on the spectral theorem. Let $N_{n,m}$ be the Normalized Counting Measure of eigenvalues of $H_{n,m}$. Denote

$$g_{n,m}(z) = \int_{\mathbb{R}} \frac{N_{n,m}(d\lambda)}{\lambda - z}, \ \Im z \neq 0$$

then

$$g_{n,m}(z) = \frac{1}{n} \mathrm{Tr}\,(H_{n,m} - z)^{-1} := \frac{1}{n} \mathrm{Tr}\, G_{H_{n,m}}(z).$$

Stieltjes transform method (2)

The mechanism of the proof is the following.

• The expectations of the the Normalized Counting Measure of eigenvalues of $H_{n,m}$ will converge weakly to the measure, whose Stieltjes transform solves

$$f(z) = -\left(z - c\int_{\mathbb{R}} \frac{\tau\sigma(d\tau)}{1+\tau f(z)}\right)^{-1}. \qquad (*)$$

(This equation has a unique solution in the class we consider).

• The Stieltjes transform is a one-to-one correspondence between probability measures and a certain well known class of analytic functions and it is continuous if one consider weak convergence of measure on one side and uniform convergence on compact subset of $\mathbb{C}\setminus\mathbb{R}$ on the other side.

Beginning the proof (1)

- Since for any interval $\Delta \subset \mathbb{R}$,

$$\mathbf{Var}\{N_{n,m}(\Delta)\} \leq 4/n \qquad (**)$$

it suffices to study the convergence of the Stieltjes transform $f_{n,m}$ of $\bar{N}_{n,m}$.

- **Notation:**

$G_{n,m} = G_{H_{n,m}}$ is the resolvent of the matrix $H_{n,m}$.
$g_{n,m}(z) = \frac{1}{n}\mathrm{Tr}\,(H_{n,m} - z)^{-1} = \frac{1}{n}\mathrm{Tr}\,G_{n,m}(z)$ is the Stietjes transform of $N_{n,m}$
$\mathbf{E}\{g_{n,m}\} = f_{n,m}$ is the Stietjes transform of $\bar{N}_{n,m}$.
$L_\alpha(X) = Y_\alpha \otimes Y_\alpha(X) = (X, Y_\alpha)Y_\alpha,\ \forall X \in \mathbb{R}^n(\mathbb{C}^n)$

Continuing the proof: induction and equation $(*)$

Forgetting the indices n, m, let G be the resolvent of $H_{n,m}$ and write $\overline{G}$ for its expectation. Denote also $G_\alpha = G|_{Y_\alpha=0}$.

Now we write the resolvent in order to fit with our equation $(*)$.

We start from

$$\overline{G} = -\frac{1}{z} + \frac{1}{z}\sum_{\alpha=1}^{m} \mathbf{E}\left\{\frac{\tau_\alpha}{1+\tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)} G_\alpha L_\alpha\right\}$$

and notice that $\mathbf{E}\{(G_\alpha Y_\alpha, Y_\alpha)\} = \mathbf{E}\{n^{-1}\mathrm{Tr}\, G_\alpha\} \sim f_{n,m} \sim f$.

Take the normalized trace of $\bar{G}$.

After all these approximations we arrive at the relation $(*)$ we want

$$f \sim -\frac{1}{z} + \frac{1}{z}\frac{m}{n}\int_{\mathbb{R}} \frac{\tau\sigma(d\tau)}{1+\tau f}\, f.$$

Continuing the proof (3)

Finally after all these approximations we arrive at the relation $(*)$ we want

$$f \sim -\frac{1}{z} + \frac{1}{z}\frac{m}{n}\int_{\mathbb{R}} \frac{\tau\sigma(d\tau)}{1+\tau f}\, f.$$

Conclusion: let

$$\overline{G} = -\frac{1}{z} + \frac{1}{z}\frac{m}{n}\int_{\mathbb{R}} \frac{\tau\sigma(d\tau)}{1+\tau f}\overline{G} + R.$$

It suffices to show that the normalized trace of R goes to 0.

For that we first trunctate the vectors to reduce to the case when all $|Y_\alpha|$ are bounded by some universal constant C.

The first term that appears in R after truncation at level C is of the form

$$R_1 = \sum_{\alpha=1}^{m} \mathbf{E}\{\tau_\alpha G(Y_\alpha \otimes Y_\alpha)\mathbf{1}_{|Y_\alpha|\geq t}\}.$$

Truncation. A fundamental result of G. Paouris.

For the truncation technic one needs a good control of quantities like

$$\mathbf{E}\{|Y_\alpha|^q \mathbf{1}_{|Y_\alpha| \geq C}\}$$

for a fixed large C. For this we use the following result:

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Theorem (G. Paouris) There exists $C > 0$ such that for any integer $n \geq 1$ and any isotropic random vector $Y \in \mathbb{R}^n$ with a log-concave distribution we have

$$\mathbf{P}\{|Y| \geq Ct\} \leq \exp(-t\sqrt{n}).$$

for every $t \geq 1$.

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As a consequence,

$$\mathbf{E}\{|Y_\alpha|^2 \mathbf{1}_{|Y_\alpha|\geq C}\} \leq c\exp(-c'\sqrt{n}).$$

Continuing the proof

An other "typical" term of R is of the form:

$$\mathbf{E}\{|\mathbf{E}_\alpha\{(G_\alpha Y_\alpha, Y_\alpha) - f_{n,m}|\}\}$$

which is

$$\leq \mathbf{E}\left\{\mathbf{Var}_\alpha^{1/2}\{(G_\alpha Y_\alpha, Y_\alpha)\}\right\} + \mathbf{E}\{|g_\alpha - g_{n,m}|\} + \mathbf{Var}_\alpha^{1/2}\{g_{n,m}\}.$$

with

$$\mathbf{E}_\alpha\{(G_\alpha Y_\alpha, Y_\alpha)\} = n^{-1}\mathrm{Tr}\, G_\alpha := g_\alpha$$

Here $\mathbf{E}_\alpha\{...\}$ denotes the expectation only with respect to Y_α.

Let us focus on the first term

$$\mathbf{E}\left\{\mathbf{Var}_\alpha^{1/2}\{(G_\alpha Y_\alpha, Y_\alpha)\}\right\}.$$

Recall: $g_{n,m}(z) = \frac{1}{n}\mathrm{Tr}\,(H_{n,m} - z)^{-1} = \frac{1}{n}\mathrm{Tr}\, G_{H_{n,m}}(z)$, $\mathbf{E}\{g_{n,m}\} = f_{n,m}$.

Kannan-Lovász-Simonovits type inequality

The first term again is

$$\mathbf{E}\left\{\mathbf{Var}_\alpha^{1/2}\{(G_\alpha Y_\alpha, Y_\alpha)\}\right\}$$

where G_α is the resolvent of an hermitian matrix. We want to show that it goes to 0 with n, m.

We can reformulate the question: estimate

$$\mathbf{Var}\{(GY, Y)\}$$

in terms of some norm of G, where G is an $n \times n$ hermitian matrix (non-random) and Y is an isotropic random vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with a log-concave law.

Kannan-Lovász-Simonovits type of inequality

$$\mathbf{Var}\, g(Y) \leq \frac{C}{n}\mathbf{E}\,|\nabla g(Y)|^2$$

for $g(Y) = (GY, Y)$ with G an Hermitian matrice.

Central limit problem: a result of B. Klartag

When G is the identity, it is the central limit problem for log-concave measure. An important breakthrough was done recently by **B. Klartag**. We state here the following reformulation which fit with our questions.

Theorem (B. Klartag) There exist positive constants C, α (with $\alpha < 1$), such that for any integer $n \geq 1$ and any isotropic random vector $Y \in \mathbb{R}^n$ with a log-concave distribution we have

$$\mathbf{Var}\{\,|Y|^2\} \leq C/n^{\alpha}.$$

The conjecture is with $\alpha = 1$. An estimate of the type

$$\mathbf{Var}\{\,|Y|^2\} \leq C/\log^{\alpha} n.$$

was first given by **B. Klartag**. Few times later, a similar estimate was proved by **B. Fleury, O. Guédon, G. Paouris** using a different approach. Recently, **B. Klartag** gave the above polynomial bound and proved the conjecture in the unconditionnal case.

End of the proof

From the result of Klartag, we deduce:

Proposition. *Let* A *be an hermitian matrix and let* $Y \in \mathbb{R}^n$ *(or* $\mathbb{C}^n$*) be an isotropic random vector with a log-concave distribution. Then*

$$\mathbf{Var}\{(AY, Y)\} \leq C\|A\|^2/n^\alpha$$

where $\|A\|$ denotes the operator norm.

To conclude our estimate, observe that the resolvent of an hermitian matrix A satisfies

$$||G_A(z)|| \leq |\Im z|^{-1}$$

which allows to show that on any compact domain of $\mathbb{C} \setminus \mathbb{R}$ one has

$$\lim_n \mathbf{Var}_\alpha\{(G_\alpha Y_\alpha, Y_\alpha)\} = 0.$$