Equitable coloring of random graphs

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Phenomena in High Dimensions, Samos 2007

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A proper k-coloring of G is a function $c : V(G) \rightarrow \{1, 2, ..., k\}$ such that if $u, v \in V(G)$ are adjacent, then $c(u) \neq c(v)$.

$\chi(G)$ is the least positive integer k for which there is a k-coloring of G (or equivalently the least positive integer k for which $V(G)$ is the union of k independent sets).

Definition

An **equitable coloring** of a graph is a proper vertex coloring such that the sizes of any two color classes differ by at most one.

Applications

- Scheduling, partitioning and load balancing problems.
- Deviation bounds for sums of random variables with limited dependence.
- H-factors in graphs
- A new proof of Blow-Up Lemma.

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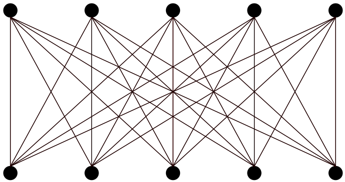

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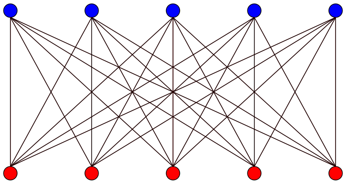

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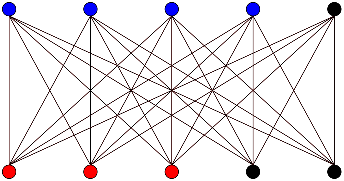

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The least positive integer k such that for any $k' \geq k$ there exists an equitable coloring of a graph G with k' colors is said to be the **equitable chromatic threshold** of G and is denoted by $\chi_=^*(G)$.

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KIERSTEAD - KOSTOCHKA 2006, polynomial time algorithm, 5 page proof

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CHEN - LIH -WU *Let G be a connected graph with maximum degree at most Δ. If G is distinct from $K_{\Delta+1}, K_{\Delta,\Delta}$ (for odd Δ) and is not an odd cycle, then $\chi^*_=(G) \leq \Delta$.*

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True if $\Delta \leq 4$ (CHEN - LIH -WU $\Delta \leq 3$, KIERSTEAD - KOSTOCHKA $\Delta = 4$)

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Theorem

Kostochka, Nakprasit, Pemmaraju *For every* $d, n \geq 1$, *if a graph* G *is* d*-degenerate, has* n *vertices and the maximum degree of* G *is at most* $n/15$, *then* $\chi^*_=(G) \leq 16d$.

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Color $G(n,p)$ equitably!

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But the color classes used for the leftover are small!

Conjecture

There exists a constant C such that if $C/n < p < 0.99$*, then almost surely*

$$\chi(G(n,p)) \le \chi_=(G(n,p)) \le \chi^*_=(G(n,p)) = (1+o(1))\chi(G(n,p))$$

holds (the first two inequalities are true by definition).

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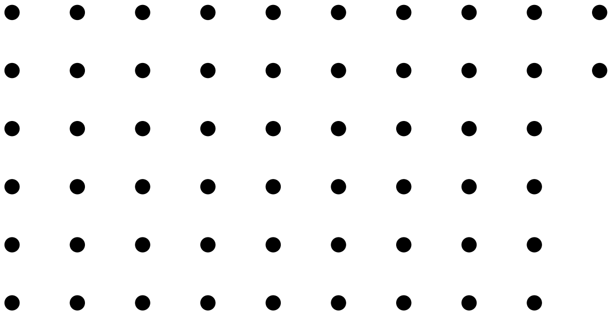

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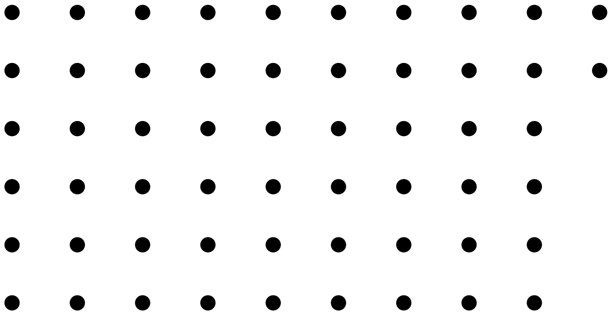

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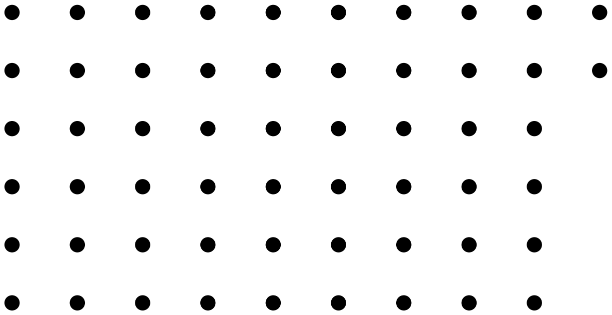

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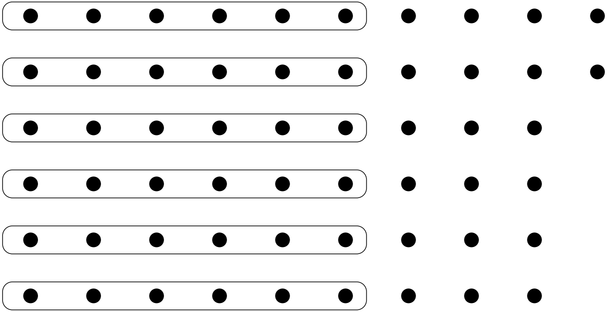
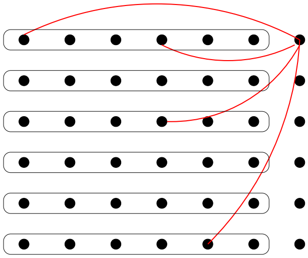
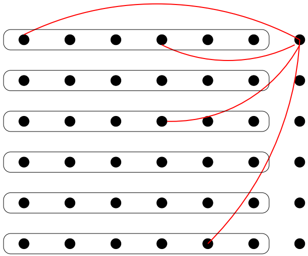
Algorithm in $\lceil n/k \rceil$ turns.
In the $l+1$st turn, we expose the edges between the $l+1$st later and the previous ones.

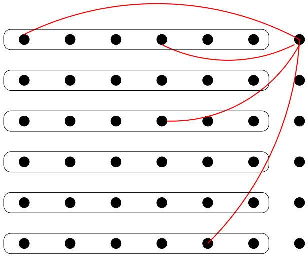


Suppose, we managed to extend each k color classes by one vertex from the "that time current" layer in all previous turns.





We define an auxiliary random bipartite graph $G(k, k, q)$:



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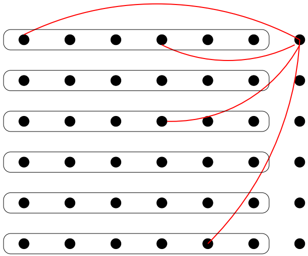

We define an auxiliary random bipartite graph $G(k, k, q)$:

$q = (1 - p)^l$

Theorem

(a) If $p < 0.99$ *and* $\log(np) \gg \log\log n$*, then almost surely we have*

$$\chi(G(n,p)) \le \chi_{=}(G(n,p)) \le \chi_{=}^{*}(G(n,p)) \le (2+o(1))\chi(G(n,p)).$$

(b) If $p \to 0$ *and there is a* $\delta > 1$ *such that* $p \ge \frac{\log^{\delta} n}{n}$*, then almost surely we have*

$$\chi_{=}^{*}(G(n,p)) = O_{\delta}(\chi(G(n,p))).$$

Theorem

If $p < 0.99$ and $p > n^{-\theta}$ for some $\theta < 1/5$, then the following holds almost surely:

$$\chi(G(n,p)) \leq \chi_=(G(n,p)) \leq (1+o(1))\chi(G(n,p)).$$

Case I: $\frac{1}{\log^8 n} \leq p \leq 0.99$

Theorem

Let G be a graph on n vertices in which every induced subgraph $G[U]$ with $|U| \geq m$ contains an independent set of size s. Suppose further that $\frac{n-\Delta(G)-m-ms^2}{s} \geq m$ holds. Then G can be properly colored using color classes only of size s and $s-1$.

Definition

An (n, d, λ)-graph is a d-regular graph on n vertices with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ such that $\lambda \geq \max\{|\lambda_i| : 2 \leq i \leq n\}$.

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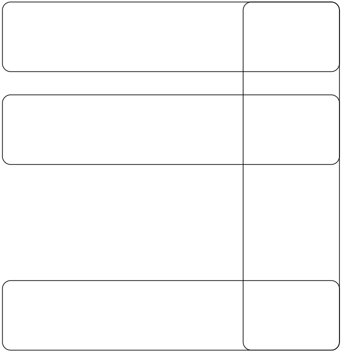
Theorem

Let G_n be a sequence of (n, d, λ)-graphs where $d(n) \leq 0.9n$ and $\frac{d^3}{n^2\lambda} = \Omega(n^\alpha)$ holds for some $\alpha > 0$. Then $\chi_=(G_n) = O(\frac{d}{\log d})$.

Case II: $n^{-1/5} \le p \le \frac{1}{\log^8 n}$

Definition

Independent (t,k)-comb



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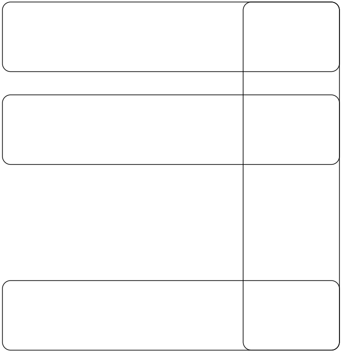

If $p > n^{-1/5}$, then almost surely every large subgraph of $G(n, p)$ contains a large (t, k)-comb.

Theorem

There exists a constant C *such that if* $\frac{C}{n} \leq p \leq \log^{-7} n$*, then a.s.* $\chi_=(G(n,p)) \leq (2+o(1))\chi(G(n,p))$.

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Find the asymptotics of $\chi_=(G(n,p))$ when p tends to 0 very quickly.

Find the asymptotics of $\chi^*_=(G(n,p))$ even only for constant p.

Thank you for your attention!