Volume thresholds for Gaussian and spherical random polytopes and their duals

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Random ± 1 Polytopes

- μ - uniform probability measure on $\{-1, 1\}^n$
- Let $Z_1, \ldots, Z_{N(n)}$ be independent random vectors $\sim \mu$ and set

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Theorem (Dyer, Füredi, McDiarmid, 1992)

Let $\varepsilon > 0$ *and* $\lambda = 2/\sqrt{e} \approx 1.213$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\,(C_N)]}{\mathrm{vol}\,([-1,1]^n)} \longrightarrow \begin{cases} 0 & \text{if } N \le (\lambda-\varepsilon)^n, \\ 1 & \text{if } N \ge (\lambda+\varepsilon)^n. \end{cases}$$

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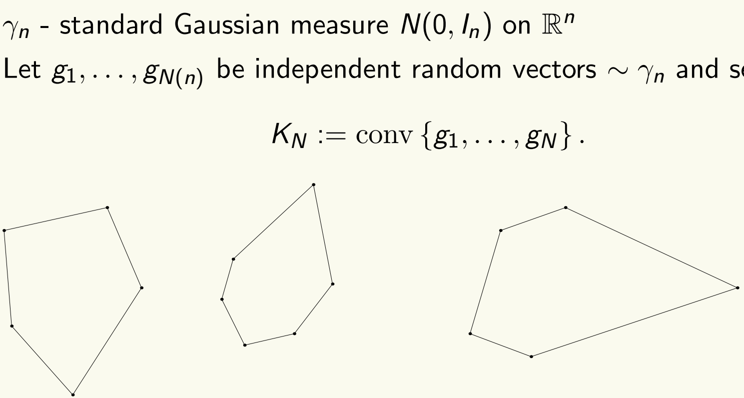
Generalized by Gatzouras & Giannopoulos in 2006 to a large class of compactly supported probability measures μ.

Gaussian Model?

- γ_n - standard Gaussian measure $N(0, I_n)$ on $\mathbb{R}^n$
- Let $g_1, \ldots, g_{N(n)}$ be independent random vectors $\sim \gamma_n$ and set

$$K_N := \operatorname{conv}\{g_1, \ldots, g_N\}.$$

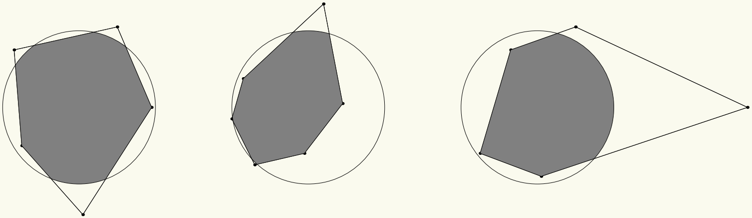


Question: What does it mean for K_N to capture significant volume?

A First Approach

- Fix $R = R(n) > 0$.
- We'll study

$$\frac{\mathbb{E}\,\mathrm{vol}\left(K_N \cap RB_2^n\right)}{\mathrm{vol}\left(RB_2^n\right)}$$



- Since $\mathbb{E}|g_1| \approx \sqrt{n}$, one natural choice for R is $R(n) = \sqrt{n}$.

As usual, let

$$\Phi(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$$

Well-known: $1 - \Phi(a) \approx \frac{1}{\sqrt{2\pi}(a+1)} e^{-a^2/2}$.

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Theorem

Let $\kappa > 0$, $c > 0$ and let $0 < \varepsilon < c$. Then, as $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\,(K_N \cap cn^{\kappa} B_2^n)]}{\mathrm{vol}\,(cn^{\kappa} B_2^n)} \longrightarrow \begin{cases} 0 & \textit{if } N \leq \Psi((c-\varepsilon)n^{\kappa}), \\ 1 & \textit{if } N \geq \Psi((c+\varepsilon)n^{\kappa}). \end{cases}$$

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(Affentranger, '91) gives an asymptotic formula for $\mathbb{E}\,\mathrm{vol}\,(K_N)$ where n is fixed and $N \to \infty$.

± 1 Case

Note that

$$\mathbb{E}\,\mathrm{vol}\,(C_N) = \mathbb{E}\int_{[-1,1]^n} 1_{\{x\in C_N\}}dx = \int_{[-1,1]^n} \mathbb{P}(x \in C_N)dx.$$

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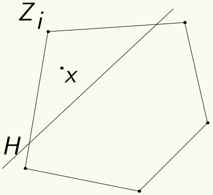
Let H be a closed halfspace with $x \in H$.



$$\Rightarrow \mathbb{P}(x \in C_N) \leq N\mathbb{P}(Z_1 \in H).$$

For $x \in [-1,1]^n$, let

$$\mathcal{H}_x := \{\text{closed halfspaces } H : x \in H\}$$

and set

$$q(x) := \inf_{H \in \mathcal{H}_x} \mathbb{P}(Z_1 \in H)$$

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For $\alpha > 0$, set

$$Q_n^\alpha := \{x \in [-1,1]^n : q(x) \geq e^{-\alpha n}\}.$$

Note that

$$x \notin Q_n^\alpha \iff \exists\, H \in \mathcal{H}_x : |H \cap \{-1,1\}^n| < 2^n e^{-\alpha n}.$$

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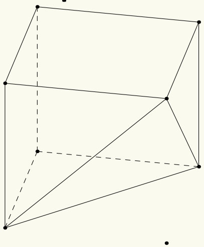

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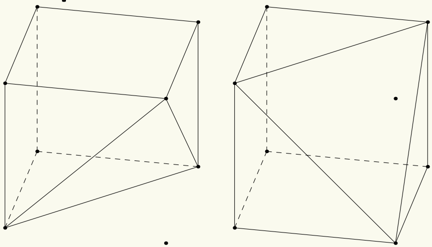

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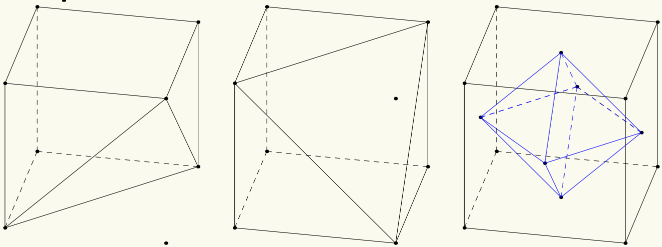

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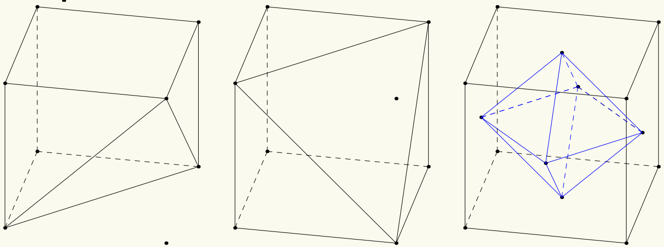

Remark: For μ uniform on $[-1,1]^n$, Q_n^{α} is just the floating body.

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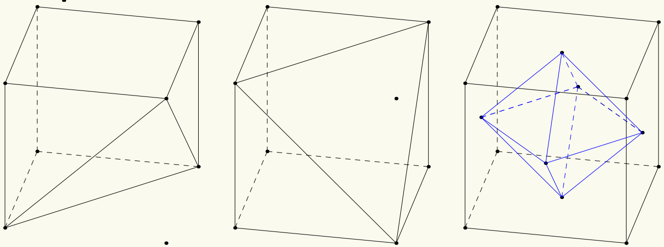


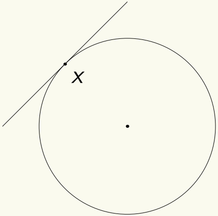
Remark: For μ uniform on $[-1,1]^n$, Q_n^α is just the floating body.

Lemma

$$\mathbb{P}(Q_n^\alpha \subset C_N) \geq 1 - o(1) - 2\binom{N}{n}(1 - e^{-\alpha n})^{N-n} \quad (as\ n \to \infty).$$

Gaussian Case

$$q(x) = 1 - \Phi(|x|).$$



Consequently $\{x \in \mathbb{R}^n : q(x) \geq 1 - \Phi(R)\} = RB_2^n$.

Lemma

Let $R > 0$. Then

$$\mathbb{P}(RB_2^n \subset K_N) \geq 1 - 2\binom{N}{n}(\Phi(R))^{N-n}.$$

A Different Take

Let ν be a log-concave probability measure on $\mathbb{R}^n$. Suppose ν is isotropic:

$$\int_{\mathbb{R}^n} x d\nu(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\nu(x) = |\theta|^2 \quad \forall\, \theta \in R^n.$$

Examples:

- standard Gaussian measure γ_n
- $\nu(A) := \mathrm{vol}\,(A \cap K)\,/\,\mathrm{vol}\,(K)$, where K is a convex body in isotropic position

Behavior of $\mathbb{E}\nu(K_N)$?

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Behavior of $\mathbb{E}\nu(K_N)$?

Theorem (Klartag, 2006)

There exist universal constants $C, c > 0$ such that for each $0 \leq \varepsilon \leq 1$,

$$\nu\{x \in \mathbb{R}^n : ||x| - \sqrt{n}| \geq \varepsilon\sqrt{n}\} \leq Cn^{-c\varepsilon^2}.$$

Theorem

Let ν *be an isotropic, log-concave probability measure on* $\mathbb{R}^n$ *and let* $0 < \varepsilon < 1$. *Then, as* $n \to \infty$,

$$\mathbb{E}\nu(K_N) \longrightarrow \begin{cases} 0 & \text{if } N \leq \Psi((1-\varepsilon)\sqrt{n}), \\ 1 & \text{if } N \geq \Psi((1+\varepsilon)\sqrt{n}). \end{cases}$$

Polytopes Generated by Random Facets

Let

$$K'_N := \{x \in \mathbb{R}^n : \langle g_i, x \rangle \leq 1 \text{ for each } i = 1, \ldots, N\}.$$

Theorem

Let $\kappa > 0$, $c > 0$ and let $0 < \varepsilon < c$. Then, as $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\left(K'_N \cap (cn^\kappa)^{-1} B_2^n\right)]}{\mathrm{vol}\left((cn^\kappa)^{-1} B_2^n\right)} \longrightarrow \begin{cases} 1 & \text{if } n < N \leq \Psi((c-\varepsilon)n^\kappa), \\ 0 & \text{if } N \geq \Psi((c+\varepsilon)n^\kappa). \end{cases}$$

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Theorem

Let ν be an isotropic log-concave prob. measure on $\mathbb{R}^n$ and let $0 < \varepsilon < 1$. Then, as $n \to \infty$,

$$\mathbb{E}\nu(nK'_N) \longrightarrow \begin{cases} 1 & \text{if } n < N \le \Psi((1-\varepsilon)\sqrt{n}), \\ 0 & \text{if } N \ge \Psi((1+\varepsilon)\sqrt{n}). \end{cases}$$

Spherical Model

σ - Haar measure on the Euclidean sphere S^{n-1}
$u_1, \ldots, u_N$ i.i.d. random vectors $\sim \sigma$
Set

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Let $0 < R < 1$ and set $r = 1/\sqrt{1 - R^2}$. Let $0 < \varepsilon < 1$. Then, as $n \to \infty$,

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Theorem

Let $0 < R < 1$ and set $r = 1/\sqrt{1-R^2}$. Let $0 < \varepsilon < 1$. Then, as $n \to \infty$,

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- (Müller, 1990) gives an asymptotic formula for

$$\mathrm{vol}\,(B_2^n) - \mathbb{E}\,\mathrm{vol}\,(L_N)\,,$$

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where n is fixed and $N \to \infty$.

- For the dual case it is tempting to consider

$$\frac{\mathrm{vol}\,(B_2^n)}{\mathbb{E}[\mathrm{vol}\,(L'_N)]}$$

but in fact

$$\mathbb{E}[\mathrm{vol}\,(L'_N)] = \infty.$$

Theorem

Let $0 < \varepsilon < 1$.

a. *There exists a sequence* $(t_n)_{n=1}^{\infty} = (t_n(\varepsilon))_{n=1}^{\infty}$ *with* $t_n > 1$ *and* $\lim_{n\to\infty} t_n = 1$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\left(B_2^n\right)}{\mathbb{E}\,\mathrm{vol}\left(L'_N \cap t_n B_2^n\right)} = 0 \qquad \text{if } n < N \leq \exp\left((1-\varepsilon)n \ln \sqrt{n}\right).$$

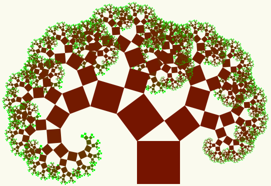
b. *There exists a sequence* $(R_n)_{n=1}^{\infty} = (R_n(\varepsilon))_{n=1}^{\infty}$ *with* $R_n > 1$ *and* $\lim_{n\to\infty} R_n = \infty$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\left(B_2^n\right)}{\mathbb{E}\,\mathrm{vol}\left(L'_N \cap R_n B_2^n\right)} = 1 \qquad \text{if } N \geq \exp\left((1+\varepsilon)n \ln \sqrt{n}\right).$$

Thank You

- I thank N. Tomczak-Jaegermann, R. Latala, P. Mankiewicz, and the conference organizers.
- Pictures were made using *Asymptote*

http://asymptote.sourceforge.net



- Slides created with Latex's Beamer class; *Edmonton U of A* theme by Josh Nault.