Hausdorff dimension of the residual set of a ball packing

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Residual set

- How small can be the residual set of a ball packing?
- Good measure: Hausdorff dimension. (Box dimension)
- Related to density of ball packings?
- Which type of ball packings?

Hausdorff dimension in the plane

- Theorem (Larman) The hausdorff dimension of the residual set of a disc packing in the unit square is at least 1.03.

Related Results about density in the plane

- The maximum density of disc packings is $\frac{\pi}{\sqrt{12}}$
- Density of disc packings with two different radii. (Heppes)
- The Maximum density of ball packings with radius from the interval $[r, 1]$ is $\rho_n(r)$ (in the n-dimensional space).
- Theorem: $\rho_2(0.743) = \rho_2(1) = \frac{\pi}{\sqrt{12}}$ (Fejes-Tóth, Florian, Böröczky)
- Theorem: $\rho_2(r) < 1 - cr$ (Florian)

Main Results

- Let $\lambda_n = \frac{196}{n+196}$ and let $d_n = n - \lambda_n$.
- Theorem: $\rho_n(r) < 1 - \overline{c_n} r^{\lambda_n}$ for all $0 < r \leq 1$.
- Theorem: The Hausdorff dimension of the residual set of a ball packing is at least d_n.

Enlarging the packing

- A Ball packing $\mathcal{B} = \{B_1 \mathcal{B}_2, \ldots\}$ is an r-packing if the radius of each ball is between r and 1.
- $B_i = B(O_i, r_i)$ is the ball with radius r_i centered at the point O_i.
- Let $\varepsilon = \frac{r}{98}$.
- There exist sets $D_1, D_2, \ldots$ such that
- $B_i \subset D_i \subset B_i^{+2\varepsilon} = B(O_i, r_i + 2\varepsilon)$
- $\mathcal{D} = \{D_1, D_2, \ldots\}$ is a packing
- $\mathrm{Vol}_n(D_i) \geq \left(1 + \frac{n\varepsilon}{r_i}\right) \mathrm{Vol}_n(B_i)$

Inequalities

- Let $A_x = \mathrm{Vol}_n(\cup_{r_i=x, i\leq l} B_i)$
- Let $A = \mathrm{Vol}_n([0,1]^n) = 1$.
- $\sum_{x \geq y} A_x \left(1 + c_n \frac{y}{x}\right) < A$.
- From these inequalities the theorem follows
- $\rho_n(r) < 1 - \overline{c_n} r^{\lambda_n}$
- where $\lambda_n = \Theta(\frac{1}{n})$.

Covering

- Let K be a cube and let $l(K)$ denote the side length of K.
- A set of axis parallel cubes $\mathcal{K} = \{K_1, K_2, \ldots\}$ is a covering of $T \subset \mathbb{R}^n$ if $T \subset \bigcup_i K_i$.
- A covering $\mathcal{K}$ is a k-covering if $l(K_i) = k$.

Definitions

- $\rho_{\mathrm{Box}}(T, d, k) = \inf\left\{\sum_{K \in \mathcal{K}} l(K)^d \ : \ \mathcal{K} \text{ is a } k'\text{-covering of } T \text{ for some } k' \leq k\right\}$.
- $\mathrm{Vol}_{d,box}(T) = \lim_{k \to 0} \rho_{\mathrm{Box}}(T, d, k)$.

Box volume & Density

- The d_n-dimensional Box-volume of the residual set of a ball packing in the unit cube of dimension n is at least $5^{-n}\overline{c}_n$.

Hausdorff dimension

- $\rho_{\mathrm{Haus}}(T, d, k) = \inf\left\{\sum_{K \in \mathcal{K}} l(K)^d \ : \ \mathcal{K} \text{ is a covering of } T \text{ with } l(K) \leq k \text{ for all } K \in \mathcal{K}\right.$
- $\mathrm{Vol}_{d,haus}(T) = \lim_{k \to 0} \rho_{\mathrm{Haus}}(T, d, k)$.