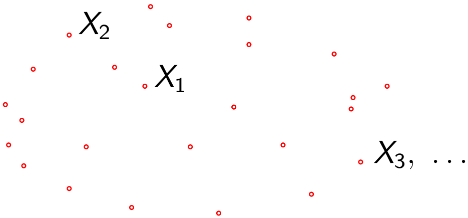
Tail Inequalities for Random Polytopes

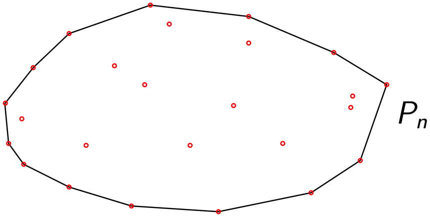
Matthias Reitzner, TU Wien

- Introduction
- Mean values
- Azuma's inequality
- Vu's inequality
- Other types of random polytopes

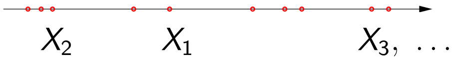
Random polytopes



Random polytopes

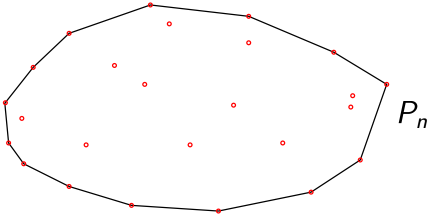


Random polytopes



Random polytopes

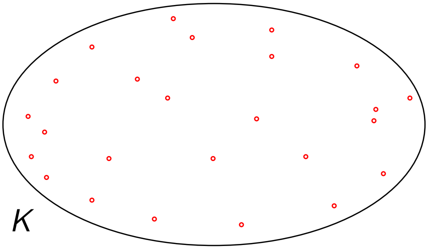
Random polytopes



Random polytopes with vertices in K

random points $X_1, \dots, X_n$ uniformly in K, $V(K) = 1$

$$P_n = [X_1, \dots, X_n] \subset K$$



Random polytopes with vertices in K

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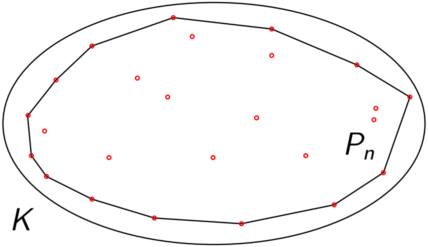

Random polytopes with vertices in K

- $$\mathbf{f}(P_n) = \begin{pmatrix} f_0(P_n) \\ f_1(P_n) \\ \vdots \\ f_{d-1}(P_n) \end{pmatrix} = \begin{pmatrix} \sharp \text{ vertices of } P_n \\ \sharp \text{ edges of } P_n \\ \vdots \\ \sharp \text{ facets of } P_n \end{pmatrix}$$

- intrinsic volumes: $$\begin{cases} V_d(K) & - & V_d(P_n) \\ V_{d-1}(K) & - & V_{d-1}(P_n) \\ & \vdots & \\ V_1(K) & - & V_1(P_n) \end{cases}$$

Random polytopes with vertices in K

- $$\mathbf{f}(P_n) = \begin{pmatrix} f_0(P_n) \\ f_1(P_n) \\ \vdots \\ f_{d-1}(P_n) \end{pmatrix} = \begin{pmatrix} \sharp \text{ vertices of } P_n \\ \sharp \text{ edges of } P_n \\ \vdots \\ \sharp \text{ facets of } P_n \end{pmatrix}$$

- intrinsic volumes: $$\begin{cases} V(K) - V(P_n) \\ V_{d-1}(K) - V_{d-1}(P_n) \\ \vdots \\ V_1(K) - V_1(P_n) \end{cases}$$

Mean values

$K \in \mathcal{K}^d_{2,+}$:

$$V(K) - \mathbb{E}V(P_n) = c_d \Omega(K)\, n^{-\frac{2}{d+1}} + \cdots$$

$$\Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx$$

Mean values

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$$V(K) - \mathbb{E}V(P_n) = c_d \Omega(K)\, n^{-\frac{2}{d+1}} + \cdots$$

$$\mathbb{E}\,\mathbf{f}(P_n) = \mathbf{c}\;\Omega(K)\, n^{\frac{d-1}{d+1}} + \cdots$$

$$\Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx$$

Mean values

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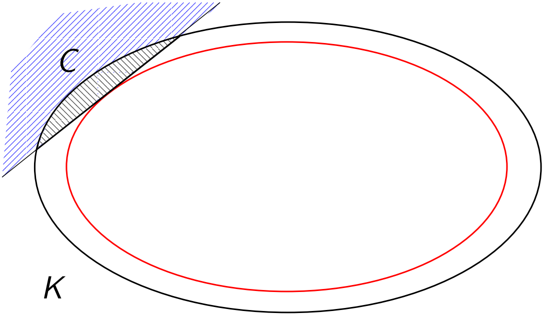
$K \in \mathcal{P}^d$:

$$V(K) - \mathbb{E}V(P_n) = c_d\, T(K)\, n^{-1} \ln^{d-1} n + \cdots$$

$$\mathbb{E}\,\mathbf{f}(P_n) = \tilde{\mathbf{c}}\ T(K) \ln^{d-1} n + \cdots$$

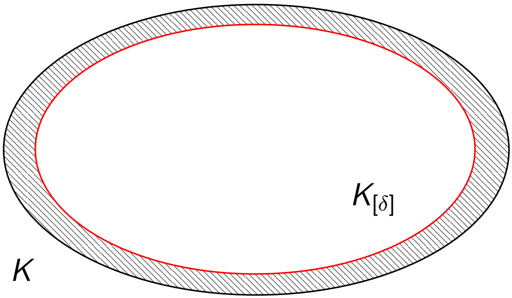
Affentranger, Bárány, Blaschke, Buchta, Dalla, Efron, Giannopoulos, Groemer, Larman, Reitzner , Rényi, Schneider, Sulanke, Wieacker, . . .

The floating body

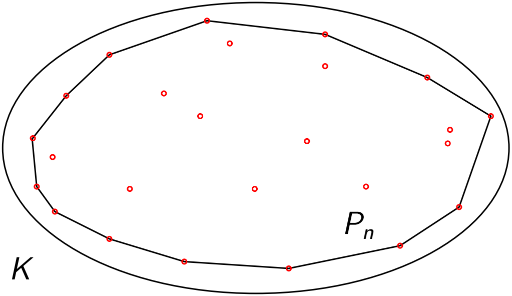


$V(C) = \delta$

The floating body



The floating body



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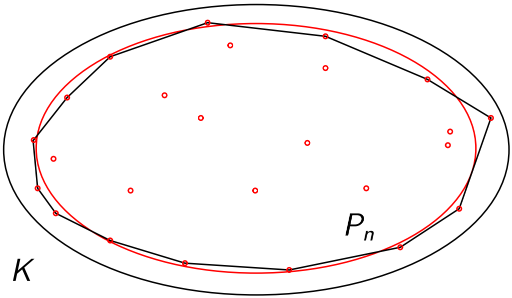
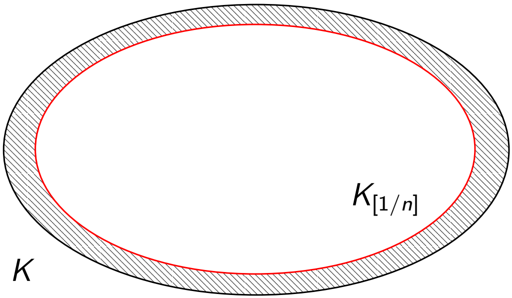


The floating body



The floating body and mean values

$K \in \mathcal{K}^d$:

$$V(K) - V(K_{[\delta]}) = c_d \Omega(K) \delta^{\frac{2}{d+1}} + \cdots$$

(Schütt, Werner)

$$V(K) - \mathbb{E}V(P_n) = c_d \Omega(K)\, n^{-\frac{2}{d+1}} + \cdots$$

(Schütt)

The floating body and mean values

$K \in \mathcal{P}^d$:

$$V(K) - V(K_{[\delta]}) = c_d T(K) \delta \ln^{d-1}(1/\delta) + \cdots$$

Bárány, Buchta, Schütt

$$V(K) - \mathbb{E}V(P_n) = c_d T(K) n^{-1} \ln^{d-1} n + \cdots$$

Bárány, Buchta

floating body and first deviation inequalities

K smooth:

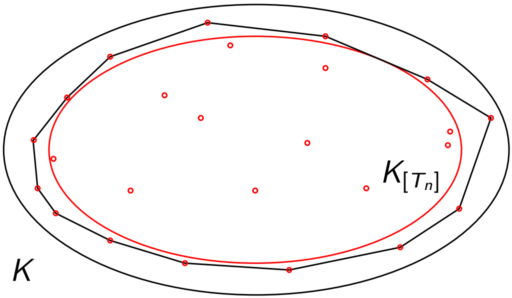
$$T_n = \frac{\alpha_n}{n}$$

with $\alpha_n \to \infty$

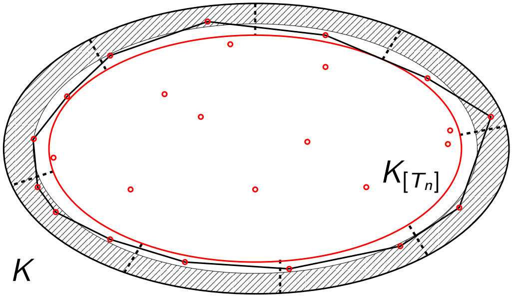
With high probability $K_{[T_n]} \subset P_n$.

Bárány, Buchta, Dalla, Larman, ...

The floating body



The floating body



floating body and first deviation inequalities

K smooth:

$$T_n = \frac{\alpha_n}{n}$$

with $\alpha_n \to \infty$

$$1 - \mathbb{P}(K_{[T_n]} \subset P_n) \quad \leq \quad m_{T_n} e^{-nT_n}$$

floating body and first deviation inequalities

K smooth:

$$T_n = \frac{\alpha_n}{n}$$

with $\alpha_n = \alpha \ln n$

$$\begin{aligned} 1 - \mathbb{P}(K_{[T_n]} \subset P_n) &\leq m_{T_n} e^{-nT_n} \\ &\lesssim n^{1-2/(d+1)} n^{-\alpha} = n^{-c} \end{aligned}$$

floating body and first deviation inequalities

K polytope:

$$T_n = \frac{\alpha_n}{n}, \;\; s_n = \frac{1}{\beta_n n}$$

with $\alpha_n, \beta_n \to \infty$

$$1 - I\!P(K_{[T_n]} \subset P_n \subset K_{[s_n]}) \;\; \leq \;\; m_{T_n} e^{-nT_n} + (1 - e^{-nV(K_{[s_n]})})$$

floating body and first deviation inequalities

K polytope:

$$T_n = \frac{\alpha_n}{n}, \; s_n = \frac{1}{\beta_n n}$$

with $\alpha_n = \alpha \, \ln\ln n, \beta_n = \ln^{\beta} n$

$$\begin{aligned} 1 - \mathbb{P}(K_{[T_n]} \subset P_n \subset K_{[s_n]}) &\leq m_{T_n} e^{-nT_n} + (1 - e^{-nV(K_{[s_n]})}) \\ &\lesssim \ln^{-c} n \end{aligned}$$

Azuma's inequality

$$f = f(T), \;\; T = (t_1, \ldots, t_m)$$

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \,|\, t_1, \ldots, t_i) - \mathbb{E}(f \,|\, t_1, \ldots, t_{i-1})$$

Azuma's inequality

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$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f | t_1, \ldots, t_i) - \mathbb{E}(f | t_1, \ldots, t_{i-1})$$

$$\mathbb{E} C_i(t_1, \ldots, t_i) = 0$$

Azuma's inequality

$$f = f(T), \;\; T = (t_1, \ldots, t_m)$$

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f | t_1, \ldots, t_i) - \mathbb{E}(f | t_1, \ldots, t_{i-1})$$

$$f(T) - \mathbb{E}f(T) = \sum_{1}^{m} C_i(t_1, \ldots, t_i)$$

Azuma's inequality

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$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

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$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \dots, t_i)|$$

Hoeffding-Azuma inequality:

For $B \geq B(t)$

$$\mathbb{P}(|f(T) - \mathbb{E}f(T)| \geq x) \leq 2e^{-x^2/(2mB^2)}$$

Azuma's inequality

$$f = f(T), \ \ T = (t_1, \ldots, t_m)$$

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f | t_1, \ldots, t_i) - \mathbb{E}(f | t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

Hoeffding-Azuma inequality:

$$\mathbb{P}(|f(T) - \mathbb{E}f(T)| \geq x) \leq 2e^{-x^2/(2mB^2)} + \mathbb{P}(B(t) \geq B)$$

Azuma's inequality

smooth K:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

Azuma's inequality

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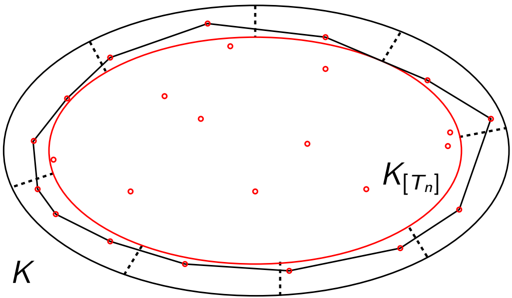

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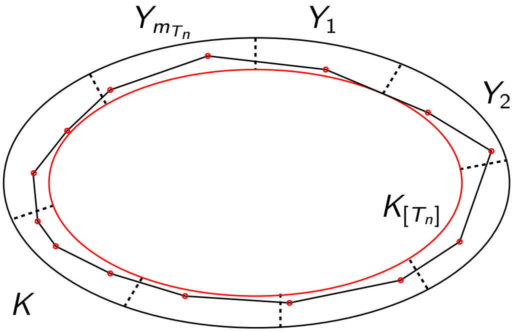

Azuma's inequality

smooth K:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|V(Y_1, \dots, Y_i, \dots, Y_{m_{T_n}}) - V(Y_1, \dots, Y_i', \dots, Y_{m_{T_n}})| \leq \frac{\ln n}{n}$$

Azuma's inequality

smooth K:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|\mathbb{E}(V|Y_1, \dots, Y_i) - \mathbb{E}(V|Y_1, \dots, Y_{i-1})| \leq \frac{\ln n}{n}$$

Azuma's inequality

smooth K:

$$V = V(X_1, \ldots X_n) = V(Y_1, \ldots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \ldots, Y_i)| \leq \frac{\ln n}{n}$$

Azuma's inequality

smooth K:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \leq \frac{\ln n}{n}, \quad m_{T_n} \leq (n/\ln n)^{1-2/(d+1)}$$

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$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x| \ldots) \leq e^{-x^2/(\ln n/n)^{1+2/(d+1)}}$$

Azuma's inequality

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$$|C_i(Y_1, \ldots, Y_i)| \leq \frac{\ln n}{n}, \quad m_{T_n} \leq (n/\ln n)^{1-2/(d+1)}$$

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x) \leq e^{-x^2/(\ln n/n)^{1+2/(d+1)}} + n^{-c}$$

Variance

smooth K:

$$\implies \quad \sigma^2(V) \leq cn^{-1-2/(d+1)} \ln^{1+2/(d+1)} n$$

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$$\sigma^2(V) \leq c n^{-1-2/(d+1)}$$

Reitzner

Azuma's inequality

smooth K:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \leq \frac{\ln n}{n}, \quad m_{T_n} \leq (n/\ln n)^{1-2/(d+1)}$$

Azuma's inequality for $V(P_n)$

For smooth K

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x) \leq 2e^{-c(x/\sigma(V))^2 \ \ln^{-1-2/(d+1)} n} + n^{-c}$$

Azuma's inequality

K polytope:

$$V = V(X_1, \ldots X_n) = V(Y_1, \ldots, Y_{m_{T_n}})$$

Azuma's inequality

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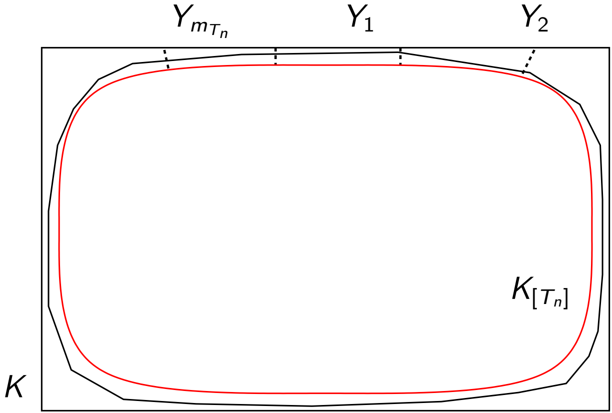
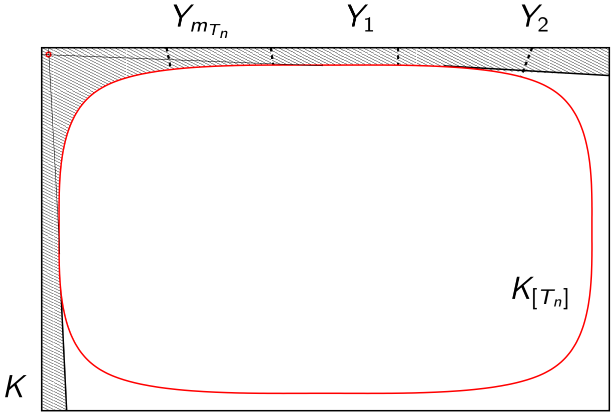
$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$



Azuma's inequality

K polytope:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$



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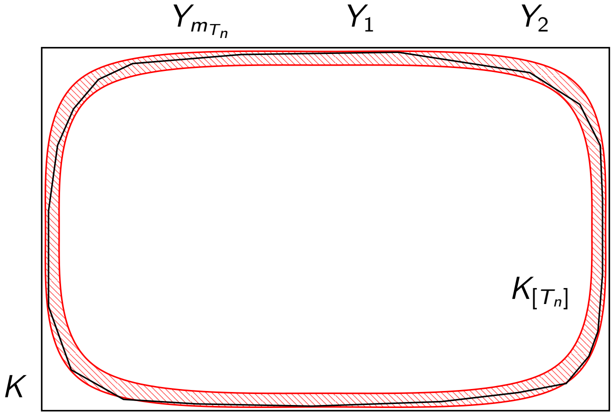

Azuma's inequality

K polytope:

$$V = V(X_1, \ldots X_n) = V(Y_1, \ldots, Y_{m_{T_n}})$$

$$|V(Y_1, \ldots, Y_i, \ldots, Y_{m_{T_n}}) - V(Y_1, \ldots, Y_i', \ldots, Y_{m_{T_n}})| \leq \frac{\mathrm{lln}^{4d-3} n}{n}$$

Bárány, Reitzner

Azuma's inequality

K polytope:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \leq \frac{\ln^{4d-3} n}{n}$$

Bárány, Reitzner

Azuma's inequality

K polytope:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \le \frac{l \ln^{4d-3} n}{n}, \quad m_{T_n} \le \ln^{d-1} n$$

Bárány, Reitzner

Azuma's inequality

K polytope:

$$V = V(X_1, \ldots X_n) = V(Y_1, \ldots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \ldots, Y_i)| \leq \frac{\ln^{4d-3} n}{n}, \quad m_{T_n} \leq \ln^{d-1} n$$

Bárány, Reitzner

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x| \ldots) \leq e^{-x^2/(n^{-2}\ln^{d-1} n \, \ln^{8d-6} n)}$$

Azuma's inequality

K polytope:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \leq \frac{\mathrm{lln}^{4d-3} n}{n}, \quad m_{T_n} \leq \ln^{d-1} n$$

Bárány, Reitzner

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x) \leq e^{-x^2/(n^{-2} \ln^{d-1} n \ \mathrm{lln}^{8d-6} n)} + \ln^{-c} n$$

Variance

K polytope:

$$\implies \quad \sigma^2(V) \leq cn^{-2} \ln^{d-1} n \, \text{lln}^{8d-6} n$$

Variance

K polytope:

$$\Longrightarrow \quad \sigma^2(V) \leq cn^{-2} \ln^{d-1} n \; \text{lln}^{8d-6} n$$

$$\sigma^2(V) \leq cn^{-2} \ln^{d-1} n$$

Bárány, Reitzner

Azuma's inequality

K polytope:

$$V = V(X_1, \dots X_n) = V(Y_1, \dots, Y_{m_{T_n}})$$

$$|C_i(Y_1, \dots, Y_i)| \le \frac{\mathrm{lln}^{4d-3} n}{n}, \quad m_{T_n} \le \ln^{d-1} n$$

Azuma's inequality for $V(P_n)$

For K a polytope

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \ge x) \le 2e^{-c(x/\sigma(V))^2 \ \mathrm{lln}^{-8d+6} n} + \ln^{-c} n$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \mid t_1, \ldots, t_i) - \mathbb{E}(f \mid t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \,|\, t_1, \ldots, t_i) - \mathbb{E}(f \,|\, t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

$$S_i^2(t_1, \ldots, t_{i-1}) = \qquad \mathbb{E}_{t_i} C_i^2(t_1, \ldots, t_i)$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = I\!E(f \,|\, t_1, \ldots, t_i) - I\!E(f \,|\, t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

$$S^2(t) = \sum_i S_i^2(t_1, \ldots, t_{i-1}) = \sum_i I\!E_{t_i} C_i^2(t_1, \ldots, t_i)$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \,|\, t_1, \ldots, t_i) - \mathbb{E}(f \,|\, t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

$$S^2(t) = \sum_i S_i^2(t_1, \ldots, t_{i-1}) = \sum_i \mathbb{E}_{t_i} C_i^2(t_1, \ldots, t_i)$$

$$\sigma^2(f) \;=\; \mathbb{E}(f - \mathbb{E}f)^2 = \mathbb{E}(\sum_i C_i)^2$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \,|\, t_1, \ldots, t_i) - \mathbb{E}(f \,|\, t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

$$S^2(t) = \sum_i S_i^2(t_1, \ldots, t_{i-1}) = \sum_i \mathbb{E}_{t_i} C_i^2(t_1, \ldots, t_i)$$

$$\sigma^2(f) \;=\; \mathbb{E}(f - \mathbb{E}f)^2 = \mathbb{E}(\sum_i C_i)^2 = \sum \mathbb{E} C_i^2$$

Vu's concentration inequality

$$C_i(t_1, \dots, t_i) = \mathbb{E}(f \,|\, t_1, \dots, t_i) - \mathbb{E}(f \,|\, t_1, \dots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \dots, t_i)|$$

$$S^2(t) = \sum_i S_i^2(t_1, \dots, t_{i-1}) = \sum_i \mathbb{E}_{t_i} C_i^2(t_1, \dots, t_i)$$

$$\begin{aligned} \sigma^2(f) &= \mathbb{E}(f - \mathbb{E}f)^2 = \mathbb{E}(\sum_i C_i)^2 = \sum \mathbb{E}C_i^2 \\ &= \mathbb{E}S^2(t) \end{aligned}$$

Vu's concentration inequality

$$C_i(t_1, \ldots, t_i) = \mathbb{E}(f \,|\, t_1, \ldots, t_i) - \mathbb{E}(f \,|\, t_1, \ldots, t_{i-1})$$

$$B(t) = \max_i \sup_{t_i} |C_i(t_1, \ldots, t_i)|$$

$$S^2(t) = \sum_i S_i^2(t_1, \ldots, t_{i-1}) = \sum_i \mathbb{E}_{t_i} C_i^2(t_1, \ldots, t_i)$$

Vu's concentration inequality

For $x \leq S/(2B)$

$$\begin{aligned} \mathbb{P}(|f(T) - \mathbb{E}f(T)| \geq x) &\leq 2e^{-x^2/(4S)} \\ &+ \mathbb{P}(S^2(t) \geq S \text{ or } B(t) \geq B) \end{aligned}$$

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Vu's concentration inequality

K smooth :

$$\mathbb{P}(B(t) \geq B) \leq m_B e^{-nB}$$

Vu's concentration inequality

K smooth , $B = n^{-1+\frac{d-1}{3d+5}}$:

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Vu's concentration inequality

Vu's inequality for $V(P_n)$

For smooth K and $x \leq cn^{-\frac{(d+3)^2}{(d+1)(3d+5)}}$.

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x) \quad \leq \quad 2e^{-c(x/\sigma(V))^2} + e^{-cn^{\frac{d-1}{3d+5}}}$$

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Azuma's inequality for $V(P_n)$

For smooth K

$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \geq x) \leq 2e^{-c(x/\sigma(V))^2 \ \ln^{-1-2/(d+1)} n} + n^{-c}$$

Other types of random polytopes: Gaussian polytopes

$X_1, \dots, X_n$ normal distributed in $I\!R^d$, $P_n = [X_1, \dots, X_n]$

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$$\sigma^2(V) \approx \ln^{\frac{d-3}{2}} n$$

Affentranger, Bárány, Baryshnikov, Hueter, Hug, Raynaud, Reitzner , Rényi, Schneider, Sulanke, Vitale, Vu, Wieacker, . . .

Gaussian polytopes and Azuma's inequality

$$V = V(X_1, \ldots X_n) = V(Y_1, \ldots, Y_m)$$

$$|C_i(Y_1, \ldots, Y_i)| \leq \frac{\mathrm{lln}^{\frac{d+1}{2}} n}{\ln^{\frac{1}{2}} n},$$

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$$\mathbb{P}(|V(P_n) - \mathbb{E}V(P_n)| \ge x) \le 2e^{-c(x/\sigma(V))^2 \, \text{lln}^{-(d+1)} n} + \ln^{-c} n$$

Random points on ∂K

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Buchta, Gruber, Müller, Tichy, Reitzner, Schneider, Schütt, Werner,

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