Twisting Schatten classes

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Definition

A short **exact sequence** is a diagram like

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0$$

where Y, Z are Banach spaces and the morphisms are such that the image of each arrow is the kernel of the next one. We say it is trivial if $j(Y)$ is complemented in X.

Examples

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

$$0 \longrightarrow c_0 \longrightarrow c_0 \oplus (\ell_\infty/c_0) \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

- Theorem (Kalton-Peck, 1979)

For $1 < p < \infty$, there exists a non trivial exact sequence

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  - The homogeneous map $\Omega : \ell_p \longrightarrow \clubsuit$ defined like $\Omega(x) = \sum x_i \log\left(\frac{|x_i|}{\|x\|_{\ell_p}}\right) e_i$ satisfies:

There exists a constant C such that for every $x, y \in \ell_p$ we have

$$\Omega(x+y) - \Omega(x) - \Omega(y) \in \ell_p$$

$$\|\Omega(x+y) - \Omega(x) - \Omega(y)\|_{\ell_p} \leq C(\|x\|_{\ell_p} + \|y\|_{\ell_p})$$

- Z_p is defined as the space of all $(x, y) \in \ell_p \oplus \ell_p$ for which the quasi-norm

$$\|(x,y)\| := \|x - \Omega(y)\|_{\ell_p} + \|y\|_{\ell_p} < \infty$$

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- Theorem (Kalton,1978)

  Let

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

  be an exact sequence. If Y, Z are B-convex Banach spaces then X is isomorphic to a B-convex Banach space.

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- A natural candidate for $\Omega : \mathcal{S}_p \longrightarrow \clubsuit$ is, for a given $T = \sum s_i(T) e_i \otimes f_i$,

$$\Omega(T) = \sum s_i(T) \log\left(\frac{s_i(T)}{\|T\|_{\mathcal{S}_p}}\right) e_i \otimes f_i$$

Complex interpolation

Given $(\mathcal{S}_{p_0}, \mathcal{S}_{p_1})$ and $S = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$, we consider holomorphic functions $F : S \to \mathcal{S}_{p_0} + \mathcal{S}_{p_1}$ such that

$$\mathbb{R} \ni t \to F(j + it) \in \mathcal{S}_{p_j}$$

is continuous and bounded for $j = 0, 1$.

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$\mathcal{F}(\mathcal{S}_{p_0}, \mathcal{S}_{p_1})$ is the set of all functions defined above with the norm

$$\|F\|_{\mathcal{F}} := \max_{j=0,1} \left(\sup_{t \in \mathbb{R}} \|F(j + it)\|_{\mathcal{S}_{p_j}} \right).$$

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$(\mathcal{S}_{p_0}, \mathcal{S}_{p_1})_\theta := \{T \in \mathcal{S}_{p_0} + \mathcal{S}_{p_1} : T = F(\theta), F \in \mathcal{F}(\mathcal{S}_{p_0}, \mathcal{S}_{p_1})\}$
with norm

$$\|T\| := \inf_{\{F : F(\theta) = T\}} \|F\|_{\mathcal{F}}$$

Using interpolation techniques

1.
$$\delta_\theta : \mathcal{F}(\mathcal{S}_{p_0}, \mathcal{S}_{p_1}) \longrightarrow (\mathcal{S}_{p_0}, \mathcal{S}_{p_1})_\theta = \mathcal{S}_p$$
for $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

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3. For $T = \sum s_i(T) e_i \otimes f_i$, we define
   $$B(T)(z) = \sum s_i(T)^{p(\frac{1-z}{p_0} + \frac{z}{p_1})} e_i \otimes f_i$$

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4. $\delta'_\theta B(T) = p\left(\frac{1}{p_1} - \frac{1}{p_0}\right) \sum s_i(T) \log\left(\frac{s_i(T)}{\|T\|_{\mathcal{S}_p}}\right) e_i \otimes f_i$

Constructing the exact sequence

1. Define Θ_p as the space of all $(S, T) \in \mathcal{S}_p \oplus \mathcal{S}_p$ such that

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What else can be said about Θ_p?

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There exists $C > 0$ such that for every $A, A' \in B(H)$ and $T \in \mathcal{S}_p$ we have

$$\|\Omega(ATA') - A\Omega(T)A'\|_{\mathcal{S}_p} \leq C\|A\|_{B(H)}\|T\|_{\mathcal{S}_p}\|A'\|_{B(H)}$$

Duality

Theorem

Given

$$0 \longrightarrow \mathcal{S}_p \longrightarrow \Theta_p \longrightarrow \mathcal{S}_p \longrightarrow 0 \equiv \Omega_p$$

$$0 \longrightarrow \mathcal{S}_q \longrightarrow \Theta_q \longrightarrow \mathcal{S}_q \longrightarrow 0 \equiv -\Omega_q$$

we have that $(\Omega_p)^* = -\Omega_q$ *for* $p^{-1} + q^{-1} = 1$.

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we have that $(\Omega_p)^* = -\Omega_q$ *for* $p^{-1} + q^{-1} = 1$.

Corollary

$(\Theta_p)^*$ *is isomorphic to* Θ_q *for* $p^{-1} + q^{-1} = 1$ *via "trace" duality.*

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is not trivial for $1 < p < \infty$

- We may assume, by duality, $1 < p \leq 2$.
- For each $N \in \mathbb{N}$, consider $(0, e_i \otimes f_i) \in \Theta_p$, $i = 1, ..., N$.

Non triviality

Θ_p has not type p

Assume $\mathbb{E}\|\sum_{i=1}^N r_i(0, e_i \otimes f_i)\| \leq C \left(\sum_{i=1}^N \|(0, e_i \otimes f_i)\|^p\right)^{1/p}$

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- $LS = \mathbb{E}\left(\|\Omega(\sum_{i=1}^{N} r_i e_i \otimes f_i)\| + \|\sum_{i=1}^{N} r_i e_i \otimes f_i\|\right)$

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- $\log N^{1/p} + 1 \leq C$

More exact sequences

Definition

$d^n\mathcal{S}_p := \left\{(T_1, ..., T_n) : T_i = \delta_\theta^{(n-i)}(F), F \in \mathcal{F}, i = 1...n\right\}$ endowed with the norm $\inf \|F\|_{\mathcal{F}}$.

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- For $n = 2$, $d^2\mathcal{S}_p = \Theta_p$.

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- For $n = 1$, $d^1\mathcal{S}_p = \mathcal{S}_p$.
- For $n = 2$, $d^2\mathcal{S}_p = \Theta_p$.

Theorem

For every $n \in \mathbb{N}$ and every choice $n = n_0 + n_1$ there exists a non trivial twisted sum of $B(H)$-modules:

$$0 \longrightarrow d^{n_0}\mathcal{S}_p \longrightarrow d^n\mathcal{S}_p \longrightarrow d^{n_1}\mathcal{S}_p \longrightarrow 0$$

Sketch of the proof

The case n=3

$$0 \longrightarrow d^2\mathcal{S}_p \longrightarrow d^3\mathcal{S}_p \longrightarrow d^1\mathcal{S}_p \longrightarrow 0$$

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$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & d^1\mathcal{S}_p & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & d^2\mathcal{S}_p & \longrightarrow & d^3\mathcal{S}_p & \longrightarrow & d^1\mathcal{S}_p & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
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 & & 0 & & & & & &
\end{array}$$

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & d^1\mathcal{S}_p & = & d^1\mathcal{S}_p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & d^2\mathcal{S}_p & \longrightarrow & d^3\mathcal{S}_p & \longrightarrow & d^1\mathcal{S}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
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 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$