Sets of constant height and PPT states in quantum information theory

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Quantum information theory

(from the geometric functional analysis angle)

A complex Hilbert space $\mathcal{H}$, usually $\mathbb{C}^d$, and the C^*-algebra $\mathcal{B}(\mathcal{H})$

The real space $\mathcal{M}_d^{sa}$ of $d \times d$ Hermitian matrices

The positive semi-definite cone $\mathcal{PSD} \subset \mathcal{M}_d^{sa}$

The base of $\mathcal{PSD}$ consisting of density matrices: $\mathcal{M}_d^{\text{tot}} := \mathcal{PSD} \cap \{\text{tr}(\cdot) = 1\}$ ($\sim$ the states of $\mathcal{B}(\mathbb{C}^d)$)

Other cones, their bases (usually convex subsets of $\mathcal{M}_d^{\text{tot}}$) and related norms on $\mathcal{M}_d$

Completely positive maps $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ etc.

Interesting convex subsets of $\mathcal{M}^{\mathrm{tot}}(\mathcal{H})$

Context: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_m$
Important cases: $m = 2$ *and* m *"moderately large"*

Separable states (or matrices) $\mathcal{M}^{\mathrm{sep}}(\mathcal{H})$
PPT (positive partial transpose) states $\mathcal{M}^{\mathrm{PPT}}(\mathcal{H})$
Euclidean (Hilbert-Schmidt) balls, other ellipsoids. . .

Precise relations between these are unclear, particularly for large m (even if all $\mathcal{H}_j$'s are equal to $\mathbb{C}^2$)

For $m = 2$, these sets of matrices are related via the Choi-Jamiołkowski isomorphism to classes of *-invariant maps $\mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ (talk of E. Werner)

Definitions

$$\mathcal{SEP}(\mathcal{H}_1 \otimes \mathcal{H}_2) := \mathrm{conv}\{\mathcal{PSD}(\mathcal{H}_1) \otimes \mathcal{PSD}(\mathcal{H}_2)\}$$

$$\begin{aligned} \mathcal{M}^{\mathrm{sep}}(\mathcal{H}_1 \otimes \mathcal{H}_2) &:= \mathcal{SEP} \cap \{\mathrm{tr}(\cdot) = 1\} \\ &= \mathrm{conv}\{\rho_1 \otimes \rho_2 : \rho_j \in \mathcal{M}^{\mathrm{tot}}(\mathcal{H}_j)\} \end{aligned}$$

$$\mathcal{PPT}(\mathcal{H}_1 \otimes \mathcal{H}_2) := \mathcal{PSD} \cap \{\rho : T_1(\rho) \in \mathcal{PSD}\}$$

$$\mathcal{M}^{\mathrm{PPT}}(\mathcal{H}_1 \otimes \mathcal{H}_2) := \mathcal{PPT} \cap \{\mathrm{tr}(\cdot) = 1\}$$

Notation: $d_1 := \dim \mathcal{H}_1$, $d_2 := \dim \mathcal{H}_2$
$d := \dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = d_1 d_2$

Links to geometry of Banach spaces

$\mathrm{conv}(-\mathcal{M}^{\mathrm{tot}}(\mathcal{H}) \cup \mathcal{M}^{\mathrm{tot}}(\mathcal{H}))$ = the unit ball in $(\mathcal{M}_d^{sa}, \|\cdot\|_1)$, where $\|\cdot\|_1$ is the trace class norm

$\mathrm{conv}(-\mathcal{M}^{\mathrm{sep}}(\mathcal{H}) \cup \mathcal{M}^{\mathrm{sep}}(\mathcal{H}))$ = the unit ball in the projective tensor product of normed spaces $(\mathcal{M}_{d_1}^{sa}, \|\cdot\|_1)$ and $(\mathcal{M}_{d_2}^{sa}, \|\cdot\|_1)$

Major role is played by duality considerations

What is partial transpose?

$$(\rho_1 \otimes \rho_2)^T = \rho_1^T \otimes \rho_2^T$$

$$T_1(\rho_1 \otimes \rho_2) = \rho_1^T \otimes \rho_2, \quad T_2(\rho_1 \otimes \rho_2) = \rho_1 \otimes \rho_2^T$$

Note: $(T_1(\rho))^T = T_2(\rho)$, hence

$$T_1(\rho) \in \mathcal{PSD} \;\Leftrightarrow\; T_2(\rho) \in \mathcal{PSD}$$

Also:

$$\rho_1 \otimes \rho_2 \in \mathcal{PSD} \;\Leftrightarrow\; \rho_1, \rho_2 \in \mathcal{PSD} \;\Leftrightarrow\; \rho_1^T, \rho_2^T \in \mathcal{PSD}$$

Thus:

$$\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{PSD}$$

Partial transpose via block matrices

If $\dim \mathcal{H}_1 = 2$, then

$$\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ni \rho \quad \leftrightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A, B, C, D \in \mathcal{B}(\mathcal{H}_2)$

$$T_1(\rho) = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \qquad T_2(\rho) = \begin{bmatrix} A^T & B^T \\ C^T & D^T \end{bmatrix}$$

A simple example of a non-PPT state

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$ Eigenvalues: 1,0,0,0

$$\rho' := T_1(\rho) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Eigenvalues: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

$\exists$ non-PPT state $\Leftrightarrow A \to A^T$ is not completely positive

Stømer-Woronowicz theorem

Assume $\min(d_1, d_2) > 1$

$$\mathcal{SEP}(\mathcal{H}_1 \otimes \mathcal{H}_2) \subset \mathcal{PPT}(\mathcal{H}_1 \otimes \mathcal{H}_2) \subsetneq \mathcal{PSD}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

First inclusion?

Equality iff $d_1 + d_2 < 6$

Otoh, if $d_1 = d_2 \to \infty$, then (Aubrun-S., 2006)

$$\left(\frac{\mathrm{vol}\,\mathcal{M}^{\mathrm{sep}}}{\mathrm{vol}\,\mathcal{M}^{\mathrm{PPT}}}\right)^{1/(d^2-1)} \sim \left(\frac{\mathrm{vol}\,\mathcal{M}^{\mathrm{sep}}}{\mathrm{vol}\,\mathcal{M}^{\mathrm{tot}}}\right)^{1/(d^2-1)} \sim d^{-1/4}$$

PPT states and their role

$\mathcal{M}^{\text{sep}} \rightarrow$ classical behavior

$\mathcal{M}^{\text{PPT}} \rightarrow$???

Open problem: Can *PPT* states exhibit nonclassical (EPR) correlations?

An easier (to state) question:

Is $\sup_{d_1,d_2>1} \left(\frac{\text{vol}\,\mathcal{M}_d^{\text{PPT}}}{\text{vol}\,\mathcal{M}_d^{\text{tot}}}\right)^{1/(d^2-1)} < 1$?

We know from the previous slide that $\inf(\cdot) > 0$

The structure of *PPT*

$\mathcal{PPT} = \mathcal{PSD} \cap T_1(\mathcal{PSD})$, $\mathcal{M}^{\mathsf{PPT}} = \mathcal{M}^{\mathsf{tot}} \cap T_1(\mathcal{M}^{\mathsf{tot}})$

Known (Spingarn??):
$K \subset \mathbb{R}^n$ with centroid at 0 $\Rightarrow$ vol$(-K \cap K) \geq 2^{-n}$vol$K$

Probably not optimal, but exponential decrease may occur (simplex)

Here: a specific set K, a "partial" reflection T_1 with respect to $\rho_* = d^{-1}\mathrm{Id}$, which plays the role of the origin

Sets of constant height

$K \subset \mathbb{R}^n$ a convex body; r the inradius of K

K is said to be *of constant height* if
1. every point of ∂K is contained in a face tangent to the inscribed ball

This is equivalent to **2.** $\frac{r\,\mathrm{vol}_{n-1}\partial K}{\mathrm{vol}_n K} = n$ ($\leq$ always)

If the insphere of K is the unit ball, this is further equivalent to **3.** $K = L^\circ$, where $L = \bar{L} \subset S^{n-1}$ and $\mathrm{conv}\, L$ contains the origin in its interior

Why **1.** $\Leftrightarrow$ **2.** $\Leftrightarrow$ **3.**?

1. $\Leftrightarrow$ **2.** obvious for polytopes: K decomposes into a union of pyramids of height r

The general case: approximation or integral formulae; both based on the fact that K has unique tangent a.e. on ∂K

1. $\Leftrightarrow$ **3.** faces tangent to the unit ball

$\leftrightarrow$ points in $K^\circ \cap S^{n-1}$

A few simple observations

(i) $\mathcal{M}_d^{\mathsf{tot}}$ is of constant height (*center* $= \rho_*$)

(ii) Intersection of two sets of constant height with the same inscribed ball is of constant height

(iii) $\mathcal{M}_d^{\mathsf{PPT}} = \mathcal{M}^{\mathsf{tot}} \cap T_1(\mathcal{M}^{\mathsf{tot}})$ is of constant height

Why (i) and (ii)?

(i) $\mathcal{M}_d^{\text{tot}}$ is of constant height

Center $= \rho_*$, *inradius* $= (d(d-1))^{-1/2}$

Maximal faces: matrices diagonalizable in a fixed basis with specified eigenvalue equal to 0.

$d^2 - 2$-dimensional boundary of $\mathcal{M}_d^{\text{tot}}$ is a union of $d-2$-dimensional simplices tangent to the inscribed ball at their centroids

(ii) If $K_1 = L_1^\circ$ and $K_2 = L_2^\circ$, then $K_1 \cap K_2 = (L_1 \cup L_2)^\circ$ and $L_1, L_2 \subset S^{n-1} \Rightarrow L_1 \cup L_2 \subset S^{n-1}$

What about sets of separable states?

For 2×2 and 2×3 states, the set $\mathcal{M}^{\mathrm{sep}}$ is of constant height (because it equals $\mathcal{M}^{\mathrm{PPT}}$)

What about higher dimensions?

A corollary of $\mathcal{M}^{\text{PPT}}$ being of constant height

Consider two models of selecting random states mixed two qubit states:

- $\rho_1 \in \mathcal{M}^{\text{tot}}(\mathbb{C}^2 \otimes \mathbb{C}^2)$
- $\rho_2 \in \partial\mathcal{M}^{\text{tot}}(\mathbb{C}^2 \otimes \mathbb{C}^2)$

distributed uniformly according to the Lebesgue (resp., surface) measure. Then

$$\mathbb{P}(\rho_1 \text{ is separable}) = 2 \cdot \mathbb{P}(\rho_2 \text{ is separable})$$

This phenomenon was earlier observed experimentally by Slater (2005).

Why $p_1 = 2\,p_2$?

$$p_1 = \frac{\mathrm{vol}_n \mathcal{M}^{\mathrm{PPT}}}{\mathrm{vol}_n \mathcal{M}^{\mathrm{tot}}}, \quad p_2 = \frac{\mathrm{vol}_{n-1}(\partial\mathcal{M}^{\mathrm{tot}} \cap \mathcal{M}^{\mathrm{PPT}})}{\mathrm{vol}_{n-1}\partial\mathcal{M}^{\mathrm{tot}}}$$

Hence

$$\frac{p_1}{p_2} = \frac{\mathrm{vol}_{n-1}\partial\mathcal{M}^{\mathrm{PPT}}}{\mathrm{vol}_{n-1}\partial\mathcal{M}^{\mathrm{PPT}}} \cdot \frac{\mathrm{vol}_n \mathcal{M}^{\mathrm{PPT}}}{\mathrm{vol}_n \mathcal{M}^{\mathrm{tot}}} \cdot \frac{\mathrm{vol}_{n-1}\partial\mathcal{M}^{\mathrm{tot}}}{\mathrm{vol}_{n-1}(\partial\mathcal{M}^{\mathrm{tot}} \cap \mathcal{M}^{\mathrm{PPT}})}$$

$$= \frac{\mathrm{vol}_{n-1}\partial\mathcal{M}^{\mathrm{PPT}}}{\mathrm{vol}_{n-1}(\partial\mathcal{M}^{\mathrm{tot}} \cap \mathcal{M}^{\mathrm{PPT}})}$$

as $\mathcal{M}^{\mathrm{PPT}}$ and $\mathcal{M}^{\mathrm{tot}}$ are both of constant height of the same dimension and with the same inradius.

Since T_1 is an involution, half of the boundary of $\mathcal{M}^{\mathrm{PPT}}$ comes from the boundary of $\mathcal{M}^{\mathrm{tot}}$ and so the ratio must be 2. (The measure of the "corners" is 0.)