Anti-concentration Inequalities

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Concentration and Anti-concentration

- Concentration phenomena: Nice random variables X are concentrated about their means.
- Examples:
  1. Probability theory: $X =$ sum of independent random variables (concentration inequalities: Chernoff, Bernstein, Bennett, …; large deviation theory).
  2. Geometric functional analysis: $X =$ Lipschitz function on the Euclidean sphere.
- *How strong* concentration should one expect?
  No stronger than a Gaussian (Central Limit Theorem).
- Anti-concentration phenomena: nice random variables S concentrate *no stronger* than a Gaussian.
  (Locally well spread).

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- Concentration inequalities:

$$\mathbb{P}(|X - \mathbb{E}X| > \varepsilon) \leq ?$$

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Estimate *Lévy's concentration function* of a random variable X:

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1. Probability Theory.
   - For *sums of independent random variables*, studied by [Lévy, Kolmogorov, Littlewood-Offord, Erdös, Esséen, Halasz, . . .]
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2. Geometric Functional Analysis. For *Lipschitz functions*:

Small Ball Probability Theorem

Let f be a convex even function on the unit Euclidean sphere (S^{n-1}, σ), whose average over the sphere $= 1$ and Lipschitz constant $= L$. Then

$$\sigma(x : |f(x)| \leq \varepsilon) \leq \varepsilon^{c/L^2}.$$

- Conjectured by V.; [Latala-Oleszkiewicz] deduced the Theorem from the *B-conjecture*, solved by [Cordero-Fradelizi-Maurey].
- Interpretation. $K \subseteq \mathbb{R}^n$: convex, symmetric set; $f(x) = \|x\|_K$. SBPT: asymptotic "dimension" of the spikes (parts of K far from the origin) is $\gtrsim 1/L^2$.
- Applied to Dvoretzky-type thms in [Klartag-V.]

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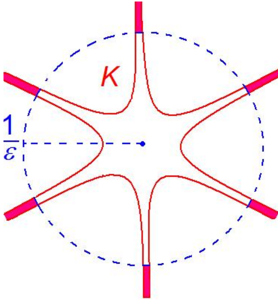

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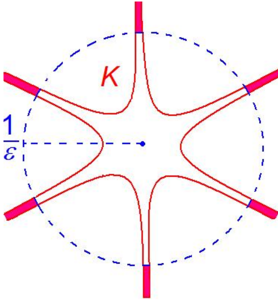

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$$p_\varepsilon(X) := \sup_{v\in\mathbb{R}} \mathbb{P}(|X - v| \le \varepsilon).$$

- What estimate can we *expect*?
- For every random variable X with density, we have

$$p_\varepsilon(X) \sim \varepsilon.$$

- If X is discrete, this fails for small ε (because of the atoms), so we can only expect

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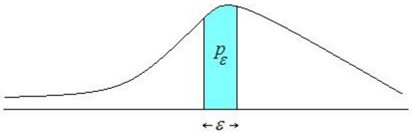

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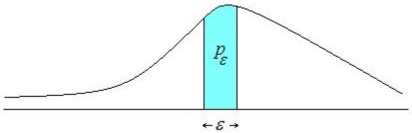

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- Classical example: Sums of independent random variables

$$S := \sum_{k=1}^{n} a_k \xi_k$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. (we can think of ± 1),
and $a = (a_1, \ldots, a_n)$ is a fixed vector of real coefficients

- An *ideal estimate* on the concentration function would be

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- Ideal estimate:

$$p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S - v| \le \varepsilon) \lesssim \varepsilon / \|a\|_2 + e^{-cn}.$$

- Trivial example: Gaussian sums,
  with ξ_k = standard normal i.i.d. random variables.
  The ideal estimate holds even without the exponential term.
- Nontrivial example: Bernoulli sums,
  with $\xi_k = \pm 1$ symmetric i.i.d. random variables.
- The problem for Bernoulli sums is nontrivial even for $\varepsilon = 0$,
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Application: Random matrices

This was our main motivation.

- A: an $n \times n$ matrix with i.i.d. entries.
  What is the probability that A is singular?
  Ideal answer: e^{-cn}.
- Geometric picture.
  Let X_k denote the column vectors of A.
  A nonsingular $\Rightarrow X_1 \notin \text{span}(X_2, \ldots, X_n) := H$
- We condition on H (i.e. on $X_2, \ldots, X_n$); let a be the normal of H.
  A nonsingular $\Rightarrow \langle a, X_1 \rangle \neq 0$.
  Write this in coordinates for $a = (a_k)_1^n$ and $X = (\xi_k)_1^n$ (i.i.d):

$$A \text{ is nonsingular} \Rightarrow \sum_{k=1}^{n} a_k \xi_k \neq 0.$$

$$\mathbb{P}(A \text{ is singular}) \geq p_0(a).$$

- Thus, in order to solve the invertibility problem, we *have* to prove an anti-concentration inequality. See Mark Ridelson's talk.

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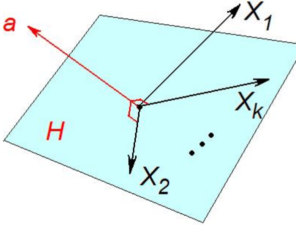

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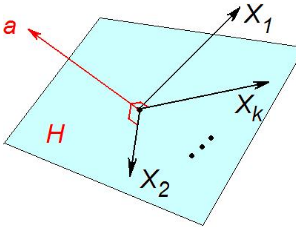

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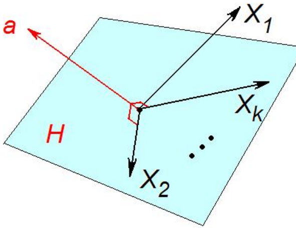

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Anti-concentration: the Littlewood-Offord Problem

Littlewood-Offord Problem.

For Bernoulli sums $S = \sum a_k \xi_k$, estimate the concentration function

$$p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S - v| \le \varepsilon).$$

- For concentrated vectors, e.g. $a = (1, 1, 0, \ldots, 0)$, $p_0(a) = \frac{1}{2} = \text{const}$.
  There are lots of cancelations in the sum $S = \pm 1 \pm 1$.
- For spread vectors, the small ball probability gets better: for $a = (1, 1, 1, \ldots, 1)$, we have $p_0(a) = \binom{n}{n/2}/2^n \sim n^{-1/2}$.
- This is a general fact:

If $a \ge 1$ pointwise, then $p_0(a) \le p_0(1, 1, \ldots, 1) \sim n^{-1/2}$.
[Littlewood-Offord], [Erdös, 1945].

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If $|a_j - a_k| \ge 1$ for $k \ne j$, then $p_1(a) \lesssim n^{-3/2}$.
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Anti-concentration: the Littlewood-Offord Problem

Littlewood-Offord Problem.

For Bernoulli sums $S = \sum a_k \xi_k$, estimate the concentration function

$$p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S - v| \le \varepsilon).$$

- Will be less cancelations if the coefficients are essentially different: For $a = (1, 2, 3, \dots)$, we have $p_0(a) \sim n^{-3/2}$.
- This is a general fact:

If $|a_j - a_k| \ge 1$ for $k \neq j$, then $p_1(a) \lesssim n^{-3/2}$.
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- The only source of cancelations in random sums $S = \sum \pm a_k$ is a rich additive structure of the coefficients a_k.
- Cancelations can only occur when the coefficients a_k are *arithmetically commensurable*. Specifically, if there are lots of cancelations, then the coefficients a_k can be embedded into a short arithmetic progression.

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If the small ball probability $p_\varepsilon(a)$ is large, then the coefficient vector a can be embedded into a short arithmetic progression.

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Theorem (Tao-Vu)

Let $a_1, \ldots, a_n$ be integers, and let $A \geq 1$, $\delta \in (0,1)$. Suppose for the random Bernoulli sums one has

$$p_0(a) \geq n^{-A}.$$

Then all except $O_{A,\varepsilon}(n^{\delta})$ coefficients a_k are contained in the Minkowski sum of $O(A/\delta)$ arithmetic progressions of lengths $n^{O_{A,\delta}(1)}$.

- Usefulness. One can reduce the small ball probability to an arbitrary polynomial order by controlling the additive structure of a.
- Shortcomings. **1.** We often have *real coefficients a_k* (not $\mathbb{Z}$). **2.** We are interested in *general small ball probabilities $p_\varepsilon(a)$* rather than the measure of atoms $p_0(a)$.
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Embedding into arithmetic progressions via LCD

- **Goal:** embed a vector $a \in \mathbb{R}^n$ into a short arithmetic progression (essentially). What is its length?
- Bounded by the essential least common denominator (LCD) of a:

$$D(a) = D_{\alpha,\kappa}(a) = \inf\{t > 0 : \ ta \text{ is a nonzero essential integer}\}$$

(all except κ coefficients of ta are of dist. α from nonzero integers).
- For $a \in \mathbb{Q}^n$, this is the usual LCD.

- The vector $D(a)a$ (and thus a itself) essentially embeds into an arithmetic progression of length $\|D(a)a\|_\infty \lesssim D(a)$.
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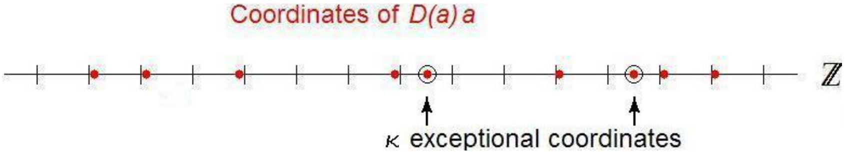

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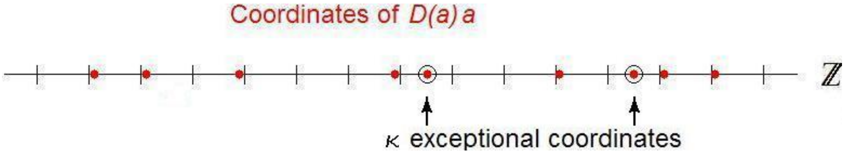

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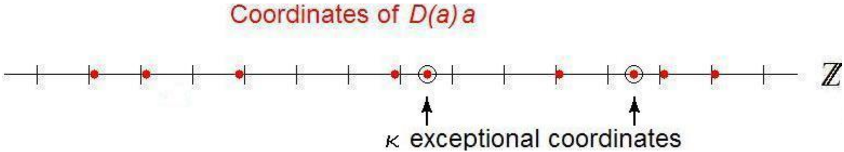

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Anti-concentration: the Littlewood-Offord Phenomenon

Theorem (Anti-Concentration)

Consider a sum of independent random variables

$$S = \sum_{k=1}^{n} a_k \xi_k$$

where ξ_k *are i.i.d. with third moments and* $C_1 \le |a_k| \le C_2$ *for all* k*.*
Then, for every $\alpha \in (0,1)$*,* $\kappa \in (0,n)$ *and* $\varepsilon \ge 0$ *one has*

$$p_\varepsilon(S) \lesssim \frac{1}{\sqrt{\kappa}}\Big(\varepsilon + \frac{1}{D_{\alpha,\kappa}(a)}\Big) + e^{-c\alpha^2\kappa}.$$

Recall: $D_{\alpha,\kappa}(a)$ is the essential LCD of a ($\pm\alpha$ and up to κ coefficients).

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- $\varepsilon = 0$; thus $p_0(a)$ is the measure of atoms
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- number of exceptional coefficients $\kappa = 0.01n$:

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99% of the coefficients of a are within 0.1 of an arithmetic progression of length $\sim n^{-1/2}/p_0(a)$.

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- $a = (1, 2, \ldots, n)$. To apply (ILO), we normalize and truncate:

$$p_0(a) = p_0\Big(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\Big) \le p_0\Big(\frac{n/2}{n}, \frac{n/2+1}{n}, \ldots, \frac{n}{n}\Big)$$

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Soft approach

- We will sketch the proof.
  There are two approaches, soft and ergodic.
- Soft approach: deduce anti-concentration inequalities from Central Limit Theorem. [Litvak-Pajor-Rudelson-Tomczak].
- By CLT, the random sum

$$S \approx \text{Gaussian}.$$

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$$\left|\mathbb{P}(S/\|a\|_2 \le t) - \mathbb{P}(g \le t)\right| \lesssim \left(\frac{\|a\|_3}{\|a\|_2}\right)^3 \qquad \text{for every } t.$$

- The more *spread* the coefficient vector a, the better (RHS smaller). RHS minimized for $a = (1, 1, \dots, 1)$, for which it is $\left(\frac{n^{1/3}}{n^{1/2}}\right)^3 = n^{-1/2}$. Thus the best bound the soft approach gives is $p_0(a) \le n^{-1/2}$.
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Soft approach

Theorem (Berry-Esséen's Central Limit Theorem)

Consider a sum of independent random variables $S = \sum a_k \xi_k$*, where* ξ_k *are i.i.d. centered with variance* 1 *and finite third moments. Let* g *be the standard normal random variable. Then*

$$\left|\mathbb{P}(S/\|a\|_2 \le t) - \mathbb{P}(g \le t)\right| \lesssim \left(\frac{\|a\|_3}{\|a\|_2}\right)^3 \qquad \text{for every } t.$$

- The more *spread* the coefficient vector a, the better (RHS smaller). RHS minimized for $a = (1, 1, \ldots, 1)$, for which it is $\left(\frac{n^{1/3}}{n^{1/2}}\right)^3 = n^{-1/2}$. Thus the best bound the soft approach gives is $p_0(a) \le n^{-1/2}$.
- Anti-concentration inequalities can not be based on ℓ_p norms of the coefficient vector a (which works nicely for the concentration inequalities, e.g. Bernstein's!).
- The ℓ_p norms do not distinguish between $(1, 1, \ldots, 1)$ and $(1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots, 1 + \frac{n}{n})$, for which concentration functions are different. The norms *feel the bulk* and *ignore the fluctuations*.

Ergodic approach

Instead of applying Berry-Esséen's CLT directly, use a tool from its proof: Esséen's inequality. This method goes back to [Halasz, 1977].

Proposition (Esséen's Inequality)

The concentration function of any random variable S is bounded by the L^1 norm of its characteristic function $\phi(t) = \mathbb{E}\exp(iSt)$:

$$p_\varepsilon(S) \lesssim \int_{-\pi/2}^{\pi/2} |\phi(t/\varepsilon)| \, dt.$$

- Proof: take Fourier transform.
- We use Esséen's Inequality for the random sum $S = \sum_1^n a_k \xi_k$. We work with the example of Bernoulli sums ($\xi_k = \pm 1$). By the independence, the characteristic function of S factors

$$\phi(t) = \prod_1^n \phi_k(t), \qquad \phi_k(t) = \mathbb{E}\exp(ia_k\xi_k t) = \cos(a_k t).$$

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$$|\phi(t)| = \prod_{1}^{n} |\cos(a_k t)| \le \exp(-f(t)),$$

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By Esséen's Inequality,

$$p_\varepsilon(S) \lesssim \int_{-\pi/2}^{\pi/2} |\phi(t/\varepsilon)|\, dt \le \int_{-\pi/2}^{\pi/2} \exp(-f(t/\varepsilon))\, dt$$
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- Ergodic approach: regard t as *time*; $\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon}$ = long term average.
- A *system of n particles* $a_k t$ that move along $\mathbb{T}$ at speeds a_k:

- The estimate is *poor* precisely when $f(t)$ is small $\Leftrightarrow$ most particles *return to the origin*, making $\sin^2(a_k t)$ small.
- We are thus interested in the recurrence properties of the system. *How often* do most particles return to the origin?

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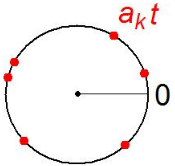

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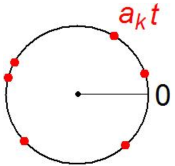

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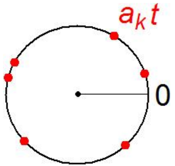

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- We need to understand *how particles can move in the system*.
- Two extreme types of systems (common in ergodic theory):

 1. Quasi-random ("mixing"). Particles move as if independent.
 2. Quasi-periodic. Particles "stick together".

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1. Quasi-random systems.

- By "independence", the event that *most particles are near the origin* is exponentially rare (frequency e^{-cn}).
- Away from the origin, $\sin^2(a_k t) \geq$ const, thus $f(t) \sim cn$.
- This leads to the bound

$$p_\varepsilon(S) \lesssim \varepsilon + e^{-cn}.$$

  (ε is due to a constant initial time to depart from the origin).
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- Example. $a = (1, 1, \ldots, 1)$. Move as one particle.
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- More general example. Rational coefficients with small LCD. Then ta_k often becomes an integer, i.e. the particles often return to the origin together.
- Main observation. Small LCD is the *only* reason for the almost periodicity of the system:

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Observation (Quasi-periodicity and LCD)

If a system (ta_k) is quasi-periodic then essential LCD of (a_k) is small.

- Proof. Assume most of ta_k *often* return near the origin together – say, with frequency ω (i.e. spend portion of time ω near the origin).
- Equivalently, ta becomes an *essential integer* with frequency ω.
- Thus ta becomes essential integer twice within time $\sim \frac{1}{\omega}$.
  $\exists$ two instances $0 < t_1 - t_2 < 1/\omega$ in which $t_1 a$ and $t_2 a$ are different essential integers.
- Subtract $\Rightarrow$ $(t_2 - t_1)a$ is also an essential integer.
  By the definition of the essential LCD,

$$D(a) \le t_2 - t_1 < \frac{1}{\omega}. \qquad \square$$

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Improvements

[O.Friedland-S.Sodin] recently simplified the argument:

- Used a more convenient notion of essential integers as *vectors in* $\mathbb{R}^n$ *that can be approximated by integer vectors within* $\alpha\sqrt{n}$ *in Euclidean distance*.
- Bypassed *Halasz's regularity argument* (which I skipped) using a direct and simple analytic bound.

Using the anti-concentration inequality

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- In order to use the anti-concentration inequality, we need to know that LCD of a is *large*.
- Is LCD large for *typical* (i.e. random) coefficient vectors a?
- For random matrix problems, a = normal to the random hyperplane spanned by $n-1$ i.i.d. vectors X_k in $\mathbb{R}^n$:
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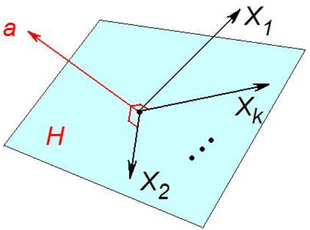

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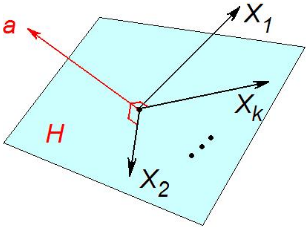

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