Anti-concentration Inequalities

Roman Vershynin    Mark Rudelson

University of California, Davis
University of Missouri-Columbia

Phenomena in High Dimensions
Third Annual Conference
Samos, Greece
June 2007
Concentration and Anti-concentration

- **Concentration phenomena**: Nice random variables $X$ are concentrated about their means.

- **Examples**:
  1. **Probability theory**: $X = \text{sum of independent random variables}$ (concentration inequalities: Chernoff, Bernstein, Bennett, \ldots; large deviation theory).
  2. **Geometric functional analysis**: $X = \text{Lipschitz function on the Euclidean sphere}$.

- **How strong** concentration should one expect? No stronger than a Gaussian (Central Limit Theorem).

- **Anti-concentration phenomena**: nice random variables $S$ concentrate no stronger than a Gaussian. (Locally well spread).
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- Concentration inequalities:
  \[ \Pr(|X - \mathbb{E}X| > \varepsilon) \leq ? \]

- Anti-concentration inequalities: for a given (or all) \( v \),
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**Anti-concentration**

**Problem**

Estimate *Lévy’s concentration function* of a random variable $X$:

$$p_\varepsilon(X) := \sup_{v \in \mathbb{R}} \mathbb{P}(|X - v| \leq \varepsilon).$$

1. Probability Theory.
   - For *sums of independent random variables*, studied by [Lévy, Kolmogorov, Littlewood-Offord, Erdös, Esséen, Halasz, ...]
   - For *random processes* (esp. Brownian motion), see the survey [Li-Shao]
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2. Geometric Functional Analysis. For \textit{Lipschitz functions}:

\textbf{Small Ball Probability Theorem}

Let $f$ be a convex even function on the unit Euclidean sphere $(S^{n-1}, \sigma)$, whose average over the sphere $= 1$ and Lipschitz constant $= L$. Then

$$\sigma(x : |f(x)| \leq \varepsilon) \leq \varepsilon^{c/L^2}.$$ 

- Conjectured by V.; [Latala-Oleszkiewicz] deduced the Theorem from the \textit{B-conjecture}, solved by [Cordero-Fradelizi-Maurey].

- Interpretation. $K \subseteq \mathbb{R}^n$: convex, symmetric set; $f(x) = \|x\|_K$.
SBPT: asymptotic “dimension” of the spikes (parts of $K$ far from the origin) is $\gtrsim 1/L^2$.

- Applied to Dvoretzky-type thms in [Klartag-V].
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\[ p_\varepsilon(X) := \sup_{v \in \mathbb{R}} \mathbb{P}(|X - v| \leq \varepsilon). \]

- What estimate can we expect?
- For every random variable \( X \) with density, we have
  \[ p_\varepsilon(X) \sim \varepsilon. \]

- If \( X \) is discrete, this fails for small \( \varepsilon \) (because of the atoms), so we can only expect
  \[ p_\varepsilon(X) \lesssim \varepsilon + \text{measure of an atom}. \]
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- Classical example: Sums of independent random variables

\[ S := \sum_{k=1}^{n} a_k \xi_k \]

where \( \xi_1, \ldots, \xi_n \) are i.i.d. (we can think of \( \pm 1 \)), and \( a = (a_1, \ldots, a_n) \) is a fixed vector of real coefficients.

An ideal estimate on the concentration function would be

\[ p_{\varepsilon}(a) := p_{\varepsilon}(S) \lesssim \varepsilon/\|a\|_2 + e^{-cn}, \]

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○ Ideal estimate:

\[ p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S - v| \leq \varepsilon) \lesssim \varepsilon / \|a\|_2 + e^{-cn}. \]

○ Trivial example: Gaussian sums,
  with \( \xi_k = \) standard normal i.i.d. random variables.
The ideal estimate holds even without the exponential term.

○ Nontrivial example: Bernoulli sums,
  with \( \xi_k = \pm 1 \) symmetric i.i.d. random variables.
The problem for Bernoulli sums is nontrivial even for \( \varepsilon = 0 \),
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This is the most studied case in the literature.
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Application: Random matrices

This was our main motivation.

- **A:** an $n \times n$ matrix with i.i.d. entries.
  What is the probability that $A$ is singular?
  Ideal answer: $e^{-cn}$.

- Geometric picture.
  Let $X_k$ denote the column vectors of $A$.
  $A$ nonsingular $\Rightarrow$ $X_1 \not\in \text{span}(X_2, \ldots, X_n) := H$
  We condition on $H$ (i.e. on $X_2, \ldots, X_n$); let $a$ be the normal of $H$.
  $A$ nonsingular $\Rightarrow \langle a, X_1 \rangle \neq 0$.
  Write this in coordinates for $a = (a_k)_1^n$ and $X = (\xi_k)_1^n$ (i.i.d):

    $$A \text{ is nonsingular} \Rightarrow \sum_{k=1}^n a_k \xi_k \neq 0.$$ 

    $$\mathbb{P}(A \text{ is singular}) \geq p_0(a).$$

- Thus, in order to solve the invertibility problem, we have to prove an anti-concentration inequality. See Mark Ridelson’s talk.
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**Littlewood-Offord Problem.**

For Bernoulli sums $S = \sum a_k \xi_k$, estimate the concentration function

$$p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S - v| \leq \varepsilon).$$

- For **concentrated vectors**, e.g. $a = (1, 1, 0, \ldots, 0)$, $p_0(a) = \frac{1}{2} = \text{const.}$
  There are lots of cancelations in the sum $S = \pm 1 \pm 1$.
- For **spread vectors**, the small ball probability gets better:
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- This is a general fact:

  If $a \geq 1$ pointwise, then $p_0(a) \leq p_0(1, 1, \ldots, 1) \sim n^{-1/2}$.

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- Will be less cancelations if the coefficients are essentially different: For $a = (1, 2, 3, \ldots)$, we have $p_0(a) \sim n^{-3/2}$.
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- Still lots of cancelations in the sum $S = \pm 1 \pm 2 \cdots \pm n$.
- Question. How to prevent cancelations in random sums? For what vectors $a$ is the concentration function $p_0(a)$ small? E.g. exponential rather than polynomial.
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- [Tao-Vu, 2006] proposed an explanation for cancelations, which they called the *Inverse Littlewood-Offord Phenomenon*:

  - The only source of cancelations in random sums $S = \sum \pm a_k$ is a rich additive structure of the coefficients $a_k$.
  - Cancelations can only occur when the coefficients $a_k$ are arithmetically commensurable. Specifically, if there are lots of cancelations, then the coefficients $a_k$ can be embedded into a short arithmetic progression.

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If the small ball probability $p_\epsilon(a)$ is large, then the coefficient vector $a$ can be embedded into a short arithmetic progression.
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Theorem (Tao-Vu)

Let $a_1, \ldots, a_n$ be integers, and let $A \geq 1$, $\delta \in (0, 1)$. Suppose for the random Bernoulli sums one has

$$p_0(a) \geq n^{-A}.$$ 

Then all except $O_{A,\varepsilon}(n^\delta)$ coefficients $a_k$ are contained in the Minkowski sum of $O(A/\delta)$ arithmetic progressions of lengths $n^{O_{A,\delta}(1)}$.

- **Usefulness.** One can reduce the small ball probability to an arbitrary polynomial order by controlling the additive structure of $a$.
- **Shortcomings.** 1. We often have real coefficients $a_k$ (not $\mathbb{Z}$).
  2. We are interested in general small ball probabilities $p_\varepsilon(a)$ rather than the measure of atoms $p_0(a)$.
- **Problem.** Develop the Inverse L.-O. Phenomenon over $\mathbb{R}$. 
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Essential integers

For real coefficient vectors $a = (a_1, \ldots, a_n)$, the embedding into an arithmetic progression must clearly be approximate (near an arithmetic progression).

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Embedding into arithmetic progressions via LCD

- **Goal:** embed a vector \( a \in \mathbb{R}^n \) into a short arithmetic progression (essentially). What is its length?
- Bounded by the essential least common denominator (LCD) of \( a \):

\[
D(a) = D_{\alpha, \kappa}(a) = \inf\{ t > 0 : ta \text{ is a nonzero essential integer} \}
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(all except \( \kappa \) coefficients of \( ta \) are of dist. \( \alpha \) from nonzero integers).
- For \( a \in \mathbb{Q}^n \), this is the usual LCD.

The vector \( D(a)a \) (and thus \( a \) itself) essentially embeds into an arithmetic progression of length \( \|D(a)a\|_\infty \lesssim D(a) \).
So, \( D(a) \) being small means that \( a \) has rich additive structure.
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Theorem (Anti-Concentration)

Consider a sum of independent random variables

\[ S = \sum_{k=1}^{n} a_k \xi_k \]

where \( \xi_k \) are i.i.d. with third moments and \( C_1 \leq |a_k| \leq C_2 \) for all \( k \).

Then, for every \( \alpha \in (0, 1) \), \( \kappa \in (0, n) \) and \( \varepsilon \geq 0 \) one has

\[
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Recall: \( D_{\alpha, \kappa}(a) \) is the essential LCD of \( a \) (\( \pm \alpha \) and up to \( \kappa \) coefficients).
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Partial case:
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Partial case:

- \( \varepsilon = 0 \); thus \( p_0(a) \) is the measure of atoms
- accuracy \( \alpha = 0.1 \)
- number of exceptional coefficients \( \kappa = 0.01n \):

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99\% of the coefficients of \( a \) are within 0.1 of an arithmetic progression of length \( \sim n^{-1/2}/p_0(a) \).

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Examples:
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- \( a = (1, 2, \ldots, n) \). To apply (ILO), we normalize and truncate:
  \[ p_0(a) = p_0 \left( \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} \right) \leq p_0 \left( \frac{n/2}{n}, \frac{n/2+1}{n}, \ldots, \frac{n}{n} \right) \]
  The LCD of such vector is \( \gtrsim n \). Then (ILO) gives
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Soft approach

- We will sketch the proof.
- There are two approaches, soft and ergodic.
- **Soft approach:** deduce anti-concentration inequalities from Central Limit Theorem. [Litvak-Pajor-Rudelson-Tomczak].
- By CLT, the random sum

\[ S \approx \text{Gaussian}. \]

Hence can approximate the concentration function

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Consider a sum of independent random variables $S = \sum a_k \xi_k$, where $\xi_k$ are i.i.d. centered with variance 1 and finite third moments. Let $g$ be the standard normal random variable. Then

$$|\Pr(S/\|a\|_2 \leq t) - \Pr(g \leq t)| \lesssim \left(\frac{\|a\|_3}{\|a\|_2}\right)^3$$

for every $t$.

- The more spread the coefficient vector $a$, the better (RHS smaller).
  RHS minimized for $a = (1, 1, \ldots, 1)$, for which it is $\left(\frac{n^{1/3}}{n^{1/2}}\right)^3 = n^{-1/2}$.
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- The \( \ell_p \) norms do not distinguish between \((1, 1, \ldots, 1)\) and \((1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots, 1 + \frac{n}{n})\), for which concentration functions are different. The norms feel the bulk and ignore the fluctuations.
Ergodic approach

Instead of applying Berry-Esséen’s CLT directly, use a tool from its proof: Esséen’s inequality. This method goes back to [Halasz, 1977].

**Proposition (Esséen’s Inequality)**

The concentration function of any random variable $S$ is bounded by the $L^1$ norm of its characteristic function $\phi(t) = \mathbb{E} \exp(iSt)$:

$$p_\varepsilon(S) \lesssim \int_{-\pi/2}^{\pi/2} |\phi(t/\varepsilon)| \; dt.$$ 

- **Proof:** take Fourier transform.
- We use Esséen’s Inequality for the random sum $S = \sum_{k=1}^n a_k \xi_k$. We work with the example of Bernoulli sums ($\xi_k = \pm 1$). By the independence, the characteristic function of $S$ factors

$$\phi(t) = \prod_{k=1}^n \phi_k(t), \quad \phi_k(t) = \mathbb{E} \exp(i a_k \xi_k t) = \cos(a_k t).$$
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- A system of $n$ particles $a_k t$ that move along $\mathbb{T}$ at speeds $a_k$:

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  ![Diagram of a system of particles moving along a circle]

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![Circle with points at \( a_k t \) and 0](image)

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- Two extreme types of systems (common in ergodic theory):
  1. Quasi-random ("mixing"). Particles move as if independent.
  2. Quasi-periodic. Particles "stick together".
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- By “independence”, the event that most particles are near the origin is exponentially rare (frequency \( e^{-cn} \)).
- Away from the origin, \( \sin^2(a_k t) \geq \text{const} \), thus \( f(t) \sim cn \).
- This leads to the bound

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- **Example.** \( a = (1, 1, \ldots, 1) \). Move as one particle. Thus \( f(t) \sim n \sin^2 t \), and integration gives \( p_\varepsilon(S) \lesssim n^{-1/2} \).

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Observation (Quasi-periodicity and LCD)

If a system \((ta_k)\) is quasi-periodic then essential LCD of \((a_k)\) is small.

- Proof. Assume most of \(ta_k\) often return near the origin together – say, with frequency \(\omega\) (i.e. spend portion of time \(\omega\) near the origin).
- Equivalently, \(ta\) becomes an essential integer with frequency \(\omega\).
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- By the definition of the essential LCD,

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- **Conclusion of the proof.**
  1. If the essential LCD \( D(a) \) is large, then the system is *not* quasi-periodic \( \Rightarrow \) closer to *quasi-random*.
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[O.Friedland-S.Sodin] recently simplified the argument:

- Used a more convenient notion of essential integers as vectors in $\mathbb{R}^n$ that can be approximated by integer vectors within $\alpha \sqrt{n}$ in Euclidean distance.
- Bypassed Halasz’s regularity argument (which I skipped) using a direct and simple analytic bound.
Using the anti-concentration inequality

\[ p_\varepsilon(a) \lesssim \frac{1}{\sqrt{n}} \left( \varepsilon + \frac{1}{D(a)} \right) + e^{-cn}. \]

- In order to use the anti-concentration inequality, we need to know that LCD of \( a \) is \textit{large}.
- Is LCD large for typical (i.e. random) coefficient vectors \( a \)?
- For random matrix problems, \( a = \) normal to the random hyperplane spanned by \( n - 1 \) i.i.d. vectors \( X_k \) in \( \mathbb{R}^n \):

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  ![Diagram of a random hyperplane spanned by vectors]

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![Diagram of normal distribution]

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