Projections and Liftings on the Sphere

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Motivation 1: Projection Mean Bodies

$K \subset \mathbb{R}^d$ a convex body,
$h(K, \cdot)$ support function of K,
$\mathcal{L}_k$ the Grassmannian of k-dimensional subspaces, $k \in \{1, ..., d-1\}$,
$K|L$ orthogonal projection of K onto $L \in \mathcal{L}_k$.

The k-th **projection mean body** $P_k(K)$ of K is the Minkowski average of the projections $K|L$ (averaged with the invariant probability measure dL on $\mathcal{L}_k$):

$$h(P_k(K), \cdot) = \int_{\mathcal{L}_k} h(K|L, \cdot) dL.$$

For $k = d-1$ this body was first considered by **Schneider (1977)**, who showed:

$$P_{d-1}(K) \text{ homothetic to } K \iff K \text{ is a ball.}$$

The injectivity of the operator $p_k : K \mapsto P_k(K)$, $k \geq 2$, was investigated by several authors **(Spriestersbach, Goodey, Jiang, Kiderlen)**.

Current state:

p_k is injective for $k = 3$ and $k \geq d/2$,

p_2 is injective $\iff d \neq 14$,

the cases $3 < k < d/2$ are open.

Analytic interpretation of

$$h(K,\cdot) \mapsto \int_{\mathcal{L}_k} h(K|L,\cdot)dL :$$

The function $f = h(K,\cdot)$ on S^{d-1} is
a) restricted (projected) to $S^{d-1} \cap L$,
b) then the restriction is lifted (by linear extension) back to S^{d-1} and
c) the result is averaged over all L.

Thus p_k appears as a (special case of a) **mean lifted projection** on S^{d-1} (applied to $h(K,\cdot)$).

Motivation 2: Directional Distribution

X spatially homogeneous random field of convex particles in $\mathbb{R}^d$ (a **stationary point process** on $\mathcal{K}^d$),
$\Xi_{d-1}(K,\cdot)$ support measure on $\mathbb{R}^d \times S^{d-1}$ (concentrated on the normal bundle of K),
$\overline{S}_{d-1}(X,\cdot)$ the **specific surface area measure** of X is the derivative of

$$\mathbb{E} \sum_{K \in X} \Xi_{d-1}(K,\cdot)$$

w.r.t. λ_d.

After normalization, $\overline{S}_{d-1}(X,\cdot)$ describes the distribution of the outer normal in a typical boundary point of X.

Suppose, the section process $X \cap L$ is observed, for randomly chosen $L \in \mathcal{L}_k$.

Can $\overline{S}_{k-1}(X \cap L, \cdot)$ be used to determine (estimate) $\overline{S}_{d-1}(X, \cdot)$?

If $\overline{S}_{k-1}(X \cap L, \cdot)$ is seen as a kind of projection of $\mu := \overline{S}_{d-1}(X, \cdot)$ onto $S^{d-1} \cap L$, we can use the trivial extension of this measure to S^{d-1}, average over all L and obtain a transformed image $b_k(\mu)$.

Is the resulting operator b_k injective?

Is it worth to lift the measures $\overline{S}_{k-1}(X \cap L, \cdot)$ in a different way to S^{d-1} before averaging (in order to improve the injectivity properties)?

Spherical Projections and Liftings

For $f \in \mathcal{C}(S^{d-1})$, $j \in (-k, \infty]$ and $L \in \mathcal{L}_k$, we define the ***j*-weighted spherical projection** $\pi_{L,j} f$ of f:

$$(\pi_{L,j} f)(u) = \int_{H^{d-k}(L,u)} f(v) \langle u, v \rangle^{k+j-1} dv, \quad u \in S^{d-1} \cap L,$$

where integration is over the half-sphere

$$H^{d-k}(L,u) = \{x \in S^{d-1} \setminus L^\perp : \frac{x|L}{\|x|L\|} = u\}.$$

$\pi_{L,\infty} f$ is the restriction of f to $S^{d-1} \cap L$.

Similarly, for a measure μ on $S^{d-1} \cap L$ and $m \in (-k, \infty]$, the **m-weighted spherical lifting** $\pi^*_{L,m}\mu$ of μ is defined as

$$(\pi^*_{L,m}\mu)(A) = \int_{S^{d-1}\setminus L^\perp} \int_{H^{d-k}(L,u)\cap A} \langle v, w\rangle^{k+m-1} dw\, \mu(dv).$$

$\pi^*_{L,\infty}\mu$ is the (trivial) extension of μ to S^{d-1}.

Finally, we put

$$\pi^{(k)}_{m,j} f = \int_{\mathcal{L}_k} \pi^*_{L,m} \pi_{L,j} f \, dL,$$

and call this the **mean lifted projection** (with weights m, j) of the function f on S^{d-1}. Similarly: $\pi^{(k)}_{m,j}\mu$ for measures μ.

The operators $\pi^{(k)}_{m,j}$ have interesting properties:

$\pi^{(k)}_{m,j}$ is a continuous endomorphism on the Banach space $\mathcal{C}(S^{d-1})$

$\pi^{(k)}_{m,j}$ is intertwining (commutes with rotations)

$\pi^{(k)}_{m,j}$ is self-adjoint

$\pi^{(k)}_{m,j} = \pi^{(k)}_{j,m}$

Examples

1) For the projection mean body $P_k(K)$:

$$h(P_k(K), \cdot) = \pi^{(k)}_{1,\infty} h(K, \cdot).$$

2) For the sectional mean normal measure:

$$\overline{S}_{k-1}(X \cap L, \cdot) = \pi_{L,1} \overline{S}_{d-1}(X, \cdot).$$

If we use the trivial lifting, the resulting average is

$$b_k(S_{d-1}(X, \cdot)) = \pi^{(k)}_{\infty,1} S_{d-1}(X, \cdot).$$

Since $\pi^{(k)}_{\infty,1} = \pi^{(k)}_{1,\infty}$, we obtain $b_k = p_k$.

3) The averaged directed section function $\overline{s}_k(K, \cdot)$ of a star body K (with radial function $\rho(K, \cdot)$), considered in **Goodey-W. (2006)** satisfies

$$\overline{s}_k(K, \cdot) = \pi_{k-d,\infty}^{(d-k+1)} \frac{1}{k} \rho^k(K, \cdot).$$

General Results

Theorem.

$$\pi_{m,j}^{(k)}\mu = c\int_{S^{d-1}\cap(\cdot)}\int_{S^{d-1}} K_{m,j}^{(k)}(\langle u,v\rangle)\,\mu(du)\,dv,$$

with a constant c and

$$K_{m,j}^{(k)}(t) = (1-t^2)^{(1-k)/2} \times \int_0^{\pi-\arccos(t)} \sin^{k+m-1}(s)\sin^{k+j-1}(s+\arccos(t))\,ds,$$

for $-1 \le t \le 1$.

A similar result holds for $m = \infty$.

Since $\pi_{m,j}^{(k)}$ is intertwining, it is injective iff all multipliers w.r.t. spherical harmonics are non-zero.

Theorem. *The n-th multiplier $a_{d,k,m,j,n}$ of $\pi_{m,j}^{(k)}$ satisfies*

$$a_{d,k,m,j,n} = \frac{\varpi_k}{\varpi_d} \sum_{t=0}^{[\frac{n}{2}]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t} \beta_{d-k+1,k+m-1,n-2t}$$

with

$$\beta_{d,p,n} = \begin{cases} \frac{p!}{2^n (p-n)! \left(\frac{d-1}{2}\right)_n} \frac{\varpi_{d-1} \varpi_{d+p+n}}{\varpi_{d+2n-1} \varpi_{p-n+1}}, & \text{if } 0 \le n \le p, \\ \frac{p!}{2^{p+1} \left(\frac{d-1}{2}\right)_{p+1}} \varpi_{d-1} P_{n-p-1}^{d+2p+2}(0), & \text{if } p < n \end{cases}$$

for $p > 0$ and

$$\beta_{d,0,n} = \frac{\varpi_{d-1}}{d-1} P_{n-1}^{d+2}(0).$$

Corollary. *$\pi^{(k)}_{j,j}$ and $\pi^{(k)}_{j,j+1}$ are injective.*

In particular, $\pi^{(k)}_{1,1}$ and $\pi^{(k)}_{1,2}$ are injective.

Kiderlen (2002) showed that also $\pi^{(k)}_{1,1-k}$ is injective.

Applications

(1) We recall that

$$h(P_k(K), \cdot) = \pi_{1,\infty}^{(k)} h(K, \cdot).$$

Theorem. *For $m > -k$, $\pi_{1,m}^{(k)} h(K, \cdot)$ is the support function of a convex body $P_{k,m}(K)$.*

K is determined by $P_{k,m}(K)$, if $m \in \{1, 2, 1-k\}$.

We call $P_{k,m}(K)$ the k-th **m-weighted projection mean body** of K.

(2) For the stationary random particle field X, we have seen that $\overline{S}_{k-1}(X \cap L, \cdot)$ can be observed.

If we spread these data for each $v \in S^{d-1} \cap L$ over the orthogonal half-sphere $H^{d-k}(L, v)$ with weight function $\langle v, \cdot \rangle^{k+m-1}$ and then average over all L, the resulting measure is

$$\pi_{1,m}^{(k)} \overline{S}_{d-1}(X, \cdot).$$

Due to the previous results, for $m \in \{1, 2, 1-k\}$, the measure $\overline{S}_{d-1}(X, \cdot)$ is uniquely determined.

Final remark:

There are some further results on injectivity, but also non-injectivity can occur.

E.g., $\pi_{k-d,\infty}^{(d-k+1)}$ is injective for

$$2 \leq k < (2d-3)/5 \quad \text{and} \quad (d-2)/2 \leq k \leq d-1,$$

but not injective for

$$k = (2d-3)/5.$$