Geometry of quantum maps and super operators

Stanislaw Szarek
Elisabeth Werner
Karol Zyczkowski

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  for a state ρ, $\Phi(\rho)$ is again a state

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$\Phi : \mathcal{M}_N \to \mathcal{M}_N$ is completely positive iff for all $k \geq 0$

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Choi Theorem

$\Phi : \mathcal{M}_N \to \mathcal{M}_N$ is $\mathcal{CP}$ iff the Choi matrix

$$D_\Phi = \begin{pmatrix} \Phi(E_{11}) \ldots \Phi(E_{1N}) \\ \ldots \\ \Phi(E_{N1}) \ldots \Phi(E_{NN}) \end{pmatrix}$$

is positive semidefinite

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conservation of probability in physical processes

$$\Longrightarrow$$

trace preserving (TP) property:

$$\mathrm{Tr}\Phi(\rho) = \mathrm{Tr}\rho$$

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i.e.

$$D_\Phi \in \mathcal{CP}_N^b = \mathcal{CP}_N \cap H^b$$

$$H^b = \{\Phi : \mathrm{Tr} D_\Phi = N\}$$

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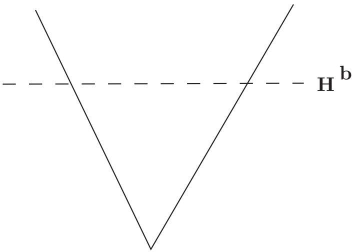

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or

$$\mathcal{P}_N \supset \mathcal{D}_N \supset \mathcal{CP}_N \supset \mathcal{T}_N \supset \mathcal{SP}_N$$

and

$$\mathcal{P}_N^b \supset \mathcal{D}_N^b \supset \mathcal{CP}_N^b \supset \mathcal{T}_N^b \supset \mathcal{SP}_N^b$$

$$\mathcal{P}_N^{TP} \supset \mathcal{D}_N^{TP} \supset \mathcal{CP}_N^{TP} \supset \mathcal{T}_N^{TP} \supset \mathcal{SP}_N^{TP}$$

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$$(\mathcal{CP}_N^b)^\circ = -\mathcal{CP}_N^b \qquad (\mathcal{SP}_N^b)^\circ = -\mathcal{P}_N^b$$

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$\circ$ and $-$ with respect to $\Phi_* = \frac{\mathbb{1}_{N^2}}{N}$

Corollary

For each of the sets $\mathcal{P}^b,\ \mathcal{D}^b,\ \mathcal{CP}^b,\ \mathcal{T}^b,\ \mathcal{SP}^b$ the Euclidean inradius is $\frac{1}{(N^2-1)^{\frac{1}{2}}}$ and the Euclidean outradius is $(N^2-1)^{\frac{1}{2}}$

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Proof

$d = N^2$.

enough to look at Hilbert Schmidt balls centered at $\Phi_* = \frac{\mathbb{1}_d}{\sqrt{d}}$

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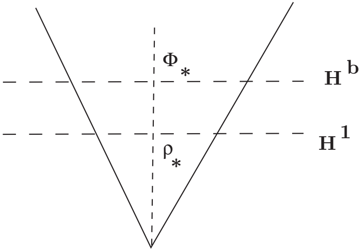

pure states p = 1-dimensional projections are in $\mathcal{CP}^1 = \mathcal{CP} \cap H^1$

$$\|\rho_* - p\|_{HS} = \| \begin{pmatrix} \frac{1}{d} \dots 0 \\ \dots \\ 0 \dots \frac{1}{d} \end{pmatrix} - \begin{pmatrix} 0 \dots 0 \\ \dots \\ 0 \dots 1 \end{pmatrix} \|_{HS} = (1 - \frac{1}{d})^{\frac{1}{2}}$$

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duality $\implies$

inradius of $(\mathcal{CP}^b)^\circ = -\mathcal{CP}^b$ = inradius of $\mathcal{CP}^b = \frac{1}{(d-1)^{\frac{1}{2}}}$

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Barnum-Gurvitz

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- $\frac{1}{2} \leq \mathrm{vrad}\big(\mathcal{CP}_{\mathrm{N}}^{\mathrm{b}}\big) \leq 1, \quad \lim_{\mathrm{N}\to\infty} \mathrm{vrad}\big(\mathcal{CP}_{\mathrm{N}}^{\mathrm{b}}\big) = \mathrm{e}^{-1/4}$

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- $1 \leq \frac{\mathrm{vrad}\left(\mathcal{D}_{\mathrm{N}}^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_{\mathrm{N}}^{\mathrm{b}}\right)} \leq 8, \quad \limsup_N \frac{\mathrm{vrad}\left(\mathcal{D}_{\mathrm{N}}^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_{\mathrm{N}}^{\mathrm{b}}\right)} \leq 2e^{1/4}$

Proof

(iii) and (iv) Aubrun-Szarek

(i) lower bound: Szarek

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- Santalo and inverse Santalo(Bourgain-Milman): for any K in $\mathbb{R}^m$

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asymptotic estimate:

Zyczkowski-Sommers: $d = N^2$

$$|\mathcal{CP}_d \cap H^1| = \sqrt{d}\big(2\pi\big)^{\frac{d(d-1)}{2}} \frac{\Gamma(1)\Gamma(2)\ldots\Gamma(d)}{\Gamma(d^2)}$$

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$$6\ N^{\frac{1}{2}} \geq \frac{1}{\mathrm{vrad}\left(\mathcal{SP}_N^{\mathrm{b}}\right)} \geq \mathrm{vrad}\left(\mathcal{P}_N^{\mathrm{b}}\right) \geq \frac{c}{\mathrm{vrad}\left(\mathcal{SP}_N^{\mathrm{b}}\right)} \geq \frac{e}{4}\ N^{\frac{1}{2}}$$

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Aubrun-Szarek

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Proposition

Let K be a convex body in an m-dimensional Euclidean space with centroid at a, and let H be a k-dimensional affine subspace passing through a. Let $r = r_K$ and $R = R_K$ be the in-radius and out-radius of K. Then

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