On strict inclusions in hierarchies of convex bodies

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Intersection bodies

Lutwak:
D, L - origin-symmetric star bodies in $\mathbb{R}^n$.
D is the intersection body of L if for every $\xi \in S^{n-1}$,

$$\rho_D(\xi) = \mathrm{vol}_{n-1}(L \cap \xi^\perp).$$

The closure in the radial metric of the class of intersection bodies of star bodies gives the class of intersection bodies.

k-intersection bodies

Koldobsky:
D, L - origin-symmetric star bodies in $\mathbb{R}^n$.
D is the k-intersection body of L if for every $(n-k)$-dimensional subspace $H \subset \mathbb{R}^n$

$$\mathrm{Vol}_k(D \cap H^\perp) = \mathrm{Vol}_{n-k}(L \cap H).$$

The closure in the radial metric gives the class of k-intersection bodies, which will be denoted by $\mathcal{I}_k$.

k-intersection bodies

Another generalization: Zhang.

See also [Koldobsky], [E.Milman] for the relationship between these two generalizations and their connection to the lower dimensional Busemann-Petty problem.

Theorem [Koldobsky]

Let D be an origin-symmetric star body in $\mathbb{R}^n$, $1 \le k < n$. The following are equivalent:

(i) D is a k-intersection body;
(ii) $\|x\|_D^{-k}$ is a positive definite distribution;
(iii) $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}.

Embeddings in L_p

A well-known fact: for any $0 < p < q \leq 2$, the space L_q embeds isometrically in L_p.

Koldobsky extended this result to negative p:
Every n-dimensional subspace of L_q, $0 < q \leq 2$, embeds in L_p for every $-n < p < 0$.

Embeddings in L_p

However, it is an open problem, whether $X = (\mathbb{R}^n, \|\cdot\|_D)$ being embedded in L_{-p} for some $0 < p < n-3$ implies that X embeds in L_{-q} for all $p < q < n$.

In particular, is it true that every k-intersection body is also an m-intersection body for $1 < k < m < n-3$?

In some cases this statement is true. Since the product of positive definite distributions is also positive definite, one immediately obtains that if X embeds in L_{-p}, $0 < p < n$, and p divides q, $p < q < n$, then X also embeds in L_{-q} (see e.g. [E.Milman]).

Embeddings in L_p

Koldobsky: There is an n-dimensional Banach subspace ($n \geq 3$) of $L_{1/2}$ that does not embed in L_1, (also a subspace of $L_{1/4}$ but not $L_{1/2}$).

Borwein and the Center for Computational Mathematics at Simon Fraser University: showed (by computer methods) that there is a Banach space that embeds in $L_{a/64}$ but not in $L_{(a+1)/64}$ for $a = 1, 2$,..., 63.

Kalton, Koldobsky: there is a Banach space embedding in L_p, $0 < p < 1$, but not in L_q, $p < q \leq 1$.

Schlieper: there is a normed space that embeds in L_{-4} but not in L_{-2} (note that the converse always holds). Also in $L_{-1/3}$ but not in $L_{-1/6}$.

Theorem 1.
For every $1 \leq k < m < n - 3$ there is an origin-symmetric convex body $K \in \mathbb{R}^n$ such that $K \notin \mathcal{I}_k$, but $K \in \mathcal{I}_m$.

Weil constructed a convex body in $\mathbb{R}^n$ $(n \geq 3)$ that is not a zonoid but all its projections onto hyperplanes are zonoids.

Neyman showed that there are n-dimensional normed spaces that do not embed in L_p, but all their $(n-1)$-dimensional subspaces embed in L_p for $p > 0$.

Yaskina constructed a convex body in $\mathbb{R}^n$ $(n \geq 5)$, which is not an intersection body, but all of its central hyperplane sections are intersection bodies.

Note that all central sections of an intersection body are intersection bodies (Fallert, Goodey and Weil).

Let $\mathcal{I}_k^m$ be the class of convex bodies all of whose m-dimensional central sections are k-intersection bodies.

Theorem 2.

Let $k+3 \le m < n$. There is an origin-symmetric convex body $K \subset \mathbb{R}^n$ such that $K \in \mathcal{I}_k^m$, but $K \notin \mathcal{I}_k^{m+1}$.

Remark. Note that $\mathcal{I}_k^{m+1} \subset \mathcal{I}_k^m$, (see e.g. [E.Milman]).

Proof of Theorem 2

For a small $\epsilon > 0$ define a body K by

$$\|x\|_K^{-k} = |x|_2^{-k} - 2\epsilon^{m-k}\|x\|_E^{-k}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $|x|_2$ is the Euclidean norm and E is the ellipsoid:

$$\|x\|_E = \left(x_1^2 + \cdots + x_m^2 + \frac{x_{m+1}^2 + \cdots + x_n^2}{\epsilon^2} \right)^{1/2}.$$

One can check that the body K is well defined and convex.

Proof of Theorem 2

Lemma 1.
For every m-dimensional subspace H of $\mathbb{R}^n$, the body $K \cap H$ is a k-intersection body.

Proof.
We have

$$\|x\|_{K\cap H}^{-k} = |x|_{B_2\cap H}^{-k} - 2\epsilon^{m-k}\|x\|_{E\cap H}^{-k}.$$

Since E is an ellipsoid with semiaxes ϵ and 1, $E \cap H$ is also an ellipsoid with semiaxes a_1, ..., a_m such that $\epsilon \leq a_i \leq 1$, $\forall i = 1, ..., m$.

Proof of Theorem 2

There is a coordinate system in H such that

$$\|y\|_{K\cap H}^{-k} = \left(y_1^2 + \cdots + y_m^2\right)^{-k/2} - 2\epsilon^{m-k}\left(\frac{y_1^2}{a_1^2} + \cdots + \frac{y_m^2}{a_m^2}\right)^{-k/2}.$$

Taking the Fourier transform of $\|y\|_{K\cap H}^{-k}$ in the plane H we get

$$(\|y\|_{K\cap H}^{-k})^\wedge(\xi)$$

$$= C_{m,k}\left(|\xi|_2^{-m+k} - 2\epsilon^{m-k}\prod_{i=1}^{m} a_i \cdot \left(a_1^2\xi_1^2 + \cdots + a_m^2\xi_m^2\right)^{(-m+k)/2}\right)$$

Proof of Theorem 2

Let a_j be the smallest semiaxis. Then for some $\lambda \geq 1$ we have $a_j = \lambda\epsilon$. Therefore

$$\prod_{i=1}^{m} a_i \leq \lambda\epsilon.$$

On the other hand if $\xi \in S^{m-1} \subset H$, then

$$\left(a_1^2\xi_1^2 + \cdots + a_m^2\xi_m^2\right)^{(-m+k)/2} \leq a_j^{-m+k} = (\lambda\epsilon)^{-m+k}.$$

Proof of Theorem 2

Therefore,

$$2\epsilon^{m-k} \prod_{i=1}^{m} a_i \cdot \left(a_1^2\xi_1^2 + \cdots + a_m^2\xi_m^2\right)^{(-m+k)/2} \leq 2\epsilon^{m-k}\lambda\epsilon(\lambda\epsilon)^{-m+k}$$

$$\leq 2\epsilon$$

So, if $\epsilon \leq 2^{-1}$, then $(\|y\|_{K\cap H}^{-k})^\wedge(\xi) \geq 0$ for all $\xi \in S^{n-1} \cap H$ and all H.

Therefore, $K \in \mathcal{I}_k^m$ (i.e. all m-dimensional sections of K are k-intersection bodies).

Proof of Theorem 2

Lemma.
There exists an $(m+1)$-dimensional section of K which is not a k-intersection body.

Proof.
Let $H = \{x \in \mathbb{R}^n : x_{m+2} = \cdots = x_n = 0\}$. Then

$$\|x\|_{K\cap H}^{-k} =$$

$$= \left(x_1^2 + \cdots + x_{m+1}^2\right)^{-k/2} - 2\epsilon^{m-k}\left(x_1^2 + \cdots + x_m^2 + \frac{x_{m+1}^2}{\epsilon^2}\right)^{-k/2}$$

Proof of Theorem 2

The Fourier transform in the variables x_1, ..., x_{m+1} equals

$$\begin{aligned}\left(\|x\|_{K\cap H}^{-k}\right)^{\wedge}(\xi) &= C_{m+1,k}\Big(\left(\xi_1^2+\cdots+\xi_{m+1}^2\right)^{(-m+k-1)/2} - \\ &- 2\epsilon^{m-k}\epsilon\left(\xi_1^2+\cdots+\xi_m^2+\epsilon^2\xi_{m+1}^2\right)^{(-m+k-1)/2}\Big)\end{aligned}$$

If $\xi = (0, ..., 0, 1) \in S^m \subset H$, then

$$\left(\|x\|_{K\cap H}^{-k}\right)^{\wedge}(\xi) = C_{m+1,k}\left(1 - 2\epsilon^{m-k}\epsilon\epsilon^{-m+k-1}\right) = -C_{m+1,k} < 0$$

Q.E.D.

Proof of Theorem 1

Theorem 1.
For every $1 \le k < m < n-3$ there is a symmetric convex body $K \in \mathbb{R}^n$ that does not belong to $\mathcal{I}_k$, but belongs to $\mathcal{I}_m$.
Proof.
For a small $\epsilon > 0$ define a body K by

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^{n-k-3/2}\|x\|_E^{-1}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where E is the ellipsoid with the norm

$$\|x\|_E^{-1} = \left(x_1^2 + \cdots + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2}\right)^{-1/2}.$$

One can show that K is convex.

Proof of Theorem 1

Consider the $-m$th power of the norm.

$$\|x\|_K^{-m} = |x|_2^{-m} - \epsilon^{n-k-3/2} m |x|_2^{-m+1} \|x\|_E^{-1} + \cdots$$

Applying the Fourier transform we get

$$(\|x\|_K^{-m})^\wedge(\xi) = C_{n,m} - \epsilon^{n-k-3/2} m (|x|_2^{-m+1} \|x\|_E^{-1})^\wedge(\xi) + \cdots$$

Proof of Theorem 1

Lemma.
Let p, $q > 0$ be integers, $p + q \leq n - 1$.

1. If $n - p - q - 1$ is even, then for all $\xi \in S^{n-1}$

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) \leq C\epsilon^{-n+p+q+1}.$$

   If $n - p - q - 1$ is odd, then for every $\alpha > 0$ there exists C_α such that for all $\xi \in S^{n-1}$,

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) \leq C_\alpha \epsilon^{-n+p+q+1/(1+\alpha)}.$$

2. Moreover, in both cases

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(e_n) \sim C\epsilon^{-n+p+q+1}$$

Proof of Theorem 1

We have

$$(\|x\|_K^{-m})^\wedge(\xi) = C_{n,m} - \epsilon^{n-k-3/2} m(|x|_2^{-m+1} \|x\|_E^{-1})^\wedge(\xi) + \cdots$$

By Lemma the order of the second term is at most

$$\epsilon^{n-k-3/2}\epsilon^{-n+m+1/(1+\alpha)} = \epsilon^{m-k-1/2-\alpha/(1+\alpha)} \longrightarrow 0, \text{ as } \epsilon \longrightarrow 0$$

The terms in $\cdots$ are even better:

$$\epsilon^{i(n-k-3/2)}\epsilon^{-n+m+1/(1+\alpha)}, \quad i \geq 2.$$

Therefore if ϵ is small, then $(\|x\|_K^{-m})^\wedge(\xi) \geq 0$, so $K \in \mathcal{I}_m$.

Proof of Theorem 1

Now consider

$$\|x\|_K^{-k} = |x|_2^{-k} - \epsilon^{n-k-3/2} k |x|_2^{-k+1} \|x\|_E^{-1} + \cdots$$

Computing the Fourier transform in the direction of $\xi = e_n$, we have

$$(\|x\|_K^{-k})^\wedge(e_n) = C_{n,k} - \epsilon^{n-k-3/2} k (|x|_2^{-k+1} \|x\|_E^{-1})^\wedge(e_n) + \cdots$$

The terms in $\cdots$ are small since they have order at most

$$\epsilon^{i(n-k-3/2)} \epsilon^{-n+k+1}, \quad i \geq 2.$$

Proof of Theorem 1

We will pay attention only to the second term. By Lemma

$$\epsilon^{n-k-3/2}(|x|_2^{-k+1}\|x\|_E^{-1})^\wedge(e_n) \sim C\epsilon^{n-k-3/2}\epsilon^{-n+k+1} = C\epsilon^{-1/2}$$

If we choose $\epsilon > 0$ small enough so that the latter is greater than $C_{n,k}$, then $(\|x\|_K^{-k})^\wedge(e_n) < 0$.

So $K \notin \mathcal{I}_k$.