Μάθημα 24

Ασκήσεις

A1(iii) $I=\displaystyle\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx$. Πρώτα παραγοντοποιούμε τον παρονομαστή. Το -1 είναι ρίζα

$$
\begin{array}{r|l}
x^3+2x^2+2x+1 & x+1 \\ \cline{2-2}
-x^3-x^2 \qquad\quad & x^2+x+1 \\ \cline{1-1}
x^2+2x+1 & \\
-x^2-x \quad\; & \\ \cline{1-1}
x+1 & \\
-x-1 & \\ \cline{1-1}
0 &
\end{array}
$$

Άρα $I=\displaystyle\int \frac{3x^2+3x+1}{(x+1)(x^2+x+1)}\,dx$. Το x^2+x+1 δεν έχει ρίζες

Αναλύουμε σε απλά κλάσματα $\dfrac{3x^2+3x+1}{(x+1)(x^2+x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+\Gamma}{x^2+x+1}$

$$\Leftrightarrow\ 3x^2+3x+1 = A(x^2+x+1)+(Bx+\Gamma)(x+1)$$

$$= Ax^2+Ax+A+Bx^2+Bx+\Gamma x+\Gamma$$

$$= (A+B)x^2+(A+B+\Gamma)x+(A+\Gamma)$$

Συνεπώς

$$\left\{\begin{array}{l} A+B=3 \\ A+\Gamma=3 \\ A+B+\Gamma=1 \end{array}\right. \Rightarrow \left.\begin{array}{l} B=\Gamma \\ \\ \end{array}\right\} \quad \left.\begin{array}{l} A+B=3 \\ A+2B=1 \end{array}\right\} \xrightarrow{(-)} \boxed{B=-2} \quad \boxed{\Gamma=-2}$$

$$A=3-B \Rightarrow \boxed{A=5}$$

Άρα

$$I=5\ln|x+1|-\int\frac{2x+2}{x^2+x+1}=5\ln|x+1|-\int\frac{2x+1+1}{x^2+x+1}\,dx$$

$$=5\ln|x+1|-\ln|x^2+x+1|-\int\frac{1}{x^2+x+1}\,dx$$

$$\int\frac{1}{x^2+x+1}\,dx=\int\frac{1}{x^2+2\frac{1}{2}x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dx=\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{1}{\frac{3}{4}\left(\left(\frac{x+1/2}{\sqrt{3/4}}\right)^2+1\right)}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{x+1/2}{\sqrt{3/4}}\right)^2+1}\,dx \quad (1)$$

$$u = \frac{x+1/2}{\sqrt{3/4}} \Rightarrow \qquad \frac{4}{3}\int \frac{1}{u^2+1}\sqrt{\frac{3}{4}}\,du = \quad (2)$$

$$\boxed{\Rightarrow \sqrt{\tfrac{3}{4}}\,u = x+\tfrac{1}{2} \Rightarrow \Rightarrow dx = \sqrt{\tfrac{3}{4}}\,du}$$

$$= \sqrt{\frac{4}{3}}\arctan(u) = \sqrt{\frac{4}{3}}\arctan\left(\frac{x+1/2}{\sqrt{3/4}}\right) \quad (3)$$

Άρα
$$I = \ln\frac{|x+1|^5}{|x^2+x+1|} - \sqrt{\frac{4}{3}}\arctan\left(\frac{x+1/2}{\sqrt{3/4}}\right) + c \quad (4)$$

A 2 ii

$$I = \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \xrightarrow[dx = 6t^5\,dt]{x = t^6} \int \frac{6t^5}{t^3+t^2}\,dt \quad (5)$$

$$= \int \frac{6t^5}{t^2(t+1)}\,dt = \int \frac{6t^3}{t+1}\,dt \qquad \deg(6t^3)=3 > \det(t+1) \quad (6)$$

$$\begin{array}{l|l} t^3 & t+1 \\ \hline -t^3-t^2 & t^2-t+1 \\ \hline 0 \; -t^2 & \\ +t^2+t & \\ \hline 0 \; +t & \\ -t-1 & \\ \hline 0 \; -1 & \end{array}$$

$$= 6\int\left(t^2-t+1-\frac{1}{t+1}\right)dt \quad (7)$$

$$= 6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right)+c \quad (8)$$

$$= 2t^3-3t^2+6t-6\ln|t+1|+c \quad (9)$$

$$= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left|\sqrt[6]{x}+1\right|+c \quad (10)$$

A 2 iii

$$\int \frac{1}{\sqrt{1+e^x}}\,dx \xrightarrow{t=e^x} \quad (11)$$

(+ παραγοντική ολοκλήρωση)

A 3(v) $\int \sqrt{\tan x}\,dx \xrightarrow[u^2=\tan x]{u=\sqrt{\tan x}} \int u \frac{2u}{u^4+1}\,du =$ (1)

$u^2 = \tan x$ (2)

$\arctan(u^2) = x$ (3)

$dx = \frac{2u}{u^4+1}\,du$ (4)

$$= \int \frac{2u^2}{u^4+1}\,du \overset{!!!}{=} \int \frac{u^2+1}{u^4+1}\,du + \int \frac{u^2-1}{u^4+1}\,du \quad (5)$$

$$\overset{!!!}{=} \int \frac{1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}}\,du + \int \frac{1-\frac{1}{u^2}}{u^2+\frac{1}{u^2}}\,du = \quad (6)$$

$$\overset{!!!}{=} \int \frac{\left(u-\frac{1}{u}\right)'}{\left(u-\frac{1}{u}\right)^2+2}\,du + \int \frac{\left(u+\frac{1}{u}\right)'}{\left(u+\frac{1}{u}\right)^2-2}\,du \quad (7)$$

$t = u - \frac{1}{u}$, $dt = \left(u-\frac{1}{u}\right)'du$ (8)

$t = u + \frac{1}{u}$, $dt = \left(u+\frac{1}{u}\right)'du$ (9)

$$\int \frac{1}{t^2+2}\,dt + \int \frac{1}{t^2-2}\,dt \quad (10)$$

$$\xrightarrow[dt=\sqrt{2}dv]{t=\sqrt{2}v} \int \frac{1}{2(v^2+1)}\sqrt{2}\,dv + \frac{1}{2\sqrt{2}}\int \left(\frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\right)dt \quad (11)$$

$$= \frac{1}{\sqrt{2}}\arctan(v) + \frac{1}{2\sqrt{2}}\left(\ln|t-\sqrt{2}| - \ln|t+\sqrt{2}|\right) + c \quad (12)$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}^{-1}t\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c = \quad (13)$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}^{-1}\left(u-\frac{1}{u}\right)\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}}\right| + c \quad (14)$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}^{-1}\left(\sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}\right)\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right| + c \quad (15)$$

A5(vi) $I = \int x \sin^2 x\, dx = \int \left(\frac{x^2}{2}\right)' \sin^2 x\, dx =$ (1)

$= \frac{x^2 \sin^2 x}{2} - \int \frac{x^2}{2}\, 2 \sin x \cos x\, dx =$ (2)

$= \frac{1}{2} x^2 \sin^2 x - \frac{1}{2} \int x^2 \sin 2x\, dx$ (3)

$\int x^2 \sin 2x\, dx = \int x^2 \left(\frac{-\cos 2x}{2}\right)' dx = -\frac{x^2 \cos 2x}{2} + \int 2x \frac{\cos 2x}{2} dx$ (4)

$= -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x\, dx =$ (5)

$= -\frac{1}{2} x^2 \cos 2x + \int x \left(\frac{\sin 2x}{2}\right)' dx =$ (6)

$= -\frac{1}{2} x^2 \cos 2x + x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx =$ (7)

$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x$. Άρα (8)

$I = \frac{1}{2} x^2 \sin^2 x - \frac{1}{2} \left(-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x\right) + c$ (9)

Αλλιώς $I = \int x \sin^2 x\, dx = \int x \frac{1 - \cos 2x}{2} dx =$ (10)

$= \frac{1}{2} \int x - \frac{1}{2} \int x \cos 2x\, dx = \frac{1}{2} \frac{x^2}{2} - \frac{1}{2} \int x \left(\frac{\sin 2x}{2}\right)' dx$ (11)

$= \frac{1}{4} x^2 - \frac{1}{2} \left(x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx\right) =$ (12)

$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x + \frac{1}{4} \cos 2x + c.$ (13)

(σελ 5

A 5 (vii) $\int \log(x+\sqrt{x})\,dx \quad \dfrac{x=t^2}{dx=2t\,dt}$ (1)

$= \int \log(t^2+t)\;2t\,dt = \int \log(t^2+t)\cdot (t^2)'\,dt$ (2)

$= t^2\log(t^2+t) - \int \dfrac{1}{t^2+t}(2t+1)\,t^2\,dt$ (3)

$= t^2\log(t^2+t) - \int \dfrac{(2t+1)t}{t+1}\,dt =$ (4)

$= t^2\log(t^2+t) - \int \left(2t-1+\dfrac{1}{t+1}\right)dt$ (5)

(διότι $\dfrac{(2t+1)t}{t+1} = \dfrac{2(t+1)-1}{t+1}\,t = \left(2-\dfrac{1}{t+1}\right)t =$ (6)

$= 2t - \dfrac{t}{t+1} = 2t - \dfrac{t+1-1}{t+1} = 2t-1+\dfrac{1}{t+1}$) (7)

$= t^2\log(t^2+t) - t^2 + t - \log|t+1| + c$ (8)

$= x\log(x+\sqrt{x}) - x + \sqrt{x} - \log|\sqrt{x}+1| + c.$ (9)