Μάθημα 24

Ασκήσεις

A1(iii) $I=\int \dfrac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx$. Πρώτα παραγοντοποιούμε τον παρονομαστή. Το -1 είναι ρίζα

$$x^3+2x^2+2x+1 \;\Big|\; \underline{x+1}$$

Άρα $I=\displaystyle\int \frac{3x^2+3x+1}{(x+1)(x^2+x+1)}\,dx$ Το x^2+x+1 δεν έχει ρίζες

Ανάλυση σε απλά κλάσματα $\dfrac{3x^2+3x+1}{(x+1)(x^2+x+1)} = \dfrac{A}{x+1}+\dfrac{Bx+\Gamma}{x^2+x+1}$

$$\Leftrightarrow \quad 3x^2+3x+1 = A(x^2+x+1)+(Bx+\Gamma)(x+1)$$

$$= Ax^2+Ax+A+Bx^2+B+\Gamma x+\Gamma$$

$$=(A+B)x^2+(A+\Gamma)x+(A+B+\Gamma)$$

Συνεπώς $\left\{\begin{array}{l} \left.\begin{array}{l} A+B=3 \\ A+\Gamma=3 \end{array}\right\} \Rightarrow B=\Gamma \\ A+B+\Gamma=1 \end{array}\right. \quad \left.\begin{array}{l} A+B=3 \\ A+2B=1 \end{array}\right\} \xrightarrow{(-)} \boxed{B=-2} \quad \boxed{\Gamma=-2}$

$$A=3-B \Rightarrow \boxed{A=5}$$

Άρα $I = 5\ln|x+1| - \displaystyle\int \frac{2x+2}{x^2+x+1} = 5\ln|x+1| - \int \frac{2x+1+1}{x^2+x+1}\,dx$

$$= 5\ln|x+1| - \ln|x^2+x+1| - \int \frac{1}{x^2+x+1}\,dx$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+2\frac{1}{2}x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{1}{\frac{3}{4}\left(\left(\frac{x+1/2}{\sqrt{3/4}}\right)^2+1\right)}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{x+1/2}{\sqrt{3/4}}\right)^2+1}\,dx$$ (1)

$u = \frac{x+1/2}{\sqrt{3/4}} \Rightarrow$

$$\frac{4}{3}\int \frac{1}{u^2+1}\sqrt{\frac{3}{4}}\,du =$$ (2)

$\Rightarrow \sqrt{\frac{3}{4}}\,u = x+\frac{1}{2} \Rightarrow$

$\Rightarrow dx = \sqrt{\frac{3}{4}}\,du$

$$= \sqrt{\frac{4}{3}}\arctan(u) = \sqrt{\frac{4}{3}}\arctan\left(\frac{x+1/2}{\sqrt{3/4}}\right)$$ (3)

Άρα $$I = \ln\frac{|x+1|^5}{|x^2+x+1|} - \sqrt{\frac{4}{3}}\arctan\left(\frac{x+1/2}{\sqrt{3/4}}\right) + c$$ (4)

A 2 ii

$$I = \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \;\xrightarrow[dx = \qquad dt]{x =}\; \int \frac{6t^5}{t^3+t^2}\,dt$$ (5)

$$= \int \frac{6t^5}{t^2(t+1)}\,dt = \int \frac{6t^3}{t+1}\,dt$$ (6)

$\deg(6t^3) = 3 > \det(t+1)$

$$\begin{array}{rr|l} t^3 & & t+1 \\ \cline{3-3} -t^3 & -t^2 & t^2-t-1 \\ \hline 0 & -t^2 & \\ & +t^2-t & \\ \hline & 0\;-t & \\ & +t\;+1 & \\ \hline & 0\;+1 & \end{array}$$

$$= 6\int\left(t^2-t-1+\frac{1}{t+1}\right)dt$$ (7)

$$= 6\left(\frac{t^3}{3}-\frac{t^2}{2}-t+\ln|t+1|\right)+c$$ (8)

$$= 2t^3 - 3t^2 - 6t + 6\ln|t+1| + c$$ (9)

$$= 2\sqrt{x} - 3\sqrt[3]{x} - 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x}+1| + c$$ (10)

A 2 iii

$$\int \frac{1}{\sqrt{1+e^x}}\,dx \;\xrightarrow{t = e^x}$$ (11)

(+ παραγοντική ολοκλήρωση)

$$\text{A 3(v)} \quad \int \sqrt{\tan x}\, dx \;\xrightarrow[u^2=\tan x]{u=\sqrt{\tan x}}\; \int u \,\frac{2u}{u^4+1}\, du = \qquad (1)$$

$$u^2 = \tan x \qquad (2)$$

$$\arctan(u^2) = x \qquad (3)$$

$$dx = \frac{2u}{u^4+1}\, du \qquad (4)$$

$$= \int \frac{2u^2}{u^4+1}\, du \overset{!!!}{=} \int \frac{u^2+1}{u^4+1}\, du + \int \frac{u^2-1}{u^4+1}\, du \qquad (5)$$

$$\overset{!!!}{=} \int \frac{1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}}\, du + \int \frac{1-\frac{1}{u^2}}{u^2+\frac{1}{u^2}}\, du = \qquad (6)$$

$$\overset{!!!}{=} \int \frac{\left(u-\frac{1}{u}\right)'}{\left(u-\frac{1}{u}\right)^2+2}\, du + \int \frac{\left(u+\frac{1}{u}\right)'}{\left(u+\frac{1}{u}\right)^2-2}\, du \qquad (7)$$

$$t = u - \frac{1}{u} \qquad\qquad t = u + \frac{1}{u} \qquad (8)$$

$$dt = \left(u-\frac{1}{u}\right)' du \qquad\qquad dt = \left(u+\frac{1}{u}\right)' du \qquad (9)$$

$$\int \frac{1}{t^2+2}\, dt + \int \frac{1}{t^2-2}\, dt \qquad (10)$$

$$\xrightarrow[dt=\sqrt{2}\,dv]{t=\sqrt{2}v} \int \frac{1}{2(v^2+1)} \sqrt{2}\, dv + \frac{1}{2\sqrt{2}} \int \left(\frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\right) dt \qquad (11)$$

$$= \frac{1}{\sqrt{2}} \arctan(v) + \frac{1}{2\sqrt{2}} \left(\ln|t-\sqrt{2}| - \ln|t+\sqrt{2}|\right) + c \qquad (12)$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}^{-1} t\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c = \qquad (13)$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}^{-1}\left(u-\frac{1}{u}\right)\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{u-\frac{1}{u}-\sqrt{2}}{u-\frac{1}{u}+\sqrt{2}}\right| + c \qquad (14)$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}^{-1}\left(\sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}\right)\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{\tan x} - \frac{1}{\sqrt{\tan x}} - \sqrt{2}}{\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}} + \sqrt{2}}\right| + c \qquad (15)$$

σελ 4

A 5(vi) $I = \int x \sin^2 x \, dx = \int \left(\frac{x^2}{2}\right)' \sin^2 x \, dx =$ (1)

$= \frac{x^2 \sin^2 x}{2} - \int \frac{x^2}{2} \, 2 \sin x \cos x \, dx =$ (2)

$= \frac{1}{2} x^2 \sin^2 x - \frac{1}{2} \int x^2 \sin 2x \, dx$ (3)

$\int x^2 \sin 2x \, dx = \int x^2 \left(\frac{-\cos 2x}{2}\right)' dx = -\frac{x^2 \cos 2x}{2} + \int 2x \frac{\cos 2x}{2} dx$ (4)

$= -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x \, dx =$ (5)

$= -\frac{1}{2} x^2 \cos 2x + \int x \left(\frac{\sin 2x}{2}\right)' dx =$ (6)

$= -\frac{1}{2} x^2 \cos 2x + x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx =$ (7)

$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x$. Άρα (8)

$I = \frac{1}{2} x^2 \sin^2 x - \frac{1}{2} \left(-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x\right) + c$ (9)

Αλλιώς $I = \int x \sin^2 x \, dx = \int x \, \frac{1 - \cos 2x}{2} \, dx =$ (10)

$= \frac{1}{2} \int x - \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{2} \frac{x^2}{2} - \frac{1}{2} \int x \left(\frac{\sin 2x}{2}\right)' dx$ (11)

$= \frac{1}{4} x^2 - \frac{1}{2} \left(x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right) =$ (12)

$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x + \frac{1}{4} \cos 2x + c$. (13)

A 5 (vii) $\displaystyle\int \log(x+\sqrt{x})\,dx \quad \frac{x=t^2}{dx=2t\,dt}$ (1)

$= \displaystyle\int \log(t^2+t)\;2t\,dt = \int \log(t^2+t)\cdot(t^2)'\,dt$ (2)

$= t^2\log(t^2+t) - \displaystyle\int \frac{1}{t^2+t}(2t+1)\,t^2\,dt$ (3)

$= t^2\log(t^2+t) - \displaystyle\int \frac{(2t+1)t}{t+1}\,dt =$ (4)

$= t^2\log(t^2+t) - \displaystyle\int \left(2t-1+\frac{1}{t+1}\right)dt$ (5)

(διότι $\dfrac{(2t+1)t}{t+1} = \dfrac{2(t+1)-1}{t+1}\,t = \left(2-\dfrac{1}{t+1}\right)t =$ (6)

$= 2t - \dfrac{t}{t+1} = 2t - \dfrac{t+1-1}{t+1} = 2t-1+\dfrac{1}{t+1}$) (7)

$= t^2\log(t^2+t) - t^2 + t - \log|t+1| + c$ (8)

$= x\log(x+\sqrt{x}) - x + \sqrt{x} - \log|\sqrt{x}+1| + c.$ (9)