ELEMENTARY CHAINS AND $C^{(n)}$-CARDINALS

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ABSTRACT. The $C^{(n)}$-cardinals were introduced recently by Bagaria and are strong forms of the usual large cardinals. For a wide range of large cardinal notions, Bagaria has shown that the consistency of the corresponding $C^{(n)}$-versions follows from the existence of rank-into-rank elementary embeddings. In this article, we further study the $C^{(n)}$-hierarchies of tall, strong, superstrong, supercompact, and extendible cardinals, giving some improved consistency bounds while, at the same time, addressing questions which had been left open. In addition, we consider two cases which were not dealt with by Bagaria; namely, $C^{(n)}$-Woodin and $C^{(n)}$-strongly compact cardinals, for which we provide characterizations in terms of their ordinary counterparts. Finally, we give a brief account on the interaction of $C^{(n)}$-cardinals with the forcing machinery.

1. INTRODUCTION

The concept of *reflection*, which pervades the body of set theory, has a long history with its origins tracing back to Gödel and, indeed, Cantor himself. Perhaps its most renowned formal manifestation is the well-known *Reflection Principle*, which is due to Lévy and Montague from the early 1960's.

For any fixed natural number n, we let $C^{(n)}$ denote the collection of ordinals which are Σ_n-correct in the universe, that is, $C^{(n)} = \{\alpha : V_\alpha \prec_{\Sigma_n} V\}$. In other words, $\alpha \in C^{(n)}$ if and only if V_α "reflects" all true Σ_n-statements, with parameters in V_α. Already by the work of Lévy and Montague it follows that, for any natural number n, $C^{(n)}$ is a closed and unbounded proper class. For example, $C^{(0)} = \mathbf{ON}$ while $C^{(1)}$ is precisely the class of those uncountable cardinals α for which $H_\alpha = V_\alpha$; notably, such a local characterization is not available for the classes $C^{(n)}$, when $n > 1$.

Blending this aspect of reflection with the usual large cardinal notions, Bagaria recently introduced the various hierarchies of $C^{(n)}$-(large) cardinals (cf. [1]), and underlined their significance by revealing intimate connections between such principles and the concept of *structural reflection* for the set-theoretic universe. For instance, Bagaria showed that the existence of $C^{(n)}$-*extendible* cardinals is equivalent to fragments of *Vopěnka's Principle*, in

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a way which generalizes the ordinary notions of supercompactness and extendibility.

To define the $C^{(n)}$-cardinals, we work in the usual framework of elementary embeddings j for the ordinary large cardinal postulates (i.e., where the critical point $\text{cp}(j) = \kappa$ is the cardinal in question) and we require, in addition to the standard definition for the postulate at hand and for any given n, that the image $j(\kappa)$ belongs to the class $C^{(n)}$, as the latter is computed in V. This modification gives rise to the corresponding $C^{(n)}$-(large) cardinal. For concreteness, let us give the definition of $C^{(n)}$-extendibility.

Definition 1.1 ([1]). A cardinal κ is called λ-$C^{(n)}$-**extendible**, for some $\lambda > \kappa$, if there exists a $\theta > \lambda$ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ with $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$; we say that κ is $C^{(n)}$-**extendible** if it is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$.

Bagaria studied the $C^{(n)}$-versions of large cardinals such as measurable, strong, superstrong, supercompact, extendible, huge, superhuge (and more). It is shown in [1] that the consistency of all the aforementioned $C^{(n)}$-cardinals follows from the existence of an I3 embedding (see §24 in [7]).

The structure of this article is as follows. In the rest of this section, we give some necessary preliminaries and we also introduce the notions of $C^{(n)}$-tall, $C^{(n)}$-Woodin, and $C^{(n)}$-strongly compact cardinals.

In Section 2, we use the versatile tool of elementary chains in order to derive consistency upper bounds for the $C^{(n)}$-versions of tall, strong, superstrong, supercompact and extendible cardinals. We also study the connection between $C^{(n)}$-extendibility and $C^{(n)}$-supercompactness.

In Section 3, we consider the cases of $C^{(n)}$-Woodin and of $C^{(n)}$-strongly compact cardinals, and we obtain equivalent formulations for them in terms of their ordinary counterparts.

Finally, in Section 4, we briefly study the interaction of $C^{(n)}$-cardinals with the forcing machinery.

1.1. **Preliminaries.** Our notation and terminology are mostly standard[1]. ZFC denotes the familiar first-order axiomatization of Zermelo-Fraenkel set theory, together with the Axiom of Choice. Given any function f and any $S \subseteq dom(f)$, we write $f \restriction S$ for the restriction of the function to S and also, we write $f``S$ for the pointwise image, that is, the collection $\{f(x) : x \in S\}$. For any X and Y, ${}^X Y$ is the collection of all functions f with $dom(f) = X$ and $range(f) \subseteq Y$. The class of ordinal numbers will be denoted by **ON**; given an infinite ordinal α, $\text{cof}(\alpha)$ stands for its cofinality.

For any $A \subseteq$ **ON**, $\sup(A)$ denotes the supremum of A (in case A is a set), while $\text{Lim}(A)$ denotes the collection of its limit points, that is, $\{\xi : \sup(A \cap \xi) = \xi\}$. Given a limit ordinal α with $\text{cof}(\alpha) > \omega$ and some $C \subseteq \alpha$,

[1] See [7] for an account on all undefined set-theoretic notions.

C is called *club* in α if $\sup(C) = \alpha$ and $\alpha \cap \mathrm{Lim}(C) \subseteq C$; moreover, C is called β-*club* in α, for some regular $\beta < \mathrm{cof}(\alpha)$, if $\sup(C) = \alpha$ and $\{\xi \in \alpha \cap \mathrm{Lim}(C) : \mathrm{cof}(\xi) = \beta\} \subseteq C$. Likewise, if $I \subseteq \mathrm{cof}(\alpha)$ is an ordinal interval, then C is called I-*club* in α if it is β-club, for all regular $\beta \in I$. Finally, $S \subseteq \alpha$ is called *stationary* in α if $S \cap C \neq \varnothing$ for every club $C \subseteq \alpha$.

Partial orders will be denoted by blackboard capital letters such as $\mathbb{P}$ and $\mathbb{Q}$. We write $p < q$ to mean than p is stronger than q. We use $V^{\mathbb{P}}$ to stand for the universe of $\mathbb{P}$-names. Moreover, if G is $\mathbb{P}$-generic over V and $\alpha \in \mathbf{ON}$, we let $V[G]_\alpha = (V_\alpha)^{V[G]}$ and $V_\alpha[G] = \{\tau_G : \tau \in V^{\mathbb{P}} \cap V_\alpha\}$.

Given a non-trivial elementary embedding $j : V \longrightarrow M$, where M is a transitive class model of ZFC, we denote by $\mathrm{cp}(j)$ its critical point, that is, the least ordinal moved by j. Most of the large cardinal notions that we study are standard; see [7] for more details. We now give a less popular definition, which is still interesting in its own right.

Definition 1.2 ([6]). A cardinal κ is called λ-**tall**, for some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $\mathrm{cp}(j) = \kappa$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) > \lambda$; we say that κ is **tall** if it is λ-tall for every $\lambda > \kappa$.

Tall cardinals were introduced by Hamkins. In [6], it is shown that tallness embeddings can be described by extenders, and that the existence of a tall cardinal is equiconsistent with the existence of a strong cardinal.

1.2. $C^{(n)}$**-cardinals.** Recall that, for every $n \geqslant 1$, membership in $C^{(n)}$ is expressible by a Π_n (but *not* by any Σ_n) formula. In particular, $C^{(n+1)} \subset C^{(n)}$, i.e., the inclusion is proper.

All the background definitions on $C^{(n)}$-cardinals may be found in [1]. A special case is that of $C^{(n)+}$-extendibility, defined as follows.

Definition 1.3 ([1]). A cardinal κ is called λ-$C^{(n)+}$-**extendible**, for some $\lambda > \kappa$ with $\lambda \in C^{(n)}$, if there exists a $\theta \in C^{(n)}$ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$, with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$; we say that κ is $C^{(n)+}$-**extendible** if it is λ-$C^{(n)+}$-extendible for all $\lambda > \kappa$ with $\lambda \in C^{(n)}$.

For the $C^{(n)}$-cardinals which will be of interest to us, it is shown in [1] that, for every $n \geqslant 1$, the statements "κ is λ-$C^{(n)}$-strong", "κ is $C^{(n)}$-superstrong", "κ is λ-$C^{(n)}$-supercompact", "κ is λ-$C^{(n)}$-extendible" and "κ is λ-$C^{(n)+}$-extendible" are all Σ_{n+1}-expressible. Consequently, the corresponding global versions are Π_{n+2}-expressible.[2]

Let us now introduce the $C^{(n)}$-versions of tall, Woodin, and strongly compact cardinals.

Definition 1.4. A cardinal κ is called λ-$C^{(n)}$-**tall**, for some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $\mathrm{cp}(j) = \kappa$,

[2] It should be pointed out that, e.g., in the case of λ-$C^{(n)}$-supercompactness, the expressibility can be attained using the machinery of *Martin-Steel extenders*. For more details on such extenders see [8]; for their use in the context of ($C^{(n)}$-) supercompactness, the reader may consult § 5 in [1] or the Appendix of [10].

$j(\kappa) > \lambda$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) \in C^{(n)}$; we say that κ is $C^{(n)}$-**tall** if it is λ-$C^{(n)}$-tall for all $\lambda > \kappa$.

Either using ordinary extenders or, Martin–Steel ones, for every $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-tall" is Σ_{n+1}-expressible; thus, "κ is $C^{(n)}$-tall" is Π_{n+2}-expressible.

Definition 1.5. A cardinal δ is called $C^{(n)}$-**Woodin** if for every $f \in {}^{\delta}\delta$ there exists a $\kappa < \delta$ with $f``\kappa \subseteq \kappa$, and there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$, $j(\delta) = \delta$ and $j(\kappa) \in C^{(n)}$.

Observe that the above definition is in accordance with the local character of Woodin cardinals, i.e., we demand that $j(\delta) = \delta$ so that the various embeddings may be witnessed by extenders in V_δ. It is easy to see that, for fixed n, the statement "δ is $C^{(n)}$-Woodin" is absolute for any $V_{\delta'}$ with $\delta' > \delta$ and $\delta' \in C^{(n)}$. Notice also that if the cardinal δ is $C^{(n)}$-Woodin, then δ (is of course Woodin and) belongs to $\mathrm{Lim}(C^{(n)})$. As we shall see in §3.1, this is no coincidence. Finally, we consider the following notion.

Definition 1.6. A cardinal κ is called λ-$C^{(n)}$-**compact**, some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) \in C^{(n)}$ and such that, for every $X \subseteq M$ with $|X| \leqslant \lambda$, there is a $Y \in M$ so that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$; we say that κ is $C^{(n)}$-**strongly compact** if it is λ-$C^{(n)}$-compact for all $\lambda > \kappa$.

2. Consistency bounds

We focus on the $C^{(n)}$-versions of tall, superstrong, strong, supercompact and extendible cardinals (in order of appearence). Our general method can be roughly described as follows.

Suppose that we are given such a $C^{(n)}$-cardinal κ and an elementary embedding $j : V \longrightarrow M$ witnessing this fact appropriately. Under the additional assumption that the image $j(\kappa)$ is a regular (or even inaccessible) cardinal in V, we shall construct various elementary chains of substructures of the model M, giving rise to factor elementary embeddings which have analogous strength to that of the initial j.

The aim of these constructions is to ensure that the ordinals below $j(\kappa)$ which arise as images of the large cardinal κ under embeddings of the sort in question, is a sufficiently "rich" subset of $j(\kappa)$; e.g., stationary, α-club for some $\alpha < j(\kappa)$, etc. If $j(\kappa)$ is indeed an inaccessible cardinal, we then check that all the aforementioned factor embeddings can be witnessed inside the model $V_{j(\kappa)}$ via derived extenders and we, consequently, obtain corresponding consistency upper bounds for each individual $C^{(n)}$-hierarchy.

We begin by first considering the case of tallness, where we shall describe our method in detail.

2.1. **Tallness.** Suppose that κ is λ-tall for some $\lambda > \kappa$, as witnessed by the elementary embedding $j : V \longrightarrow M$, i.e., M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\kappa}M \subseteq M$. Suppose that, in addition, $j(\kappa)$ is a regular cardinal. We pick some limit ordinal $\beta \in (\lambda, j(\kappa))$ and consider the following elementary substructure:

$$X = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_\beta^M\} \prec M.$$

To check that $X \prec M$, one uses the fact that β is a limit ordinal and verifies the Tarski–Vaught criterion. Notice that X is, in fact, the Skolem hull of the $range(j)$ together with V_β^M inside M, with respect to the functions of the form $f : V_\kappa \longrightarrow V$.

Starting with $X_0 = X$ and $\beta_0 = \beta$, we recursively build, for any $\xi < j(\kappa)$, an increasing (under $\subseteq$) sequence of elementary substructures $X_\xi \prec M$, together with a strictly increasing sequence of corresponding limit ordinals $\beta_\xi < j(\kappa)$, such that each X_ξ is of the form

$$X_\xi = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_\xi}^M\}.$$

Our aim is to show that, at appropriate ordinals $\gamma < j(\kappa)$, using the "current" substructure X_γ of this chain, we can define an elementary embedding j_γ which can be nicely represented and which, at the same time, witnesses λ-tallness for the cardinal κ.

So let $\beta_0 = \beta$ and X_0 as defined above. For any $\xi + 1 < j(\kappa)$, given β_ξ and X_ξ, we define

$$\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$$

and

$$X_{\xi+1} = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_{\xi+1}}^M\}.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_\xi}^M\},$$

which concludes the recursive definition of the elementary chain.

Remark 2.1. The ordinal β_0 serves the mere purpose of initializing the construction and is not important for later arguments. In fact, any other limit ordinal in $(\lambda, j(\kappa))$ would also be sufficient. Thus, and although –formally– our construction depends on this initial choice, we supress any further mention to β_0. In the few cases where it is relevant, we will refer to it as the "initial limit ordinal".

Moreover, for any elementary substructure which is of this particular form, i.e., the Skolem hull of the range of the embedding together with some set in M, we will frequently call the latter set (in this case, the various V_β^M's) as the set of *seeds*.[3]

[3] See [5] for a general theory of seeds.

It is clear that $\langle \beta_\xi : \xi < \gamma \rangle$ is a continuous and strictly increasing sequence of limit ordinals. Moreover, we have the following.

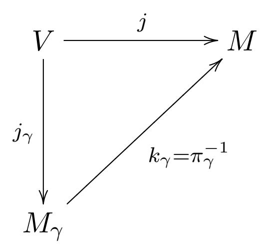
Claim 2.2. *For every $\xi < j(\kappa)$, $\beta_\xi < j(\kappa)$.*

Proof of claim. Since $j(\kappa)$ is regular, it is enough to see that, for every $\xi < j(\kappa)$, $|X_\xi \cap j(\kappa)| < j(\kappa)$. But the latter follows from a simple counting argument, using the definition of the substructures X_ξ. $\square$

Since each β_ξ is limit, $X_\xi \prec M$ and, evidently, for any $\xi < \xi' < j(\kappa)$, we have that $X_\xi \subseteq X_{\xi'}$. Therefore, an elementary chain of substructures is formed:

$$X_0 \prec X_1 \prec \ldots \prec X_\xi \prec \ldots \quad \prec M.$$

For any $\gamma < j(\kappa)$ with $\mathrm{cof}(\gamma) > \kappa$, let us consider the current substructure $X_\gamma = \bigcup_{\xi<\gamma} X_\xi$, along with the corresponding ordinal β_γ. Clearly, $\beta_\gamma = \sup\limits_{\xi<\gamma} \beta_\xi < j(\kappa)$, and $\mathrm{cof}(\beta_\gamma) = \mathrm{cof}(\gamma) > \kappa$. We then let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $\mathrm{cp}(j_\gamma) = \kappa$. This produces the following commutative diagram of elementary embeddings:



We now show that the embedding j_γ witnesses the λ-tallness of κ. The key observation is that, in this situation, $X_\gamma \cap j(\kappa)$ is in fact an ordinal; the reason is that we have "filled in all the ordinal holes below $j(\kappa)$" along our recursive construction of the substructures X_ξ. It then easily follows that

$$\mathrm{cp}(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \beta_\gamma.$$

In fact, in such constructions, $X_\gamma \cap j(\kappa)$ is an ordinal if and only if $\mathrm{cp}(k_\gamma) = j_\gamma(\kappa)$ in which case, we call the embedding j_γ an *initial factor* of j. Being slightly more general, suppose that $j : V \longrightarrow M$ is an elementary embedding with $\mathrm{cp}(j) = \kappa$ and M transitive; moreover, suppose that $X \prec M$ with $range(j) \subseteq X$, $X \cap [\kappa, j(\kappa)) \neq \varnothing$ and $X \cap j(\kappa)$ bounded in $j(\kappa)$. Let $\pi : X \cong M_0$ be the Mostowski collapse and consider, as above, the map $j_0 = \pi \circ j : V \longrightarrow M_0$ with $\mathrm{cp}(j_0) = \kappa$, which forms a commutative diagram (with $k = \pi^{-1}$). Observe that the imposed requirements on the elementary substructure X ensure that j_0 is well-defined, $M_0 \neq V$ (i.e., $j_0 \neq id$), $\mathrm{cp}(j_0) = \kappa$ and $j_0(\kappa) < j(\kappa)$. We now introduce the following notion.

Definition 2.3. Such a j_0 is called an **initial factor** of j, if $\mathrm{cp}(k) = j_0(\kappa)$.

The following facts are easily verified.

Fact 2.4. In the situation described above, j_0 is an initial factor of j if and only if $X \cap j(\kappa)$ is an ordinal. In such a case, $j_0(\kappa) = \sup(X \cap j(\kappa))$.

Fact 2.5. If j_0 is an initial factor of j (via the collapse $\pi : X \cong M_0$ with $k = \pi^{-1}$), then $V^{M_0}_{j_0(\kappa)} = V^M_{j_0(\kappa)} \subseteq range(k)$ and therefore,

$$\{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{j_0(\kappa)}\} \subseteq range(k).$$

Returning to our argument, we have that $j_\gamma(\kappa) = \beta_\gamma > \lambda$ and so, in order to conclude that this embedding witnesses λ-tallness, we only need to check that ${}^\kappa M_\gamma \subseteq M_\gamma$. This essentially comes from the fact that the set of seeds that generate M_γ is closed under κ-sequences, a fact which, in turn, follows from ${}^\kappa M \subseteq M$ and cof$(\gamma) > \kappa$. Notice that

$$M_\gamma = \pi_\gamma\text{``}X_\gamma = \{j_\gamma(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{\beta_\gamma}\}.$$

Now suppose that $\{j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa\} \subseteq M_\gamma$, where for each $\alpha < \kappa$, $x_\alpha \in V^M_{\beta_\gamma}$ and $f_\alpha \in V$. Since cof$(\gamma) > \kappa$, we clearly have that $\langle x_\alpha : \alpha < \kappa\rangle \in V^M_{\beta_\gamma} \subseteq M_\gamma$. It is also obvious that

$$\langle j_\gamma(f_\alpha) : \alpha < \kappa\rangle = j_\gamma\left(\langle f_\alpha : \alpha < \kappa\rangle\right) \restriction \kappa \in M_\gamma$$

and, hence, we can compute in M_γ the sequence $\langle j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa\rangle$ by evaluating pointwise the functions $j_\gamma(f_\alpha)$'s at the corresponding x_α's, i.e., $\langle j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa\rangle \in M_\gamma$. We have thus proved the following.

Proposition 2.6. *Suppose that $j : V \longrightarrow M$ is a λ-tall embedding for κ, with $j(\kappa)$ regular. Then, for any given (initial limit ordinal) $\beta_0 \in (\lambda, j(\kappa))$ and for any $\gamma < j(\kappa)$ with cof$(\gamma) > \kappa$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the elementary chain construction as above is an initial factor of j witnessing the λ-tallness of κ.*

In order to establish the closure under κ-sequences the only relevant information was the fact that ${}^\kappa M \subseteq M$ and that cof$(\gamma) =$ cof$(\beta_\gamma) > \kappa$. Thus, we may be slightly more general and state the following.

Corollary 2.7. *Suppose that $j : V \longrightarrow M$ is a λ-tall embedding for κ, with $j(\kappa)$ regular. Suppose that $j_0 : V \longrightarrow M_0 \cong X$ is an initial factor of j via the Mostowski collapse of*

$$X = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_\theta\} \prec M,$$

where $\theta \in (\lambda, j(\kappa))$ is such that cof$(\theta) > \kappa$. Then, j_0 is λ-tall for κ.

Our next aim is to consider the class of all images $h(\kappa)$ below $j(\kappa)$, where h is any (initial factor) tallness embedding for κ.

Proposition 2.8. *Suppose that $j : V \longrightarrow M$ witnesses the λ-tallness of κ, with $j(\kappa)$ regular. Then, the collection*

$$D = \{h(\kappa) < j(\kappa) : h \text{ is } \alpha\text{-tall for } \kappa, \text{ for some } \alpha < j(\kappa)\}$$

contains a $[\kappa^+, j(\kappa))$-club.

Proof. Let us consider the collection D' of all images $j_0(\kappa)$ below $j(\kappa)$, where j_0 is any initial factor embedding arising from a Mostowski collapse of some elementary substructure $X \prec M$ of the form

$$X = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_\theta^M\},$$

for some $\theta \in (\lambda, j(\kappa))$ with $\mathrm{cof}(\theta) > \kappa$. We show that D', which is contained in D, is in fact a $[\kappa^+, j(\kappa))$-club in $j(\kappa)$. Clearly, D' contains all the images $j_\gamma(\kappa)$ arising from initial factor embeddings coming from our elementary chain construction, for various initial limit ordinals $\beta_0 \in (\lambda, j(\kappa))$ and various lengths $\gamma < j(\kappa)$ with $\mathrm{cof}(\gamma) > \kappa$.

So suppose that $\delta \in [\kappa^+, j(\kappa))$ is regular and that $\langle j_i(\kappa) : i < \delta \rangle$ is a strictly increasing sequence below $j(\kappa)$ where, for all $i < \delta$, there is some $\theta_i \in (\lambda, j(\kappa))$ with $\mathrm{cof}(\theta_i) > \kappa$, such that $j_i : V \longrightarrow M_i$ is an initial factor of j arising via the collapse π_i of the substructure

$$X_i = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\theta_i}^M\} \prec M.$$

Recall that $X_i \cap j(\kappa)$ is an ordinal and $j_i(\kappa) = \sup(X_i \cap j(\kappa))$. Let us point out that, the fact that j_i comes from the collapse of X_i, only implies that $j_i(\kappa) \geqslant \theta_i$. On the other hand, it follows from fact 2.5 that

$$\{j(f)(x) : f \in V,\, x \in V_{j_i(\kappa)}^M\} = \{j(f)(x) : f \in V,\, x \in V_{\theta_i}^M\}$$

and thus, we may as well assume that $j_i(\kappa) = \theta_i$, for every $i < \delta$. Hence, for any $i < \ell < \delta$, we have that $X_i \subseteq X_\ell$ which, in turn, gives that $X_i \prec X_\ell$ and so, an elementary chain is formed. We may now let $\theta_\delta = \sup_{i<\delta} \theta_i$ and

$$X_\delta = \bigcup_{i<\delta} X_i = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\theta_\delta}^M\} \prec M.$$

Obviously, $\theta_\delta < j(\kappa)$ and $\mathrm{cof}(\theta_\delta) = \delta > \kappa$. Let $\pi_\delta : X_\delta \cong M_\delta$ be the Mostowski collapse and let $j_\delta = \pi_\delta \circ j : V \longrightarrow M_\delta$ with $\mathrm{cp}(j_\delta) = \kappa$ be the composed map, as usual. Then, $X_\delta \cap j(\kappa)$ is an ordinal and hence

$$j_\delta(\kappa) = \sup(X_\delta \cap j(\kappa)) = \sup_{i<\delta} \theta_i = \sup_{i<\delta} j_i(\kappa),$$

which shows the desired closure. Clearly, $j_\delta(\kappa) < j(\kappa)$ and therefore, by Corollary 2.7, j_δ is λ-tall for κ.

Finally, it is also obvious from our construction that the various images $j_0(\kappa)$ of initial factor λ-tall embeddings are unbounded in $j(\kappa)$ (by choosing a sufficiently large initial limit ordinal β_0). □

Towards obtaining our first consistency result, the next step is to consider the definability of the aforementioned collection D of images, inside $V_{j(\kappa)}$. We shall additionally assume that $j(\kappa)$ is inaccessible, and we shall show that the various tallness embeddings can be witnessed inside $V_{j(\kappa)}$ via extenders. Formally, we prove the following.

Theorem 2.9. *Suppose that, for some $\lambda > \kappa$, the elementary embedding $j : V \longrightarrow M$ witnesses the λ-tallness of κ, with $j(\kappa)$ inaccessible. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-tall".*

Proof. Given some α-tall initial factor embedding $j_0 : V \longrightarrow M_0$, where

$$M_0 \cong X = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_\theta^M\} \prec M,$$

for some $\alpha \in (\lambda, j(\kappa))$ and some ordinal $\theta < j(\kappa)$ with $\mathrm{cof}(\theta) > \kappa$, as we have already remarked, we may assume that $j_0(\kappa) = \theta$. Naturally, we may extract from it the $(\kappa, j_0(\kappa))$-extender E and construct the corresponding extender embedding $j_E : V \longrightarrow M_E$ with $\mathrm{cp}(j_E) = \kappa$ and $j_E(\kappa) = j_0(\kappa)$. We will then have that

$$M_E = \{j_E(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^{M_0}_{j_0(\kappa)}\}$$

and also, $V^{M_0}_{j_0(\kappa)} = V^{M_E}_{j_E(\kappa)}$. Furthermore $M_0 \models \mathrm{cof}(j_0(\kappa)) > \kappa$, a fact which is computed correctly since ${}^\kappa M_0 \subseteq M_0$. This means that the set of seeds $V^{M_0}_{j_0(\kappa)}$ (which generate M_E) is closed under κ-sequences and so, by arguments which we have already described, it follows that ${}^\kappa M_E \subseteq M_E$, i.e., j_E witnesses α-tallness as well.

Actually, in our case, $M_0 = M_E$ which follows from general facts regarding the commutative diagram of embeddings, for derived extenders. Notice also that, in the current setting, all derived extenders belong to $V_{j(\kappa)}$.

Then, in the ZFC model $V_{j(\kappa)}$, for any such extender E coming from a factor j_0, $V_{j(\kappa)} \models$ "j_E is α-tall for κ". This follows from the inaccessibility of $j(\kappa)$ which enables $V_{j(\kappa)}$ to faithfully verify that $j_E(\kappa) = j_0(\kappa)$ and ${}^\kappa M_E \subseteq M_E$. Therefore, the collection

$$C_{\text{tall}} = \{h_E(\kappa) < j(\kappa) : h_E \text{ is } \alpha\text{-tall extender embedding}, \alpha < j(\kappa)\},$$

which is a class in $j(\kappa)$ definable in $V_{j(\kappa)}$, contains a $[\kappa^+, j(\kappa))$-club (in particular, C_{tall} is stationary in $j(\kappa)$). Also, again by inaccessibility, for each $n \in \omega$, we have a club $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$, consisting of all ordinals below $j(\kappa)$ which are Σ_n-correct in the sense of $V_{j(\kappa)}$. Hence, $C^{(n)}_{j(\kappa)} \cap C_{\text{tall}} \neq \varnothing$, for every $n \in \omega$, and the theorem is proved. □

Remark 2.10. We could have assumed that $j(\kappa)$ is inaccessible, right at the beginning of the section, without any change in the construction of the chains. Nevertheless, we presented our construction under the minimal assumptions and this is accomplished by just requiring the regularity of $j(\kappa)$.

Remark 2.11. It is important to note that theorem 2.9 is really a theorem, and not a theorem schema, something that is indicated by the clause "for every $n \in \omega$". This is not problematic since we have a truth predicate for *set* structures like $V_{j(\kappa)}$. In subsequent sections, we shall frequently state similar (consistency) results for other large cardinal notions. Throughout, we understand that clauses of the form "for every $n \in \omega$", followed by some satisfaction in a set model, indicate a theorem instead of a schema. For example, the conclusion of theorem 2.9 can be written (slightly more) formally as: $\forall\, n \in \omega(V_{j(\kappa)} \models$ "κ is $C^{(n)}$-tall").

This remark applies to all similar results throughout this article. In other places of the text where we shall be looking at satisfaction in class models (e.g., $C^{(n)}$-cardinals in V), we will explicitly indicate any result which is a countable schema of statements, one for each (meta-theoretic) natural number $n \in \mathbb{N}$.

Notice that theorem 2.9 gives an (indirect) upper bound on the consistency strength of the theory ZFC+"κ is $C^{(n)}$-tall", for every natural number n, where the latter is indeed a countable schema of formulas. In turn, it gives rise to the following natural question.

Question 2.12. What is the consistency strength of the existence of a tallness embedding j for κ such that, additionally, $j(\kappa)$ is inaccessible?

As we shall see, 1-extendibility is an adequate upper bound answering the previous question; indeed, it will be a sufficient assumption for results concerning the cases of superstrongness and that of strongness as well (cf. § 2.2 and 2.3). On the other hand, let us mention that for supercompactness (and extendibility), we shall be using the assumption almost hugeness (cf. § 2.4), although the exact bounds for all these cases are not known.

Having dealt with tall cardinals, the essential features of our method have become apparent and we now proceed with the rest of the large cardinals. Since many of the constructions will be in a similar spirit, we will skip several details and refer to previously established facts when needed.

2.2. **Superstrongness.** Suppose that κ is a superstrong cardinal and let $j : V \longrightarrow M$ be a witnessing embedding, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. In addition, suppose that $j(\kappa)$ is regular. Bear in mind that in such a case, this is actually equivalent to requiring that $j(\kappa)$ is inaccessible, since superstrongness already implies that $j(\kappa) \in C^{(1)}$. We can thus forget about this distinction and assume that $j(\kappa)$ is inaccessible right away, proving the following.

Proposition 2.13. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, for each (initial limit ordinal) $\beta_0 \in (\kappa, j(\kappa))$ and any limit $\gamma < j(\kappa)$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the elementary chain construction as before, is an initial factor of j and superstrong for κ.*

Proof. Having fixed an initial limit ordinal $\beta_0 \in (\kappa, j(\kappa))$, we recursively construct an elementary chain of substructures X_ξ of M as before, starting with seeds in V_{β_0}. At the same time, we produce a sequence of ordinals b_ξ below $j(\kappa)$.

At any limit $\gamma < j(\kappa)$,[4] we let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse, and we let $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$ with $\mathrm{cp}(j_\gamma) = \kappa$, producing a commutative diagram of embeddings (where $k_\gamma = \pi_\gamma^{-1}$). Once again, $X_\gamma \cap j(\kappa)$ is an ordinal and hence j_γ is an initial factor of j such that

$$\mathrm{cp}(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \sup_{\xi<\gamma} \beta_\xi = \beta_\gamma.$$

In particular, $V_{j_\gamma(\kappa)} \subseteq M_\gamma$ as desired. □

Clearly, for any initial β_0, the above procedure gives a strictly increasing and continuous sequence of ordinals $\langle \beta_\xi : \xi < j(\kappa) \rangle$ below $j(\kappa)$, all of which are images of κ under initial factor superstrongness embeddings. It is easy to see that this collection of images is a (full) club in $j(\kappa)$.

Corollary 2.14. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, the collection $\{h(\kappa) < j(\kappa) : h$ is an initial factor of $j \wedge h$ is superstrong for $\kappa\}$ is a club in $j(\kappa)$.*

Exactly as in the case of tallness, for each superstrong initial factor embedding $j_0 : V \longrightarrow M_0$, the derived $(\kappa, j_0(\kappa))$-extender E belongs to $V_{j(\kappa)}$ and then, the corresponding extender embedding $j_E : V \longrightarrow M_E$ is superstrong for κ. Moreover, one again checks that, in fact, $M_0 = M_E$. Finally, by inaccessibility, $V_{j(\kappa)} \models$ "j_E is superstrong for κ" for any such extender and so, in particular, $V_{j(\kappa)} \models$ "κ is superstrong". This means that the collection $C_{\text{s-strong}}$ defined as

$$\{h_E(\kappa) < j(\kappa) : h_E \text{ is a superstrong extender embedding for } \kappa\},$$

is a definable class of $V_{j(\kappa)}$ and contains a club. Then, by considering the clubs $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$ consisting of all ordinals below $j(\kappa)$ which are Σ_n-correct in the sense of $V_{j(\kappa)}$, we get the following.

Theorem 2.15. *If $j : V \longrightarrow M$ is superstrong for κ with $j(\kappa)$ inaccessible, then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-superstrong".*

This theorem gives an (indirect) upper bound on the consistency of the theory ZFC+"κ is $C^{(n)}$-superstrong", for every natural number n, where the latter is again a countable schema.

On the other hand, for any fixed natural number n, by proposition 2.4 in [1], the existence of a $\kappa \in C^{(n)}$ which is 2^κ-supercompact implies the existence of many $C^{(n)}$-superstrong cardinals below κ. Since κ being 2^κ-supercompact implies the existence of many 1-extendible cardinals below it, the following corollary is an improvement of the aforementioned consistency bound of Bagaria.

[4] Note that in this case, as opposed to the case of tallness, the ordinal length $\gamma < j(\kappa)$ at which we take the collapse of the current substructure can be any limit ordinal below $j(\kappa)$ (i.e., even ω), since we are not interested in closure under sequences for the factor embedding.

Corollary 2.16. *If κ is 1-extendible then there exists a normal ultrafilter $\mathcal{U}$ on κ such that*

$$\{\alpha < \kappa : \forall\, n \in \omega (V_\kappa \models \text{“}\alpha \textit{ is } C^{(n)}\textit{-superstrong''})\} \in \mathcal{U}.$$

In particular, if ZFC + “$\exists\, \kappa$(κ is 1-extendible)” is consistent, then so is the theory ZFC + “κ is $C^{(n)}$-superstrong”, for every natural number n.

Proof. Suppose that $j : V_{\kappa+1} \longrightarrow V_{j(\kappa)+1}$ witnesses the 1-extendibility of κ. Let E be the derived $(\kappa, j(\kappa))$-extender from j and consider the extender embedding $j_E : V \longrightarrow M_E$. By standard arguments, j_E is superstrong for κ, $j_E(\kappa) = j(\kappa)$, and moreover,

$$V_{j(\kappa)+1} \models \text{“}j_E \text{ is superstrong for } \kappa \wedge j_E(\kappa) = j(\kappa) \text{ is inaccessible''}$$

(where note that $E \in V_{j(\kappa)+1}$). Also, $C_{\text{s-strong}}$, the definable stationary subclass of $j(\kappa)$ which was mentioned after corollary 2.14, belongs to $V_{j(\kappa)+1}$. Additionally, for every $n \in \omega$, we may consider the club class $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$ consisting of the ordinals that are Σ_n-correct in $V_{j(\kappa)}$; since $\mathcal{P}(j(\kappa)) \subseteq V_{j(\kappa)+1}$, the latter verifies the fact that $C_{\text{s-strong}}$ is stationary in $j(\kappa)$. Thus, for every $n \in \omega$,

$$\begin{aligned} V_{j(\kappa)+1} \models \quad & \exists\, (\kappa,\theta)\text{-extender } E \in V_{j(\kappa)} \text{ for some } \theta < j(\kappa), \text{ such that} \\ & V_{j(\kappa)} \models \text{“}j_E \text{ is superstrong for } \kappa \text{ and } j_E(\kappa) \in C^{(n)}\text{''} \end{aligned}$$

that is, for every $n \in \omega$,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } C^{(n)}\text{-superstrong''}.$$

Now, if we derive the usual normal ultrafilter $\mathcal{U}$ on κ from the initial embedding j, it then follows that, for every $n \in \omega$, the set

$$S_n = \{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong''}\} \in \mathcal{U}.$$

By the completeness of $\mathcal{U}$, we may now intersect all the S_n's, and the conclusion follows. □

Observe that, in the previous proof, since κ is itself inaccessible, V_κ is a model of the theory ZFC+“α is $C^{(n)}$-superstrong”, for every natural number n, in which there is actually a proper class of α's that satisfy the schema.

Let us also make a remark on the connection between the cases of superstrongness and tallness. Notice that if $j : V \longrightarrow M$ is superstrong for κ, we may as well assume that $j = j_E$, i.e., that it is an extender superstrong embedding with

$$M = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{j(\kappa)}\}.$$

So, if $j(\kappa)$ is also regular (and thus, inaccessible), then the same embedding witnesses $< j(\kappa)$-tallness for κ. Thus, everything we did for tallness can be entirely done under the context of a superstrong embedding with regular (inaccessible) target. We can therefore state the following, which immediately follows from theorem 2.9.

Corollary 2.17. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-tall".*

Consequently, 1-extendibility is an adequate (consistency) upper bound both for the $C^{(n)}$-superstrongness and for the $C^{(n)}$-tallness case.

2.3. **Strongness.** Suppose that κ is λ-strong, for some limit $\lambda > \kappa$, as witnessed by the embedding $j : V \longrightarrow M$, i.e., M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. In addition, suppose that $j(\kappa)$ is regular.

We apply the same ideas in order to build an elementary chain of substructures of M and then produce a λ-strong factor embedding from it. Two remarks are in order here. First, since –as in the case of superstrongs– we are not interested in closure under sequences for the factor embedding, the length at which we take collapses can be any limit ordinal $\gamma < j(\kappa)$. Additionally, since the crucial requirement for λ-strongness is "$V_\lambda \subseteq M$", we will start our chain by just "throwing in" all the seeds from V_λ, i.e., we let $\beta_0 = \lambda$ and define our first elementary substructure as

$$X_0 = \{j(f)(x) : f \in V,\ f : V_\kappa \longrightarrow V,\ x \in V_{\beta_0}\} \prec M.$$

From that point on, there two ways to proceed. On the one hand, we might as well proceed with the rest of the chain as usual, using seeds x from $V^M_{\beta_\xi}$ for $\xi > 0$ (where the β_ξ's are defined as in the case of superstrongness). That would result in the desired λ-strong initial factor embedding after γ steps, for any limit ordinal $\gamma < j(\kappa)$. In this case, there is really not much more to say, as the construction and the subsequent results are totally parallel to the ones of §2.2 (in particular, we will get a full club contained in the collection of images $h(\kappa) < j(\kappa)$, where h is a λ-strong factor embedding for κ).

Alternatively, one can consider only ordinal seeds, i.e., for appropriate limit ordinals[5] $\beta_\xi \in (\lambda, j(\kappa))$, we take substructures of the form

$$X_\xi = \{j(f)(\alpha) : f \in V,\ f : \kappa \longrightarrow V,\ \alpha < \beta_\xi\} \prec M$$

(note the modification in the domain of the functions). We omit more details on this approach as the results obtained are the same.

Finally, if assume that $j(\kappa)$ is also inaccessible, we then get that for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is λ-$C^{(n)}$-strong". Furthermore, as in the case of tallness and superstrongness, 1-extendibility is an adequate upper bound for the schema asserting full $C^{(n)}$-strongness, for every natural number n. Such a conclusion is hardly surprising in the light of proposition 1.2 in [1], which asserts that every λ-strong cardinal is actually λ-$C^{(n)}$-strong; this is shown in [1] by the method of iterated ultrapowers which, by the way, will be relevant for us in Section 3.

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[5] Such ordinals have to be closed under some (fixed) ordinal pairing function, so that the structures considered are actually *elementary* substructures of M.

2.4. **Supercompactness.** Suppose that κ is λ-supercompact, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$, as witnessed by the embedding $j : V \longrightarrow M$, i.e., M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. In addition, suppose that $j(\kappa)$ is regular.

In this case, as opposed to the previous ones, we build our elementary chain using a slightly different collection of seeds. Namely, we also include $j``\lambda$ which (belongs to M and) is used in order to obtain the closure under λ-sequences for the initial factor embedding that we are aiming for.

We start by picking an initial limit ordinal $\beta_0 \in (\lambda, j(\kappa))$ and by letting

$$X_0 = \{j(f)(j``\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_0}\} \prec M,$$

where note that the domain of the functions has been modified accordingly. For $\xi + 1 < j(\kappa)$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$ and

$$X_{\xi+1} = \{j(f)(j``\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_{\xi+1}}\}.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(j``\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_\xi}\},$$

which concludes the recursive definition of the elementary chain. We now proceed with the formal result.

Proposition 2.18. *Suppose that $j : V \longrightarrow M$ is λ-supercompact for κ, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$, and with $j(\kappa)$ regular. Then, for each (initial limit ordinal) $\beta_0 \in (\lambda, j(\kappa))$ and each $\gamma < j(\kappa)$ with cof$(\gamma) > \lambda$, the elementary embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the chain construction as above, is an initial factor of j witnessing λ-supercompactness of κ.*

Proof. Since M is closed under $\lambda^{<\kappa}$-sequences, any function $f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow \kappa$ belongs to M. Therefore, by the inaccessibility of $j(\kappa)$ in M, we have that $|2^{\lambda^{<\kappa}}| < j(\kappa)$. Now using the regularity of $j(\kappa)$, a counting argument shows that for each $\xi < j(\kappa)$, $\beta_\xi < j(\kappa)$.

We fix $\gamma < j(\kappa)$ with cof$(\gamma) > \lambda$, and we consider the substructure X_γ of which we take the collapse $\pi_\gamma : X_\gamma \cong M_\gamma$. We then define the map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $\mathrm{cp}(j_\gamma) = \kappa$, producing a commutative diagram as usual (with $k_\gamma = \pi_\gamma^{-1}$).

One again checks that $X_\gamma \cap j(\kappa)$ is an ordinal, which implies that j_γ is indeed an initial factor of j and that

$$\mathrm{cp}(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \sup_{\xi<\gamma} \beta_\xi = \beta_\gamma > \lambda,$$

with cof(β_γ) = cof$(\gamma) > \lambda$. Thus, in order to see that j_γ is λ-supercompact, we only need to check that ${}^{\lambda}M_\gamma \subseteq M_\gamma$. For this, note first that since $j``\lambda \cup \{j``\lambda\} \subseteq X_0 \subseteq X_\gamma$, we have that

$$M_\gamma = \{j_\gamma(f)(j_\gamma``\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_\gamma}\}.$$

Clearly, $j_\gamma``\lambda \in M_\gamma$ as well and therefore, if we consider the function $j_\gamma \restriction \lambda : \lambda \longrightarrow j_\gamma``\lambda$ (as an order-type map), we get that $j_\gamma \restriction \lambda \in M_\gamma$. We now use this function to get the closure under λ-sequences.

Let $\{j_\gamma(f_i)(j_\gamma``\lambda, x_i) : i < \lambda\} \subseteq M_\gamma$, where for $i < \lambda$, $x_i \in V^M_{\beta_\gamma}$ and $f_i \in V$. Since $\mathrm{cof}(\beta_\gamma) = \mathrm{cof}(\gamma) > \lambda$ and ${}^\lambda M \subseteq M$, we have that $\langle x_i : i < \lambda\rangle \in V^M_{\beta_\gamma} \subseteq M_\gamma$. It will be enough to show that $\langle j_\gamma(f_i) : i < \lambda\rangle \in M_\gamma$, since in that case, we can compute in M_γ the sequence $\langle j_\gamma(f_i)(j_\gamma``\lambda, x_i) : i < \lambda\rangle$ by evaluating pointwise the functions $j_\gamma(f_i)$'s at the corresponding $x_i$'s together with $j_\gamma``\lambda$.

Now, $j_\gamma(\langle f_i : i < \lambda\rangle)$ is a function $G : j_\gamma(\lambda) \longrightarrow M_\gamma$ that belongs to M_γ. Using G and $j_\gamma \restriction \lambda$, define in M_γ the function $F : \lambda \longrightarrow M_\gamma$ by letting, for every $\alpha < \lambda$, $F(\alpha) = G(j_\gamma(\alpha))$. But then, for every $\alpha < \lambda$,

$$F(\alpha) = j_\gamma(\langle f_i : i < \lambda\rangle)(j_\gamma(\alpha)) = j_\gamma(\langle f_i : i < \lambda\rangle(\alpha)) = j_\gamma(f_\alpha),$$

i.e., $F = \langle j_\gamma(f_i) : i < \lambda\rangle \in M_\gamma$ and we are done. $\square$

By our familiar methods, one obtains the following corollary.

Corollary 2.19. *If $j : V \longrightarrow M$ witnesses the λ-supercompactness of κ, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$ and with $j(\kappa)$ regular, then the collection*

$$D = \{h(\kappa) < j(\kappa) : h \text{ is } \lambda\text{-supercompact for } \kappa\}$$

contains a $[|\lambda|^+, j(\kappa))$-club.

Unlike the cases of tallness and superstrongness where the various initial factor embeddings were witnessed inside $V_{j(\kappa)}$ by short (derived) extenders, in the case of supercompactness such extenders are not sufficient for our purposes. Instead, we shall use extenders of the Martin–Steel form. The following characterization is described in [1] (see also the Appendix of [10]).

Theorem 2.20 ([1]). *A cardinal κ is λ-supercompact if and only if there exists a (κ, Y)-extender E, with Y transitive, such that $\{\kappa\} \cup [Y]^{<\omega} \cup {}^\lambda Y \cup j_E``Y \subseteq Y$ and $j_E(\kappa) > \lambda$, where j_E is the extender elementary embedding.*

In fact, given such a (κ, Y)-extender E, the corresponding embedding j_E is λ-supercompact. Conversely, given any λ-supercompact embedding j, we may extract an appropriate transitive set Y so that the derived (κ, Y)-extender meets the displayed requirements. In such a case, $j_E(\kappa) = j(\kappa)$.

Our next aim is to witness full $C^{(n)}$-supercompactness of κ inside $V_{j(\kappa)}$. Contemplating on the way we produced the chain, it is clear that the essential feature which guarrantees closure under λ-sequences is the fact that $j``\lambda \in M$ (which follows from the closure of $M$). It turns out that, to get full $C^{(n)}$-supercompactness in $V_{j(\kappa)}$, it is enough if the initial embedding is such that $j``\alpha \in M$ for every $\alpha < j(\kappa)$.

Theorem 2.21. *Suppose that $j : V \longrightarrow M$ witnesses the almost hugeness of κ. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-supercompact".*

Proof. Suppose that κ is almost huge, as witnessed by the elementary embedding $j : V \longrightarrow M$, i.e., M transitive, $\mathrm{cp}(j) = \kappa$ and ${}^{<j(\kappa)}M \subseteq M$. In particular, $V_{j(\kappa)} \subseteq M$ and $j(\kappa)$ is inaccessible. As we are aiming at full $C^{(n)}$-supercompactness below $j(\kappa)$, we may as well consider only regular $\lambda < j(\kappa)$.[6] Then, for any such fixed λ, since ${}^{\lambda}M \subseteq M$ and $j``\lambda \in M$, we may pick some initial limit ordinal $\beta_0^{(\lambda)} \in (\lambda, j(\kappa))$ and perform our construction exactly as in the λ-supercompactness case, i.e., construct, for any $\xi < j(\kappa)$, the corresponding ordinal $\beta_\xi^{(\lambda)} < j(\kappa)$ and the substructure

$$X_\xi^{(\lambda)} = \{j(f)(j``\lambda, x) : f \in V,\, f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\, x \in V_{\beta_\xi^{(\lambda)}}\} \prec M,$$

producing the elementary chain $X_0^{(\lambda)} \prec \ldots \prec X_\xi^{(\lambda)} \prec \ldots \prec M$. As before, at any limit $\gamma < j(\kappa)$ with $\mathrm{cof}(\gamma) \geqslant \lambda^+$, we may pause and take the transitive collapse of the current substructure, in order to produce a λ-supercompact initial factor embedding $j_\gamma^{(\lambda)}$ with $j_\gamma^{(\lambda)}(\kappa) = \beta_\gamma^{(\lambda)}$, where $\beta_\gamma^{(\lambda)}$ is the current ordinal of the produced sequence $\langle \beta_\xi^{(\lambda)} : \xi < j(\kappa)\rangle$. Thus, for any regular $\lambda \in (\kappa, j(\kappa))$, we have a corresponding class of ordinals in $j(\kappa)$, namely, $C_\lambda = \{\beta_\gamma^{(\lambda)} : \mathrm{cof}(\gamma) = \lambda^+\}$ which consists of the images of κ under the initial factor λ-supercompactness embeddings that are taken exactly at limits of cofinality λ^+. Clearly, C_λ is λ^+-club and, thus, stationary in $j(\kappa)$.

Now, we may let $C_{\text{s.c.}} = \bigcup\{C_\lambda : \lambda \in (\kappa, j(\kappa)),\, \lambda \text{ regular}\}$ and then $C_{\text{s.c.}}$ is a stationary subset of $j(\kappa)$ which is a disjoint union of stationary subsets. Of course, since $j(\kappa)$ is inaccessible, we may also consider the clubs $C^{(n)}_{j(\kappa)} \subseteq V_{j(\kappa)}$ of ordinals that are Σ_n-correct in the sense of $V_{j(\kappa)}$.

Then, for every $n \in \omega$ and every regular $\lambda \in (\kappa, j(\kappa))$, there is an initial factor λ-supercompact embedding j_0, with $j_0(\kappa) \in C^{(n)}_{j(\kappa)}$. We now use the characterization of theorem 2.20 in order to show that all these embeddings are witnessed by extenders inside $V_{j(\kappa)}$.

Fix some $n \in \omega$ and suppose that, for some regular $\lambda \in (\kappa, j(\kappa))$, $j_0 : V \longrightarrow M_0$ is an initial factor λ-$C^{(n)}$-supercompact embedding arising from our construction, i.e., arising via a collapse of an elementary substructure $X \prec M$ of the form above, associated with a corresponding limit ordinal $\theta = \sup(X \cap j(\kappa))$. As before, a counting argument shows that, for all $\alpha < j(\kappa)$, we have that $j_0(\alpha) < j(\kappa)$.

We now describe how to produce the appropriate transitive $Y \subseteq M_0$. The idea is simple: we start with $j_0``\lambda$ (which belongs to $M_0$) and we recursively close under all the relevant properties, repeating $\lambda^+$-many times (and taking unions at limit stages). Formally, we let $Y_0 = trcl(\{j_0``\lambda\})$ and, given any Y_α for some $\alpha < \lambda^+$, we let $Y_{\alpha+1} = trcl(Y_\alpha \cup [Y_\alpha]^{<\omega} \cup {}^{\lambda}Y_\alpha \cup j_0``Y_\alpha)$; if α is limit, then $Y_\alpha = \bigcup_{\xi<\alpha} Y_\xi$. Finally, we let $Y = Y_{\lambda^+}$; it is straightforward to check

[6] And since κ is clearly supercompact in the model $V_{j(\kappa)}$, it follows by a well-known result of Solovay that, for any regular $\lambda \in (\kappa, j(\kappa))$, we have $\lambda^{<\kappa} = \lambda$.

that $Y \subseteq M_0$, Y is transitive and $\{\kappa\} \cup [Y]^{<\omega} \cup {}^{\lambda}Y \cup j_0``Y \subseteq Y$. Moreover, since $j_0``j(\kappa) \subseteq j(\kappa)$, it follows that $Y \in V_{j(\kappa)}$. Now let E be the (κ, Y)-extender derived from the embedding j_0. Recall that any derived extender of this sort comes with a corresponding ordinal $\zeta > \kappa$ which is the least one so that $Y \subseteq j_0(V_\zeta)$ (and then, the various ultrafilters of the extender are on the sets ${}^aV_\zeta$, for $a \in [Y]^{<\omega}$). In our case, we have that $\zeta < j(\kappa)$ again due to $j_0``j(\kappa) \subseteq j(\kappa)$. We may thus conclude that $E \in V_{j(\kappa)}$.

We now consider the extender embedding $j_E : V \longrightarrow M_E$; applying theorem 2.20, it is easy to see that j_E is λ-supercompact for κ with $j_E(\kappa) = j_0(\kappa)$. Finally, the inaccessibility of $j(\kappa)$ implies that we can verify inside $V_{j(\kappa)}$ the λ-$C^{(n)}$-supercompactness of κ via the relevant extender; that is, we get that $V_{j(\kappa)} \models$ "κ is λ-$C^{(n)}$-supercompact", and the theorem follows. □

Remark 2.22. Note that, in the proof of theorem 2.21, any initial factor λ-supercompact embedding j_0 which arises from our elementary chain is, in addition, superstrong. Thus, when we recursively construct the transitive Y which serves as the support of the witnessing extender, we may as well include the whole $V_{j_0(\kappa)}$ at the very first level Y_0, together with $j_0``\lambda$. This ensures that the extender embedding j_E will be superstrong as well.

The last theorem gives an upper bound on the consistency strength of the theory $\mathsf{T} = \mathsf{ZFC} +$ "κ is $C^{(n)}$-supercompact", for every natural number n, where the latter is again a countable schema. Namely, if the theory ZFC+"$\exists\,\kappa(\kappa$ is almost huge)" is consistent, then so is T. Moreover, we now show that this bound is sharp in the following sense.

Corollary 2.23. *If the theory ZFC + "$\exists\,\kappa(\kappa$ is almost huge$)$" is consistent, then so is the theory T + "$\forall\,\lambda(\lambda$ is not almost huge$)$".*

Proof. Let κ be *the least* almost huge cardinal, witnessed by the embedding $j : V \longrightarrow M$. By theorem 2.21, $V_{j(\kappa)} \models \mathsf{T}$. Towards a contradiction, suppose that λ is the least almost huge cardinal in the sense of $V_{j(\kappa)}$. Recall that the least almost huge cardinal is strictly smaller than the least supercompact, provided that they both exist. Thus, since κ is supercompact in $V_{j(\kappa)}$, we get that $\lambda < \kappa$. But this is a contradiction since "λ is almost huge" is a Σ_2-statement and $j(\kappa)$ is inaccessible, i.e., λ would have to be an almost huge cardinal below κ. □

It is a standard fact that if α is $< \kappa$-supercompact and κ is supercompact, then α is fully supercompact. A natural question is whether this generalizes to the case of $C^{(n)}$-supercompactness, for $n > 0$. We now address this question and show that if, as mentioned in remark 2.22, the witnessing extenders are also superstrong, then it does.

In this direction, we introduce the following notion, which will be of particular interest and importance in the context of ($C^{(n)}$-) extendibility.

Definition 2.24. A cardinal κ is called **jointly λ-supercompact and θ-superstrong**, for some $\lambda, \theta \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $V_{j(\theta)} \subseteq M$.

Note that the clause "κ-superstrong" just means usual superstrongness (i.e., $V_{j(\kappa)} \subseteq M$). For the global version(s) of this notion, the absence of one of the two parameters indicates universal quantification on the parameter missing; for instance, κ is jointly supercompact and θ-superstrong, for some fixed $\theta \geqslant \kappa$, if and only if it is jointly λ-supercompact and θ-superstrong, for every $\lambda \geqslant \kappa$. On the other hand, the absence of both parameters is intended to mean that the same λ is quantified for both of them, i.e., κ is jointly supercompact and superstrong if and only if it is jointly λ-supercompact and λ-superstrong, for every $\lambda \geqslant \kappa$.[7]

We stress the fact that the above notion transcends supercompactness in the sense that if κ is the least supercompact, then it is not jointly λ-supercompact and κ-superstrong, for any λ. In fact, as we shall see in § 2.5, global joint ($C^{(n)}$-) supercompactness and κ-superstrongness is equivalent to ($C^{(n)}$-) extendibility. Having said that, and given an analogous result of Bagaria for $C^{(n)}$-extendibles (cf. proposition 3.4 in [1]), the following lemma (schema) is hardly surprising.

Lemma 2.25. *If κ is jointly $C^{(n)}$-supercompact and κ-superstrong, then $\kappa \in C^{(n+2)}$.*

Proof. The proof is an induction in the meta-theory. Alternatively, the lemma follows from proposition 3.4 in [1], and proposition 2.30 below. □

Recalling the Π_{n+2}-expressibility of $C^{(n)}$-supercompactness, the previous lemma immediately implies the following.

Corollary 2.26. *If κ is jointly $C^{(n)}$-supercompact and κ-superstrong, and $\alpha < \kappa$ is $< \kappa$-$C^{(n)}$-supercompact, then α is $C^{(n)}$-supercompact.*

Before concluding this section, let us briefly return to the model $V_{j(\kappa)}$ obtained in the proof of theorem 2.21. Clearly, the α's below κ which are $C^{(n)}$-supercompact for every n, are unbounded in κ. For any such $\alpha < \kappa$ and for any $\gamma < \kappa$, the very same extenders belonging to V_κ witness the "γ-$C^{(n)}$-supercompactness" of α either in V_κ or in $V_{j(\kappa)}$. In particular, $V_{j(\kappa)}$ thinks that κ is a limit of Σ_n-correct ordinals and thus, for every $n \in \omega$, we have that $V_{j(\kappa)} \models \kappa \in \mathrm{Lim}(C^{(n)})$. In fact, we now give a small list of the properties that κ enjoys inside that particular model $V_{j(\kappa)}$. Regarding part (*iii*) below, recall definition 1.3.

Proposition 2.27. *Suppose that κ is almost huge, as witnessed by the embedding $j : V \longrightarrow M$. Let $\mathcal{U}$ be the usual normal measure on κ derived from j. Then, for any $n \in \omega$, the following hold in the (ZFC) model $V_{j(\kappa)}$:*

[7] Similar remarks apply to the corresponding $C^{(n)}$-version of this notion, which is straightforward to state; we leave these to the reader.

(*i*) $\kappa \in Lim(C^{(n)})$.

(*ii*) *κ is $C^{(n)}$-supercompact and*

$$\{\alpha < \kappa : \alpha \textit{ is } C^{(n)}\textit{-supercompact}\} \in \mathcal{U}.$$

(*iii*) *κ is $C^{(n)+}$-extendible and*

$$\{\alpha < \kappa : \alpha \textit{ is } C^{(n)+}\textit{-extendible}\} \in \mathcal{U}.$$

Proof. Parts (*i*) and (*ii*) follow from the preceding discussion. For (*iii*), fix some n and some $\lambda \in (\kappa, j(\kappa))$ so that $V_{j(\kappa)} \models \lambda \in C^{(n)}$; now, by the closure of M and the inaccessibility of $j(\kappa)$, $j \restriction V_\lambda \in M$ and then, M thinks that $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ is elementary, has critical point κ, $(j \restriction V_\lambda)(\kappa) > \lambda$, $j(\lambda) < j(j(\kappa))$ and moreover, that $j(V_{j(\kappa)}) \models j(\lambda) \in C^{(n)}$.

Let us temporarily fix a formula $\varphi(\lambda, \mu, \kappa)$ asserting that "there exists a λ-extendibility embedding h for κ with $\mu = h(\lambda)$". From the point of view of M, we have just argued that for every $\lambda \in (\kappa, j(\kappa))$ with $V_{j(\kappa)} \models \lambda \in C^{(n)}$, there exists a $\mu < j(j(\kappa))$ such that $\varphi(\lambda, \mu, \kappa)$ holds and, moreover, such that $j(V_{j(\kappa)}) \models \mu \in C^{(n)}$. But now, by the usual reflection of the normal measure, we have that the set of ordinals $\alpha < \kappa$ such that, for all $\lambda \in (\alpha, \kappa)$, it is true that

$$V_\kappa \models \lambda \in C^{(n)} \longrightarrow \exists\, \mu < j(\kappa)(\varphi(\lambda, \mu, \alpha) \,\wedge\, V_{j(\kappa)} \models \mu \in C^{(n)}),$$

belongs to $\mathcal{U}$. Let us call this set A. Fix any $\alpha \in A$ and fix any $\lambda \in (\alpha, \kappa)$ with $V_\kappa \models \lambda \in C^{(n)}$. Furthermore, fix a $\mu < j(\kappa)$ witnessing that $\alpha \in A$, i.e., such that $\varphi(\lambda, \mu, \alpha)$ holds and, moreover, such that $V_{j(\kappa)} \models \mu \in C^{(n)}$.

Since $\mu < j(\kappa)$, by the inaccessibility of the latter we have that the witnessing λ-extendibility embedding for α belongs to $V_{j(\kappa)}$, that is,

$$V_{j(\kappa)} \models \exists\, \mu(\varphi(\lambda, \mu, \alpha) \,\wedge\, \mu \in C^{(n)}).$$

Therefore, by elementarity, for any such $\alpha \in A$ and any fixed $\lambda \in (\alpha, \kappa)$ with $V_\kappa \models \lambda \in C^{(n)}$, we have that there exists a $\mu < \kappa$ such that

$$V_\kappa \models \varphi(\lambda, \mu, \alpha) \,\wedge\, \mu \in C^{(n)},$$

i.e., such extendibility embeddings for α may be witnessed inside V_κ. Note how we have successively bounded the μ's, first below $j(\kappa)$ and now below κ, ensuring that they are also in the relative $C^{(n)}$ of these structures. This discussion shows that the set

$$B = \{\alpha < \kappa : V_\kappa \models \forall\, \lambda > \alpha(\lambda \in C^{(n)} \longrightarrow \exists\, \mu(\varphi(\lambda, \mu, \alpha) \,\wedge\, \mu \in C^{(n)}))\}$$

includes A and hence, it also belongs to $\mathcal{U}$. Consequently, by "reversing" the reflection argument of the measure, we obtain that

$$V_{j(\kappa)} \models \forall\, \lambda > \kappa(\lambda \in C^{(n)} \longrightarrow \exists\, \mu(\varphi(\lambda, \mu, \kappa) \,\wedge\, \mu \in C^{(n)})).$$

But then, if we pick any $\lambda \in (\kappa, j(\kappa))$ with $V_{j(\kappa)} \models \lambda \in C^{(n)}$ and consider any witnessing extendibility embedding $h : V_\lambda \longrightarrow V_\mu$ for κ in $V_{j(\kappa)}$, we get that, in the sense of the latter structure, all three: κ, λ and μ are in $C^{(n)}$.

Hence, $h(\kappa) \in C^{(n)}$ as well. This shows that κ is $C^{(n)+}$-extendible in $V_{j(\kappa)}$. In turn, once more by a reflection argument,

$$\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is } C^{(n)+}\text{-extendible”}\} \in \mathcal{U}$$

and then, it follows that, in $V_{j(\kappa)}$, $\{\alpha < \kappa : \alpha \text{ is } C^{(n)+}\text{-extendible}\} \in \mathcal{U}$. □

Therefore, the assumption of almost hugeness is sufficient for the consistency of all the $C^{(n)}$-cardinals considered so far, in a strong sense: for any $n \in \omega$, the model $V_{j(\kappa)}$ thinks that the cardinal κ is $C^{(n)}$-supercompact, $C^{(n)}$-superstrong and, clearly, $C^{(n)}$-strong and $C^{(n)}$-tall as well. In addition, we just showed that it is also $C^{(n)+}$-extendible. Evidently, if we consider the *least* almost huge cardinal, the corresponding versions of corollary 2.23 are obtained for all these cases.

2.5. **More extendibility.** Given the established consistency results, we now look more closely at $C^{(n)}$-extendibility and its connection with $C^{(n)}$-supercompactness. We begin with the following theorem (schema).

Theorem 2.28. *Suppose that κ is $\lambda+1$-$C^{(n)}$-extendible, for some $n > 0$ and some $\lambda = \beth_\lambda > \kappa$ with cof$(\lambda) > \kappa$. Then, κ is jointly λ-$C^{(n)}$-supercompact and λ-superstrong.*

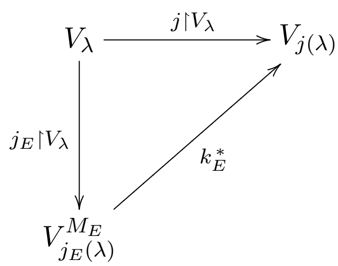
Proof. Fix some $n > 0$ and some $\lambda = \beth_\lambda > \kappa$ with $\text{cof}(\lambda) > \kappa$. Now let $j : V_{\lambda+1} \longrightarrow V_{j(\lambda)+1}$ be an elementary embedding that witnesses the $\lambda+1$-$C^{(n)}$-extendibility of κ, i.e., $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda + 1$ and $j(\kappa) \in C^{(n)}$. Let us consider E, the ordinary $(\kappa, j(\lambda))$-extender derived from j; that is, E is of the form $\langle E_a : a \in [j(\lambda)]^{<\omega}\rangle$ where, each E_a is a κ-complete ultrafilter on $[\lambda]^{|a|}$ defined as usual: for $X \subseteq [\lambda]^{|a|}$, $X \in E_a$ if and only if $a \in j(X)$.[8]

Now let $j_E : V \longrightarrow M_E$ be the extender embedding with $\text{cp}(j_E) = \kappa$. Although we do not have a “full" third factor embedding k_E commuting with j and j_E, we may nonetheless get a restricted version of it, denoted by $k^*_E : V^{M_E}_{j_E(\lambda)} \longrightarrow V_{j(\lambda)}$, by letting $k^*_E([a, f]) = j(f)(a)$, for all $[a, f] \in V^{M_E}_{j_E(\lambda)}$, where $a \in [j(\lambda)]^{<\omega}$ and $f : [\lambda]^{|a|} \longrightarrow V_\lambda$.[9] Moreover, it is easily checked that k^*_E is a well-defined $\{\in\}$-embedding and so, in particular, injective. We

[8] Note that despite the fact that the embedding j is between sets, this definition makes sense since, for any $m \in \omega$, $\mathcal{P}([\lambda]^m) \subseteq V_{\lambda+1}$. Moreover, $E \in V_{j(\lambda)+1}$ and the latter structure can faithfully verify that E is indeed a $(\kappa, j(\lambda))$-extender.

[9] Observe that any such f, representing an element in $V^{M_E}_{j_E(\lambda)}$, belongs to $V_{\lambda+1}$.

then obtain the commutative diagram



where $j \restriction V_\lambda = k^*_E \circ (j_E \restriction V_\lambda)$. By standard arguments, k^*_E is in fact surjective and, thus, it is the identity map. Consequently, $V^{M_E}_{j_E(\lambda)} = V_{j(\lambda)}$, i.e., $V_{j(\lambda)} \subseteq M_E$ and so j_E is λ-superstrong and, for every ordinal $\alpha \leqslant \lambda$, $j_E(\alpha) = j(\alpha)$. In particular, $j_E(\kappa) = j(\kappa)$ and also, since cof$(j(\lambda)) > \lambda$ (computed in V), we have that $j_E``\lambda = j``\lambda \in V_{j(\lambda)}$ and so $j_E``\lambda \in M_E$.

Thus, it will be enough to show that ${}^\lambda M_E \subseteq M_E$ in order to conclude that the embedding j_E witnesses the λ-$C^{(n)}$-supercompactness of κ as well. But this is shown exactly as in the proof of proposition 2.18. □

We immediately get the following corollary (schema). The "in particular" part answers affirmatively the corresponding question posed in [1].

Corollary 2.29. *If the cardinal κ is $C^{(n)}$-extendible, then it is jointly $C^{(n)}$-supercompact and superstrong. In particular, $C^{(n)}$-extendibility implies $C^{(n)}$-supercompactness.*

Although we do not know if the least $C^{(n)}$-supercompact is actually below the least $C^{(n)}$-extendible (for $n \geqslant 1$), we now show that, under the extra assumption of κ-superstrongness, the two (global) notions are equivalent.

Proposition 2.30. *If κ is jointly $C^{(n)}$-supercompact and κ-superstrong, then it is $C^{(n)}$-extendible.*

Proof. Fix some (meta-theoretic) $n \geqslant 1$ and suppose that κ is jointly $C^{(n)}$-supercompact and κ-superstrong. Moreover, fix some $\lambda > \kappa$ with $\lambda \in C^{(n+2)}$. Recall that $\lambda = |V_\lambda|$. Now let $j : V \longrightarrow M$ be an elementary embedding which witnesses the fact that κ is jointly λ-$C^{(n)}$-supercompact and κ-superstrong, i.e., M transitive, cp$(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$, $j(\kappa) \in C^{(n)}$ and $V_{j(\kappa)} \subseteq M$.

Recall that, by lemma 2.25, $\kappa \in C^{(n+2)}$. Thus, by elementarity, we have that $M \models j(\kappa) \in C^{(n+2)}$ and $M \models j(\lambda) \in C^{(n+2)}$. Now note that, by the closure of the target model, the restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V^M_{j(\lambda)}$ belongs to M and this witnesses the fact that

$$M \models \text{“}\kappa \text{ is } < \lambda\text{-}C^{(n)}\text{-extendible''}.$$

Moreover, since $j(\kappa) \in C^{(n)}$, we have that $V_{j(\kappa)} \models \lambda \in C^{(n+1)}$ and so, since $V_{j(\kappa)} \subseteq M$ and $M \models j(\kappa) \in C^{(n+2)}$, we consequently get that $M \models \lambda \in$

$C^{(n+1)}$. It now follows that the "$<\lambda$-$C^{(n)}$-extendibility" of κ in M, can be verified inside V_λ, that is,

$$M \models V_\lambda \models \text{“}\kappa \text{ is } C^{(n)}\text{-extendible”}.$$

Therefore, since $V_\lambda \subseteq M$, $V_\lambda \models$ "κ is $C^{(n)}$-extendible" and finally, since $\lambda \in C^{(n+2)}$, we get that κ is indeed $C^{(n)}$-extendible.

The case $n = 0$, connecting ordinary extendibility to joint supercompactness and κ-superstrongness, is similar. One here starts with a $\lambda \in C^{(3)}$, since the property of being extendible is Π_3-expressible; then, one checks that $M \models \lambda \in C^{(2)}$ and argues as above. □

We now immediately obtain the following characterization, for every natural number n.

Corollary 2.31. *A cardinal κ is $C^{(n)}$-extendible if and only if it is jointly $C^{(n)}$-supercompact and κ-superstrong.*

In particular, κ is jointly $C^{(n)}$-supercompact and superstrong if and only if it is jointly $C^{(n)}$-supercompact and κ-superstrong. In other words, for the global version, usual superstrongness of the supercompact embeddings is equivalent to the apparently stronger λ-superstrongness, for $\lambda \geqslant \kappa$.

Before concluding this section, we momentarily return to the supercompactness case. As the reader should have already noticed, the proof of theorem 2.28 gives a way of describing supercompact embeddings using ordinary (but long) extenders. Let us make this more precise here.

Corollary 2.32. *Suppose that $j : V \longrightarrow M$ is λ-supercompact for κ, for some $\lambda > \kappa$. Let $\theta = \beth_\theta \geqslant \lambda$ with $cof(\theta) > \kappa$ and let E be the $(\kappa, j(\theta))$-extender derived from j. Then, the extender embedding $j_E : V \longrightarrow M_E$ is λ-supercompact for κ with $j_E(\kappa) = j(\kappa)$.*

Proof. Fix $\lambda > \kappa$ and $j : V \longrightarrow M$, a λ-supercompact embedding for κ. Let $\theta = \beth_\theta \geqslant \lambda$ with $\mathrm{cof}(\theta) > \kappa$ and let E be the $(\kappa, j(\theta))$-extender derived from j. Then, recalling the proof of theorem 2.28 one easily checks that a similar idea goes through; namely, we consider the (in this case, full) third factor embedding k_E, arguing that $k_E \restriction V^{M_E}_{j_E(\theta)} = id$ and $V^{M_E}_{j_E(\theta)} = V^M_{j(\theta)}$. Moreover, for all $\alpha \leqslant \theta$, $j_E(\alpha) = j(\alpha)$ and also, since $\mathrm{cof}(j(\theta)) > \lambda$ (computed in V), we get that $j_E``\lambda = j``\lambda \in V^M_{j(\theta)}$ and $j_E``\lambda \in M_E$ as well. To conclude, we show closure under λ-sequences for M_E using the arguments of § 2.4, noting that the assumption $\mathrm{cof}(\theta) > \kappa$ implies that $\mathrm{cof}(j(\theta)) > \lambda$, which is exactly what is needed for these arguments to work in the current setting. □

Observe once more that if j happens to have some degree of superstrongness, then this is carried over to the extender embedding; e.g., in the above proof as it stands, if j was also θ-superstrong, then the same would be true for j_E. Moreover, since the arising extender embedding is such that $j_E(\kappa) = j(\kappa)$, the previous proof works for $C^{(n)}$-supercompactness as well, i.e., if $j : V \longrightarrow M$ witnessed the λ-$C^{(n)}$-supercompactness of κ, then j_E would also be λ-$C^{(n)}$-supercompact for κ.

3. Woodinness and strong compactness

We treat the $C^{(n)}$-versions of Woodin and strongly compact cardinals together, for two reasons: firstly, they were not considered in [1] where the rest of the $C^{(n)}$-cardinals were introduced; but most importantly, because they both admit similar constructions using iterated ultrapowers, completely characterizing them in terms of their ordinary counterparts.

3.1. $C^{(n)}$**-Woodins.** Recalling definition 1.5, we start by showing that $C^{(n)}$-Woodins form a large cardinal hierarchy of increasing strength.

Lemma 3.1. *For $n \geqslant 1$, if δ is $C^{(n+1)}$-Woodin then there are unboundedly many $C^{(n)}$-Woodins below δ. Hence, if δ is the least $C^{(n)}$-Woodin then it is not $C^{(n+1)}$-Woodin.*

Proof. Fix some $n \geqslant 1$ and suppose that δ is $C^{(n+1)}$-Woodin. Fix some $\alpha < \delta$ and some $f \in {}^{\delta}\delta$ with $f``\delta \cap \alpha = \varnothing$, and let $\kappa < \delta$ and $j : V \longrightarrow M$ witness $C^{(n+1)}$-Woodinness with respect to $f$, i.e., $f``\kappa \subseteq \kappa$, $\text{cp}(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$, $j(\delta) = \delta$ and $j(\kappa) \in C^{(n+1)}$. Clearly, $\alpha < \kappa < j(\kappa) < \delta$ and, moreover, the following is a true Σ_{n+1}-statement in the parameter α:

$$\exists\, \delta'(\delta' \in C^{(n)} \wedge V_{\delta'} \models \text{“}\exists \text{ some } C^{(n)}\text{-Woodin above } \alpha\text{”}),$$

since it holds for any $\delta' \in C^{(n)}$ above δ. Therefore, it must hold in $V_{j(\kappa)}$. But then, and since $j(\kappa) \in C^{(n+1)}$, any $\delta' < j(\kappa)$ which is a witness to this statement has to be Σ_n-correct and, consequently, there exists some $C^{(n)}$-Woodin cardinal above α (and below δ). □

The reader has probably noticed that the case $n = 0$ is conspicuously missing from this lemma. The following proposition explains why.

Proposition 3.2. *If δ is Woodin then it is $C^{(1)}$-Woodin.*

Proof. Suppose that δ is Woodin and fix some function $f \in {}^{\delta}\delta$. We further fix $\kappa < \delta$ with $f``\kappa \subseteq \kappa$ and a (κ, β)-extender $E \in V_\delta$ (for some $\beta < \delta$), such that $j = j_E : V \longrightarrow M_E$ has $\text{cp}(j) = \kappa$, $\beta \leqslant j(\kappa) < \delta$, $j(\delta) = \delta$ and $V_{j(f)(\kappa)} \subseteq M_E$. Working in M_E, since $j(\kappa)$ is measurable, let $\mathcal{U} \in M_E$ be a normal, $j(\kappa)$-complete measure on $j(\kappa)$ and let $j_{\mathcal{U}} : M_E \longrightarrow M$ be the ultrapower embedding. Then, $2^{j(\kappa)} < j_{\mathcal{U}}(j(\kappa)) < \left(2^{j(\kappa)}\right)^+ < \delta$, where the last inequality comes from $j(\kappa) < \delta = j(\delta)$ and the fact that δ is inaccessible.

Now let $\alpha \in (j_{\mathcal{U}}(j(\kappa)), \delta)$ be a true $C^{(1)}$-cardinal above $\left(2^{j(\kappa)}\right)^{M_E}$ (that is, $\alpha \in C^{(1)}$ in V). Still working in M_E, we now iterate the map $j_{\mathcal{U}}$ α-many times and let $j_\alpha : M_E \longrightarrow M_\alpha \cong \text{Ult}(M_E, \mathcal{U}_\alpha)$ be the resulting embedding. By standard facts, we get that $\text{cp}(j_\alpha) = j(\kappa)$, $V^{M_E}_{j(\kappa)} \subseteq M_\alpha$, $j_\alpha(j(\kappa)) = \alpha$, $j_\alpha(\delta) = \delta$ and the iterates are well-founded, so M_α is (taken to be) transitive. We then let $i = j_\alpha \circ j : V \longrightarrow M_\alpha$ be the composed elementary embedding with $\text{cp}(i) = \kappa$, $i(\kappa) = j_\alpha(j(\kappa)) = \alpha \in C^{(1)}$ and $i(\delta) = \delta$. Let us see that i indeed witnesses Woodinness for δ with respect to the given f. For this, observe that $i(f)(\kappa) = j_\alpha(j(f))(\kappa) = j_\alpha(j(f))(j_\alpha(\kappa)) = j_\alpha(j(f)(\kappa))$. But since $j(f)(\kappa) < j(\kappa)$, we get that $i(f)(\kappa) = j(f)(\kappa)$; therefore, $V_{i(f)(\kappa)} \subseteq M_\alpha$, since $V^{M_E}_{j(\kappa)} \subseteq M_\beta$, for all $\beta \leqslant \alpha$ along the iteration. $\square$

By a straightforward modification of the previous proof, we immediately get the following (schema).

Corollary 3.3. *If δ is Woodin and $\delta \in Lim(C^{(n)})$, then δ is $C^{(n)}$-Woodin.*

This corollary provides us with a characterization of $C^{(n)}$-Woodin cardinals. In fact, we shall give several formulations of $C^{(n)}$-Woodinness; before that, though, one more definition is in order. As usual, n is a fixed natural number.

Definition 3.4. Let κ be a cardinal, let $\lambda \geqslant \kappa$, and let A be any set. We say that κ is λ-$C^{(n)}$-**strong for** A if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, such that $\text{cp}(j) = \kappa$, $\lambda < j(\kappa)$, $V_\lambda \subseteq M$, $A \cap V_\lambda = j(A) \cap V_\lambda$ and $j(\kappa) \in C^{(n)}$.

The next theorem is based on Woodin's original result (see §26 in [7]).

Theorem 3.5. *The following are equivalent:*

(*i*) *δ is a $C^{(n)}$-Woodin cardinal.*
(*ii*) *δ is Woodin and $\delta \in Lim(C^{(n)})$.*
(*iii*) *For every $A \subseteq V_\delta$, the set*

$$S^{(n)}_A = \{\alpha < \delta : \alpha \text{ is } \gamma\text{-}C^{(n)}\text{-strong for } A, \text{ for every } \gamma < \delta\}$$

is stationary in δ.
(*iv*) *For every $f \in {}^\delta\delta$, there is a $\kappa < \delta$ with $f``\kappa \subseteq \kappa$ and an extender $E \in V_\delta$, so that $cp(j_E) = \kappa$, $j_E(f)(\kappa) = f(\kappa)$, $V_{f(\kappa)} \subseteq M_E$ and $j_E(\kappa) \in C^{(n)}$.*

Proof. The equivalence of (*i*) and (*ii*) follows from corollary 3.3. Let us first deal with the implication (*i*)$\Longrightarrow$(*iii*).

Suppose that δ is a $C^{(n)}$-Woodin cardinal (for some fixed n) and let $A \subseteq V_\delta$ be given. By Woodin's theorem, we know that the set

$$S_A = \{\alpha < \delta : \alpha \text{ is } \gamma\text{-strong for } A, \text{ for every } \gamma < \delta\}$$

is stationary in δ; we show that $S_A \subseteq S^{(n)}_A$. So fix some $\alpha \in S_A$ and some $\gamma < \delta$, and let $j : V \longrightarrow M$ witness the γ-strongness for A of α, i.e., $\text{cp}(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$ and $A \cap V_\gamma = j(A) \cap V_\gamma$. We may assume that $j(\alpha) < \delta = j(\delta)$ because if not, we may derive some $(\alpha, |V_\beta|^+)$-extender E (for some sufficiently large $\beta \in (\gamma, \delta)$) and work with j_E in place of j.

We now use again an iterated ultrapower argument. Since $\delta \in \text{Lim}(C^{(n)})$, let $\lambda > \left(2^{j(\alpha)}\right)^M$ with $\lambda \in C^{(n)}$ and –working in M– consider the ultrapower map $j_\lambda : M \longrightarrow M_\lambda$ with $\text{cp}(j_\lambda) = j(\alpha)$, $j_\lambda(j(\alpha)) = \lambda$, $j_\lambda(\delta) = \delta$ and with the iterates being well-founded, so M_λ is (taken to be) transitive. We then let $i = j_\lambda \circ j : V \longrightarrow M_\lambda$ be the composed elementary embedding with $\mathrm{cp}(i) = \alpha$, $\gamma < i(\alpha) = \lambda \in C^{(n)}$ and $i(\delta) = \delta$. Moreover, we have that $V_\gamma \subseteq M_\lambda$ because $V_\gamma \subseteq V^M_{j(\alpha)} \subseteq M_\lambda$. In order to check that this embedding witnesses γ-$C^{(n)}$-strongness for A, it remains to see that $A \cap V_\gamma = i(A) \cap V_\gamma$. For this, we have the following equalities:

$$i(A) \cap V_\gamma = j_\lambda(j(A)) \cap V_\gamma = j_\lambda(j(A) \cap V_\gamma) = j_\lambda(A \cap V_\gamma) = A \cap V_\gamma,$$

where $j_\lambda(V_\gamma) = V_\gamma$ because $V_\gamma \in V^M_{j(\alpha)}$ and similarly for the last equality. This concludes the proof of (*i*)$\Longrightarrow$(*iii*). Note that in this argument, given any $A \subseteq V_\delta$, any $\gamma < \delta$ and any $\alpha \in S_A$, the embedding i which witnesses the γ-$C^{(n)}$-strongness for A of α, can be taken to be an extender embedding and with $i(\alpha) < \delta = i(\delta)$. Given that, the final implication (*iii*)$\Longrightarrow$(*iv*) is an immediate consequence of the corresponding one in Woodin's theorem. □

3.2. $C^{(n)}$**-strongly compacts.** Recall that a cardinal κ is γ-compact, for some $\gamma \geqslant \kappa$, if and only if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$ and such that, for any $X \subseteq M$ with $|X| \leqslant \gamma$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$ (see §22 in [7]). We now show that γ-compact cardinals are actually γ-$C^{(n)}$-compact.

Theorem 3.6. *Suppose that, for some $\gamma \geqslant \kappa$, κ is γ-compact. Then, κ is γ-$C^{(n)}$-compact.*

Proof. We fix some (meta-theoretic) n and some $\gamma \geqslant \kappa$ and we then let $j : V \longrightarrow M$ be an embedding witnessing the γ-compactness of κ. Clearly, $j(\kappa) > \gamma$ and $j(\kappa)$ is measurable in M, so let $\mathcal{U} \in M$ be an M-normal measure on $j(\kappa)$. Fix some ordinal $\alpha \in C^{(n)}$ such that $\mathrm{cof}(\alpha) > \gamma$ and $\alpha > \left(2^{j(\kappa)}\right)^M$. We iterate the ultrapower construction inside M, starting with $\mathcal{U}$ and repeating for α-many steps. Let $j_\alpha : M \longrightarrow M_\alpha$ be the resulting embedding with $\mathrm{cp}(j_\alpha) = j(\kappa)$, where M_α is (taken to be) transitive. Again, by known facts, we have that $j_\alpha(j(\kappa)) = \alpha$ and we let $h : V \longrightarrow M_\alpha$ be the composed elementary embedding, i.e., $h = j_\alpha \circ j$, with $\mathrm{cp}(h) = \kappa$ and with $h(\kappa) = \alpha \in C^{(n)}$. We now check that h is γ-compact for κ.

Suppose that $X \subseteq M_\alpha$, and assume that $|X| = \gamma$ (in particular, γ is a cardinal). By the representation of iterated utrapowers we may assume that each $z \in X$ is of the form $j_\alpha(f)(\kappa_{\xi_1}, \ldots, \kappa_{\xi_m})$, where $f : [j(\kappa)]^m \longrightarrow M$ belongs to M and, for each $1 \leqslant i \leqslant m$, κ_{ξ_m} belongs to the critical sequence $\langle \kappa_\xi : \xi < \alpha \rangle$ (where $\kappa_0 = j(\kappa)$). Thus, $X = \{j_\alpha(f_i)(\vec{\kappa}_i) : i < \gamma\}$ where each $\vec{\kappa}_i$ is a finite tuple from the critical sequence. Now, since $\mathrm{cof}(\alpha) > \gamma$, there exists some ordinal $\delta < \alpha$ such that, for every $i < \gamma$, $\max(\vec{\kappa}_i) < \kappa_\delta$. Notice that $[\kappa_\delta]^{<\omega} \in M_\alpha$ and $M_\alpha \models |[\kappa_\delta]^{<\omega}| < \alpha$.

Moreover, since $\{f_i : i < \gamma\} \subseteq M$ and j is γ-compact, there is some $Y_0 \in M$ such that $\{f_i : i < \gamma\} \subseteq Y_0$ and $M \models |Y_0| < j(\kappa)$. It then follows that $\{j_\alpha(f_i) : i < \gamma\} \subseteq j_\alpha(Y_0) \in M_\alpha$ and $M_\alpha \models |j_\alpha(Y_0)| < \alpha$. Therefore, in M_α, we may use the set $j_\alpha(Y_0)$ and the set $[\kappa_\delta]^{<\omega}$ in order to define the desired Y that covers X. We let

$$Y = \{g(\vec{s}) : \exists\, m \in \omega(\text{“}g \in j_\alpha(Y_0) \text{ is a function on } [\alpha]^m\text{”} \ \wedge\ \vec{s} \in [\kappa_\delta]^m)\}.$$

We then have that $Y \in M_\alpha$, $X \subseteq Y$ and $M_\alpha \models |Y| < \alpha$. □

From the previous theorem, we immediately get the following (schema).

Corollary 3.7. *κ is strongly compact $\Longleftrightarrow$ κ is $C^{(n)}$-strongly compact.*

The obtained characterizations of $C^{(n)}$-Woodin and $C^{(n)}$-strongly compact cardinals do not seem to leave much space for investigating further these notions in their own right. On the other hand, for the rest of the notions that we have considered, the relation between the various large cardinals and their $C^{(n)}$-versions has not been fully clarified. For instance:

Question 3.8. Given a $C^{(n)}$-supercompact cardinal, can we force to “kill" its $C^{(n)}$-supercompactness while preserving its supercompactness?

Even for $n = 1$, it is unclear if this can be done. In addition, we may also wonder if the usual indestructibility results apply to our new setting.

Question 3.9. Can the $C^{(n)}$-supercompactness be made indestructible under various classes of forcing notions?[10]

4. $C^{(n)}$-CARDINALS AND FORCING

In this section, we briefly explore the interaction of $C^{(n)}$-cardinals with the forcing machinery. It will be convenient to work with *flat pairing functions*.[11] We first give a folklore result regarding names constructed by such functions.

Lemma 4.1 (Folklore). *Let $\mathbb{P}$ be a poset with rank$(\mathbb{P}) = \gamma \geqslant \omega$, and suppose that $\mathbb{P}$-names are constructed using a flat pairing function. Then, for any $G \subseteq \mathbb{P}$-generic over V and for any $\alpha \geqslant \gamma \cdot \omega$, we have that $V[G]_\alpha = V_\alpha[G]$.*

Proof. Let f be a fixed flat pairing function; we build the universe $V^{\mathbb{P}}$ recursively, as follows. We initially let $V_0^{\mathbb{P}} = \varnothing$. Given $V_\alpha^{\mathbb{P}}$ for some $\alpha \in \mathbf{ON}$, we let $V_{\alpha+1}^{\mathbb{P}} = V_\alpha^{\mathbb{P}} \cup \mathcal{P}(V_\alpha^{\mathbb{P}} \times \mathbb{P})$, where we use f in order to compute (the pairs in) the set $V_\alpha^{\mathbb{P}} \times \mathbb{P}$. For limit λ, we let $V_\lambda^{\mathbb{P}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{P}}$. Finally, the universe of $\mathbb{P}$-names is

$$V^{\mathbb{P}} = \bigcup_{\alpha \in \mathbf{ON}} V_\alpha^{\mathbb{P}}.$$

[10] This is hopeless in the case of ($C^{(n)}$-) superstrong or extendible cardinals; recent results show that such cardinals (among others) are never Laver indestructible; see [2].

[11] Such a function f plays the rôle of the Kuratowski pairing, with the additional property that for infinite α, if $x, y \in V_\alpha$ then the pair computed by f belongs to V_α; that is, unlike the Kuratowski pair, f does not increase the rank, except for the finite case.

An easy induction shows that, for every ordinal α, $V_\alpha^{\mathbb{P}} \subseteq V_{\gamma+\alpha}$. Now fix any $G \subseteq \mathbb{P}$-generic over V. Another inductive argument shows that, for every $\alpha \in \mathbf{ON}$, $V_\alpha^{\mathbb{P}}[G] = V[G]_\alpha$. Then, combining the two conclusions, for every ordinal α, $V[G]_\alpha \subseteq V_{\gamma+\alpha}[G]$ and the lemma follows. □

Let $\alpha \in C^{(n)}$, for some $n \geqslant 1$; suppose that $\mathbb{P} \in V_\alpha$ is a poset, fix some Σ_n-formula φ and let $p \in \mathbb{P}$. Then, the statement "$p \Vdash \varphi$" is Σ_n-expressible using $\mathbb{P}$ as a parameter, in V_α.[12]

Lemma 4.2. *Let $\mathbb{P}$ be a forcing notion and κ a cardinal.*

(*i*) *Suppose that $|\mathbb{P}| < \kappa$ and $\kappa \in C^{(n)}$, for some $n \geqslant 1$. Then, we have that $\mathbb{P} \Vdash \check{\kappa} \in C^{(n)}$.*

(*ii*) *Suppose that $\mathbb{P}$ does not change V_κ. Then, $\mathbb{P} \Vdash \check{\kappa} \in C^{(1)}$ if and only if $\kappa \in C^{(1)}$.*

(*iii*) *Suppose that $\mathbb{P}$ does not change V_κ and $\kappa \in C^{(2)}$. Then, for every $\lambda < \kappa$, if $\mathbb{P} \Vdash \check{\lambda} \in C^{(2)}$ then $\lambda \in C^{(2)}$.*

Proof. For (*i*), we may assume that rank$(\mathbb{P}) = \gamma < \kappa$ and that a flat pairing function has been used to construct the $\mathbb{P}$-names. Let G be $\mathbb{P}$-generic over V; by lemma 4.1, we have that $V_\kappa[G] = V[G]_\kappa$.

Fix some $n \geqslant 1$, some Π_{n-1}-formula $\phi(y, v_1, \ldots, v_k)$ and some $\mathbb{P}$-names $\dot{x}_1, \ldots, \dot{x}_k$, such that $V[G] \models \exists y \phi(y, (\dot{x}_1)_G, \ldots, (\dot{x}_k)_G)$, where $(\dot{x}_i)_G \in V[G]_\kappa$ for each $1 \leqslant i \leqslant k$. Then, there is $p \in G$ such that "$p \Vdash \exists y \phi(y, \dot{x}_1, \ldots, \dot{x}_k)$" holds. Since $V_\kappa[G] = V[G]_\kappa$, all the $\dot{x}_i$'s may be assumed to belong to V_κ. Therefore, as $\kappa \in C^{(n)}$, we get that $V_\kappa \models$ "$p \Vdash \exists y \phi(y, \dot{x}_1, \ldots, \dot{x}_k)$" and then, since $\mathbb{P} \in V_\kappa$,

$$V_\kappa[G] \models \exists y \phi(y, (\dot{x}_1)_G, \ldots, (\dot{x}_k)_G).$$

The same argument going backwards, shows that if $V_\kappa[G] = V[G]_\kappa$ satisfies a Σ_n-formula then the same is true for $V[G]$.

For (*ii*), we recall that membership in $C^{(1)}$ is Π_1-expressible and hence, the forward implication is true anyway. For the converse, assume that $\kappa \in C^{(1)}$, i.e., κ is an uncountable (strong limit) cardinal such that $V_\kappa = H_\kappa$. But if $V_\kappa = V[G]_\kappa$, it easily follows that $(V_\kappa)^{V[G]} = (H_\kappa)^{V[G]}$. Moreover, κ remains uncountable in $V[G]$ and thus, $V[G] \models \kappa \in C^{(1)}$.

For (*iii*), suppose that $\kappa \in C^{(2)}$ and that $\mathbb{P}$ does not change V_κ. Fix $G \subseteq \mathbb{P}$-generic over V and fix some $\lambda < \kappa$ so that $V[G] \models \lambda \in C^{(2)}$. By part (*ii*), $\lambda \in C^{(1)}$ and $V[G] \models \kappa \in C^{(1)}$. Now let ψ be a Σ_2-formula, whose parameter, if any, belongs to V_λ, and suppose that ψ holds in V. Then, $V[G]_\kappa = V_\kappa \models \psi$ and so, since κ is $C^{(1)}$ in the extension, $V[G] \models \psi$. Going downwards, we now get $V[G]_\lambda \models \psi$ by the correctness of λ in $V[G]$ and, then, $V_\lambda \models \psi$ by assumption on $\mathbb{P}$. □

[12] This fact requires that our model satisfies (Kripke–Platek and) Σ_n-collection along with Σ_n-separation. But this is true of the model V_α whenever $\alpha \in C^{(n)}$, for $n \geqslant 1$.

For $n > 1$, the lack of a (local) "combinatorial" characterization of membership in the class $C^{(n)}$ is a serious obstacle in showing the converse of (*i*), or of generalizing (*ii*) of lemma 4.2. Note that (*iii*) is only a partial result in this direction. The converse of (*iii*) does not necessarily hold since the following situation is consistent.

Relative to assumptions at the level of *hyper-measurability*, it is consistent that GCH holds at successor cardinals but fails at limits.[13] Suppose we are in such a model V and consider any $\kappa \in C^{(2)}$. Now force the GCH at κ by the canonical poset to add one subset to κ^+. This does not change V_κ. Then, κ and, in fact, all cardinals below it which were Σ_2-correct in V, are no longer Σ_2-correct in the extension, although all $C^{(1)}$'s below κ are preserved.

Let us finally turn to the preservation of (some of) the $C^{(n)}$-large cardinals considered so far, under appropriate forcings.

Lemma 4.3. *Suppose that κ is $C^{(n)}$-tall (superstrong, supercompact, extendible), for some $n \geqslant 1$, and let $\mathbb{P}$ be a poset with $|\mathbb{P}| < \kappa$. Then, κ remains $C^{(n)}$-tall (resp. superstrong, supercompact, extendible) in $V^{\mathbb{P}}$.*

Proof. Assume that $\mathbb{P} \in V_\kappa$ and consider the case of supercompactness. Fix an $n \geqslant 1$ and a (cardinal) $\lambda > \kappa$ and let $j : V \longrightarrow M$ witness the λ-$C^{(n)}$-supercompactness of κ in V. We let G be $\mathbb{P}$-generic over V and, by standard arguments, the embedding lifts to $j : V[G] \longrightarrow M[G]$ with $V[G] \models {}^{\lambda}M[G] \subseteq M[G]$. Moreover, by lemma 4.2(*i*), $V[G] \models j(\kappa) \in C^{(n)}$ and hence κ remains λ-$C^{(n)}$-supercompact in the extension. For tallness, the argument is essentially the same. For superstrongness, considering again a flat pairing function, we have that $V[G]_{j(\kappa)} = V_{j(\kappa)}[G]$ and hence, $V[G]_{j(\kappa)} \subseteq M[G]$ by virtue of $V_{j(\kappa)} \subseteq M \subseteq M[G]$.

Finally, by corollary 2.31, extendibility is a straightforward combination of the cases of supercompactness and superstrongness. □

We now consider preservation of $C^{(n)}$-tall and $C^{(n)}$-supercompact cardinals under sufficiently distributive forcing notions.

Lemma 4.4. *Suppose that κ is $C^{(n)}$-tall and let $\mathbb{P}$ be a $\leqslant\kappa$-distributive forcing. Then, κ remains $C^{(n)}$-tall in $V^{\mathbb{P}}$.*

Proof. Fix any $\lambda > \max\{\kappa, |\mathbb{P}|\}$ and let $j : V \longrightarrow M = \{j(f)(x) : f \in V,\ f : V_\kappa \longrightarrow V,\ x \in V^M_{j(\kappa)}\}$ be a λ-$C^{(n)}$-tallnness embedding for κ. By standard facts, j lifts through any $\leqslant\kappa$-distributive forcing (see, e.g., § 15 in [4]). Thus, and using lemma 4.2, κ remains λ-$C^{(n)}$-tall in the extension. □

Lemma 4.5. *Suppose that κ is $C^{(n)}$-supercompact and supppose that $\mathbb{P}$ is a $\leqslant \lambda^{<\kappa}$-distributive forcing, for some (cardinal) $\lambda > \kappa$. Then, κ remains λ-$C^{(n)}$-supercompact in $V^{\mathbb{P}}$.*

[13] More precisely, given a $\mathcal{P}^3(\lambda)$-hyper-measurable cardinal λ, there is a model of ZFC in which $2^\alpha = \alpha^{++}$ for every limit cardinal α, whereas GCH holds everywhere else; see [3].

Proof. Fix some n and some $\theta > \max\{\lambda, |\mathbb{P}|\}$ and let $j : V \longrightarrow M$ witness the θ-$C^{(n)}$-supercompactness of κ. Now consider the substructure

$$X_0 = \{j(f)(j``\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{j(\kappa)}\} \prec M,$$

which gives rise to a (not necessarily initial) factor λ-supercompact embedding $j_0 : V \longrightarrow M_0$, via the Mostowski collapse $\pi_0 : X_0 \cong M_0$ as usual. Since $j_0(\kappa) = j(\kappa)$, j_0 is moreover λ-$C^{(n)}$-supercompact for κ.

Let G be $\mathbb{P}$-generic over V. Again, using the representation of M_0 and the distributivity of $\mathbb{P}$, standard arguments show that j_0 lifts through the forcing, witnessing the θ-$C^{(n)}$-supercompactness of κ in $V[G]$. □

We conclude with a (somewhat informal) question.

Question 4.6. Suppose that κ is $C^{(n)}$-extendible. Under what types of forcing notions is the (local or global) $C^{(n)}$-extendibility of κ preserved? For instance, can we force the GCH while preserving such cardinals?[14]

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[14] For $n \in \{0, 1\}$, i.e., for ordinary extendibility, the answer is affirmative: the canonical (class-length) iteration which forces global GCH preserves every extendible cardinal of the universe (see [9]).