INTRODUCTION TO EXTENDERS

KONSTANTINOS TSAPROUNIS

Abstract. In these notes we present the basic theory of extenders; our exposition is based on standard text references, such as Kanamori [4] and the Martin-Steel classics [7] and [8]. In particular, we develop the basic theory and applications of ordinary extenders and, moreover, we present an account of (generalized) Martin-Steel extenders, together with one important application of the latter objects in the context of supercompact cardinals.

1. Introduction

The idea of an extender arose from a work of Mitchell; the name and the current formulation, though, were first introduced by Dodd and Jensen (see [2]) who simplified Mitchell's notion. The basic motivation for considering such objects was the desire to "combinatorially approximate" a given elementary embedding $j : V \longrightarrow M$ between inner models, in a similar way in which usual ultrapowers capture measurability embeddings. Indeed, the notion of an extender generalizes that of a normal measure and is devised for embeddings which have strength (typically) at the level of strong, superstrong, and Woodin cardinals.

As it turns out, elementary embeddings for such large cardinal notions can be approximated via suitable *sequences of measures* which are extracted from the given $j : V \longrightarrow M$. Any such sequence, which is called an extender and is usually denoted by E, enables us to construct a model M_E and an elementary embedding $j_E : V \longrightarrow M_E$ in a way which is nicely definable from E; moreover, if E is chosen carefully, then j_E and M_E closely resemble the initial j and M (in particular, they witness the same large cardinal strength for $\kappa = \mathrm{cp}(j) = \mathrm{cp}(j_E)$).

Once one becomes familiar with the aforementioned procedure of "extracting" such an E from a given embedding, (s)he may abstract the essential features of it and arrive at a general definition of an extender, the study of which has proved to be rather central in inner model theory. Although in this introductory exposition we do not deal with any deep applications, we nevertheless give some of the most basic uses of such objects. In particular, we provide equivalent characterizations of various large cardinals (e.g.,

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strong, superstrong, and Woodin) in terms of extenders, which render these notions formalizable in the language of ZFC.

The structure of this document is as follows. In Section 2 we introduce extenders derived from given elementary embeddings, and then present some of their basic properties. Everything is done from scratch and in detail, aiming at a better understanding of the underlying concepts.

The token is then passed on to Section 3, where the general definition of an extender is given, followed by an extensive discussion regarding the importance and consequences of the defining clauses. The parallel between the two sorts of extenders is drawn throughout this part, until we eventually establish their formal connection towards the end of the section. Having built the whole "extender machinery", we then continue with Section 4 where we deal with standard applications regarding large cardinals. In all of the first four sections we follow closely the corresponding material from § 26 of [4], filling in many details in the various proofs.

Finally, a not-so-standard discussion of Martin-Steel extenders is the content of Section 5. We introduce such generalized extenders and their properties, and we then concern ourselves with the problem of "capturing" a (λ-)supercompact embedding via an extender construction; once this is accomplished, we close the current notes with some concluding remarks. Our notation and terminology are mostly standard; see [3] or [4] for an account of any undefined set-theoretic concept.

It should be emphasized that none of the results and techniques presented in these notes are due to the author.

2. Extenders derived from an embedding

Before continuing any further, the reader is advised to review the construction and the basic properties of the ultrapower of the universe via a given normal measure on a measurable cardinal κ (see, e.g., § 17 in [3]).

Suppose that $j : V \longrightarrow M$ is an elementary embedding into a transitive (inner model) M with $\mathrm{cp}(j) = \kappa$. Let us pick some λ with $\kappa < \lambda \leqslant j(\kappa)$. For each $a \in [\lambda]^{<\omega}$, we define an ultrafilter E_a on $[\kappa]^{|a|}$ by letting:

$$X \in E_a \longleftrightarrow a \in j(X).$$

Note that if $X \subseteq [\kappa]^{|a|}$ then $j(X) \subseteq [j(\kappa)]^{|a|}$, so this definition makes sense. It is easy to check that E_a is a κ-complete ultrafilter and that E_a is principal (exactly) when $a \in [\kappa]^{<\omega}$.

Definition 2.1. In the setting described above, $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ is called **the (κ, λ)-extender derived from** j.

Using the extender ultrafilters, we may construct the corresponding ultrapowers as usual. Note that κ-completeness implies that these ultrapowers will be well-founded and thus, for each $a \in [\lambda]^{<\omega}$, we let $M_a \cong \mathrm{Ult}_{E_a}(V)$ be the transitive collapse as usual. Each such construction comes along with a pair of elementary embeddings that make the following diagram to commute:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_a"'] & M \\ M_a \arrow[ur, "k_a"'] & \end{tikzcd}$$

$$j_a(x) = [c^a_x]_{E_a}, \text{ for each } x \in V$$

$$k_a([f]_{E_a}) = j(f)(a), \text{ for each } f : [\kappa]^{|a|} \longrightarrow V$$

where $c^a_x : [\kappa]^{|a|} \to \{x\}$ is the constant function. For the sake of completeness, let us check that for each $a \in [\lambda]^{<\omega}$, the embedding k_a is well-defined, elementary and that it commutes:

Well-defined : If $[f]_{E_a} = [g]_{E_a} \in M_a$, i.e., $\{s \in [\kappa]^{|a|} : f(s) = g(s)\} \in E_a$, then by the definition of the ultrafilter, we equivalently have that $a \in \{s \in [j(\kappa)]^{|a|} : j(f)(s) = j(g)(s)\}$, i.e., $j(f)(a) = j(g)(a)$ and thus, $k_a([f]_{E_a}) = k_a([g]_{E_a})$.

Elementary : Let $\varphi(x_1, \ldots, x_n)$ be a formula and $[f_1]_{E_a}, \ldots, [f_n]_{E_a} \in M_a$, where for each $1 \leqslant i \leqslant n$, $f_i : [\kappa]^{|a|} \longrightarrow V$. Then,

$$\begin{aligned} M_a \models \varphi([f_1]_{E_a}, \ldots, [f_n]_{E_a}) \quad &\leftrightarrow \quad \{s : \varphi(f_1(s), \ldots, f_n(s)) \text{ holds}\} \in E_a \\ &\leftrightarrow \quad a \in \{s : \varphi^M(j(f_1)(s), \ldots, j(f_n)(s))\} \\ &\leftrightarrow \quad M \models \varphi(j(f_1)(a), \ldots, j(f_n)(a)) \\ &\leftrightarrow \quad M \models \varphi(k_a([f_1]_{E_a}), \ldots, k_a([f_n]_{E_a})). \end{aligned}$$

Commutes : This is immediate since, for $x \in V$, $j(c^a_x) : [j(\kappa)]^{|a|} \to \{j(x)\}$ is the constant function and thus, $k_a \circ j_a(x) = k_a([c^a_x]_{E_a}) = j(c^a_x)(a) = j(x)$.

So far, we have not done anything essentially different from the standard ultrapower construction in the case of measurable cardinals. The power of the new concept of an extender comes from the way in which the M_a's are interrelated, and to which we now turn our attention.

For every $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we define a "projection" function π_{ba} in the following manner: let $b = \{\xi_1, \ldots, \xi_n\}$, where we always assume that $\xi_1 < \ldots < \xi_n$, and $a = \{\xi_{i_1}, \ldots, \xi_{i_m}\}$ where again $1 \leqslant i_1 < \ldots < i_m \leqslant n$. Now, define $\pi_{ba} : [\kappa]^{|b|} \longrightarrow [\kappa]^{|a|}$ by:

$$\pi_{ba}(\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\},$$

i.e., we "project down" to a subset, according to the relation between the finite sets a and b.

Using the projections, we are about to establish the way in which the ultrapowers interact. Not surprisingly, something about the corresponding ultrafilters has to be said first.

Coherence property

For all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we have that:

$$X \in E_a \longleftrightarrow \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b.$$

To see that the coherence property holds in the case of a derived extender E, it is enough to notice that $j(\pi_{ba})(b) = a$, which follows from the definition of the projection function π_{ba}.

We are now in position to give the elementary embeddings which relate the various models M_a. So, for every $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we define the map $i_{ab} : M_a \longrightarrow M_b$ by letting:

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}, \text{ for all } f : [\kappa]^{|a|} \longrightarrow V.$$

Naturally, a corresponding commutative diagram is formed:

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[d, "j_a"'] \arrow[ddr, "j_b"] & & M \\
M_a \arrow[urr, "k_a"] \arrow[dr, "i_{ab}"'] & & \\
& M_b \arrow[uur, "k_b"'] &
\end{tikzcd}$$

$$k_a \circ j_a = k_b \circ j_b = j$$

$$k_b \circ i_{ab} = k_a$$

$$i_{ab} \circ j_a = j_b$$

In order to illustrate the effect of the coherence property on the interaction between the ultrapowers, we show that the embeddings i_{ab} are well-defined, elementary and commute:

Well-defined : If $[f]_{E_a} = [g]_{E_a} \in M_a$, i.e., $\{s \in [\kappa]^{|a|} : f(s) = g(s)\} \in E_a$, then by coherence of E_a and E_b (since $a \subseteq b$), we equivalently have that $\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) = g(\pi_{ba}(s))\} \in E_b$, i.e., $[f \circ \pi_{ba}]_{E_b} = [g \circ \pi_{ba}]_{E_b}$ and thus, $i_{ab}([f]_{E_a}) = i_{ab}([g]_{E_a})$.

Elementary : Let $\varphi(x)$ be a formula (a single free variable is taken for simplicity; the argument is the same for the general case) and $[f]_{E_a} \in M_a$, where $f : [\kappa]^{|a|} \longrightarrow V$. Then,

$$\begin{array}{rcl}
M_a \models \varphi([f]_{E_a}) & \leftrightarrow & \{s \in [\kappa]^{|a|} : \varphi(f(s)) \text{ holds}\} \in E_a \\
& \leftrightarrow & \{s \in [\kappa]^{|b|} : \varphi(f(\pi_{ba}(s))) \text{ holds}\} \in E_b \\
& \leftrightarrow & M_b \models \varphi([f \circ \pi_{ba}]_{E_b}) \\
& \leftrightarrow & M_b \models \varphi(i_{ab}([f]_{E_a})).
\end{array}$$

Commutes : As we have already observed, $j(\pi_{ba})(b) = a$ and hence, using the definition of the k_a's, we have that for every $f : [\kappa]^{|a|} \longrightarrow V$,

$$\begin{array}{rcl}
k_b \circ i_{ab}([f]_{E_a}) & = & k_b([f \circ \pi_{ba}]_{E_b}) \\
& = & j(f \circ \pi_{ba})(b) \\
& = & j(f)(j(\pi_{ba})(b)) \\
& = & j(f)(a) = k_a([f]_{E_a}).
\end{array}$$

Finally, $i_{ab} \circ j_a(x) = i_{ab}([c^a_x]_{E_a}) = [c^a_x \circ \pi_{ba}]_{E_b} = [c^b_x]_{E_b} = j_b(x)$, for every $x \in V$.

At this point we can form $\langle\langle M_a : a \in [\lambda]^{<\omega}\rangle; \langle i_{ab} : a \subseteq b \in [\lambda]^{<\omega}\rangle\rangle$, which is easily seen to be a *directed system*. Consequently, we can construct the corresponding *direct limit*, $\widetilde{M_E} = \langle D_E, \in_E\rangle$. This is a standard procedure and may be described as follows:

- We define the equivalence relation $\sim_E$ on $\displaystyle\bigcup_{a\in[\lambda]^{<\omega}} \{a\} \times M_a$ by:

$$\langle a, [f]_{E_a}\rangle \sim_E \langle b, [g]_{E_b}\rangle \longleftrightarrow \exists\, c \supseteq a \cup b \text{ s.t. } i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b}).$$

  The (Scott) equivalence class of the pair $\langle a, [f]_{E_a}\rangle$, will be denoted by $[\langle a, [f]_{E_a}\rangle]_E$. We then let $D_E = \left(\displaystyle\bigcup_{a\in[\lambda]^{<\omega}} \{a\} \times M_a\right) \Big/ \sim_E$, which is the domain of the direct limit.
- In a similar manner, we define membership $\in_E$ by:

$$[\langle a, [f]_{E_a}\rangle]_E \in_E [\langle b, [g]_{E_b}\rangle]_E \longleftrightarrow \exists\, c \supseteq a \cup b \text{ s.t. } i_{ac}([f]_{E_a}) \in i_{bc}([g]_{E_b}).$$

It is obvious from the construction that every element $x \in \widetilde{M_E}$ is of the form $x = [\langle a, [f]_{E_a}\rangle]_E$, for some $a \in [\lambda]^{<\omega}$ and some $[f]_{E_a} \in M_a$, where $f : [\kappa]^{|a|} \longrightarrow V$.

Let us also remark that, by what we have previously shown, one obtains the following equivalents for equality and membership in the direct limit structure:

$$\begin{aligned}
[\langle a, [f]_{E_a}\rangle]_E =_E [\langle b, [g]_{E_b}\rangle]_E \quad &\leftrightarrow \quad \exists\, c \supseteq a \cup b : i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b}) \\
&\leftrightarrow \quad \exists\, c \supseteq a \cup b : [f \circ \pi_{ca}]_{E_c} = [g \circ \pi_{cb}]_{E_c} \\
&\leftrightarrow \quad \exists\, c \supseteq a \cup b : j(f \circ \pi_{ca})(c) = j(g \circ \pi_{cb})(c) \\
&\leftrightarrow \quad j(f)(a) = j(g)(b)
\end{aligned}$$

and similarly, $[\langle a, [f]_{E_a}\rangle]_E \in_E [\langle b, [g]_{E_b}\rangle]_E \leftrightarrow j(f)(a) \in j(g)(b)$.

In order to avoid unnecessary formalistic complication, in what follows we feel free to supress continued brackets and subscripts of the form E_a, E. Thus, when we write, e.g., $[a, [f]] \in [b, [g]]$ in $\widetilde{M_E}$, what we really mean is that $[f] = [f]_{E_a} \in M_a$, $[g] = [g]_{E_b} \in M_b$ and $[\langle a, [f]_{E_a}\rangle]_E \in_E [\langle b, [g]_{E_b}\rangle]_E$. At any rate, the intended meaning should always be clear from the context.

Our next goal is to define elementary embeddings interconnecting all the structures we have considered so far and then, establish some basic properties of the direct limit structure. Before we do this, though, we show that the constructed direct limit is well-founded which will enable us to work with its transitive collapse and, at the same time, justifies some of the formalistic simplifications mentioned above.

Lemma 2.2. *The direct limit $\widetilde{M_E}$ is well-founded.*

Proof. Suppose that in $\widetilde{M_E}$ there are elements $x_n = [a_n, [f_n]]$ which form an $\in_E$-descending chain, i.e., $x_{n+1} \in_E x_n$, for all $n \in \omega$.

By the equivalent of membership that we stated above, we have that for all $n \in \omega$, $j(f_{n+1})(a_{n+1}) \in j(f_n)(a_n)$, which is an infinite descending chain in the well-founded transitive model M. Contradiction. $\square$

Therefore, we may conveniently work with M_E, the transitive collapse of $\widetilde{M_E}$. We can now define the desired elementary embeddings k_{aE}, j_E and k_E, as shown in the following commutative diagrams:

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[d, "j_a"'] \arrow[ddr, "j_E"] & & M \\
M_a \arrow[urr, "k_a"] \arrow[dr, "k_{aE}"'] & & \\
& M_E \arrow[uur, "k_E"'] &
\end{tikzcd}$$

$$j_E(x) = [a, [c^a_x]], \text{ for each } x \in V, \text{ and for some (any) } a \in [\lambda]^{<\omega}$$

$$k_{aE}([f]) = [\, a, [f]\,], \text{ for each } a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \longrightarrow V$$

$$k_E([\, a, [f]\,]) = j(f)(a), \text{ for } a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \longrightarrow V$$

$$\begin{tikzcd}
M_a \arrow[rr, "k_{aE}"] \arrow[dr, "i_{ab}"'] & & M_E \\
& M_b \arrow[ur, "k_{bE}"'] &
\end{tikzcd}$$

For all $a \subseteq b \in [\lambda]^{<\omega}$, $k_{bE} \circ i_{ab} = k_{aE}$

Let us first point out that in the definition of $j_E(x)$ it really does not matter which $a \in [\lambda]^{<\omega}$ we choose, since, if $a \neq a'$ and we let $b = a \cup a'$, then $[c^a_x \circ \pi_{ba}] = [c^{a'}_x \circ \pi_{ba'}] = [c^b_x]$ in M_b and thus, $[a, [c^a_x]] = [a', [c^{a'}_x]]$. (Equivalently, the latter holds because $j(c^a_x)(a) = j(c^{a'}_x)(a') = j(x)$).

The previous remarks show, in addition, that the embeddings k_E and k_{aE} are well-defined (for the latter, recall also the definition of the ultrafilter E_a). The fact that all the aforementioned embeddings commute, comes from straightforward computations, similar to the ones we have already done. Hence, we will not repeat them here.

We now establish the elementarity of k_{aE} from which, the elementarity of j_E and k_E will follow.

Lemma 2.3. *For every $a \in [\lambda]^{<\omega}$, k_{aE} is an elementary embedding.*

Proof. We proceed inductively on the complexity of the formulas (for all $a \in [\lambda]^{<\omega}$ simultaneously). Let $[f], [g] \in M_a$, for some $a \in [\lambda]^{<\omega}$, where both functions f and g are on $[\kappa]^{|a|}$. By the previous discussion,

$$M_a \models [f]=[g] \longleftrightarrow j(f)(a)=j(g)(a) \longleftrightarrow M_E \models [a,[f]]=[a,[g]]$$

(and similarly for memberhip), i.e., elementarity holds for the atomic formulas. Moreover, the cases of negation and conjuction are immediate. So, suppose that elementarity holds for $\varphi(x,y)$ and let $[g] \in M_a$, some a. We have the following:
On the one hand,

$$\begin{array}{rcl} M_a \models \exists\, x\, \varphi(x,[g]) & \longrightarrow & M_a \models \varphi([f],[g]) \text{ some } [f] \in M_a \\ & \stackrel{\text{I.H.}}{\longrightarrow} & M_E \models \varphi(k_{aE}([f]),k_{aE}([g])) \\ & \longrightarrow & M_E \models \exists\, x\, \varphi(x,k_{aE}([g])). \end{array}$$

Conversely, suppose that $M_E \models \exists\, x\, \varphi(x,k_{aE}([g]))$, i.e., there exists some $[b,[f]] \in M_E$ such that $M_E \models \varphi([b,[f]],k_{aE}([g]))$. Let $c=a\cup b$. Using that $k_{cE}\circ i_{ac}=k_{aE}$, $k_{cE}\circ i_{bc}=k_{bE}$, our inductive hypothesis and the fact that i_{ac} is elementary, we have the following:

$$\begin{array}{rcl} M_E \models \varphi([b,[f]],k_{aE}([g])) & \longrightarrow & M_E \models \varphi(k_{bE}([f]),k_{aE}([g])) \\ & \longrightarrow & M_E \models \varphi\left(k_{cE}\circ i_{bc}([f]),k_{cE}\circ i_{ac}([g])\right) \\ & \stackrel{\text{I.H.}}{\longrightarrow} & M_c \models \varphi\left(i_{bc}([f]),i_{ac}([g])\right) \\ & \longrightarrow & M_c \models \exists\, x\, \varphi(x,i_{ac}([g])) \\ & \longrightarrow & M_a \models \exists\, x\, \varphi(x,[g]). \end{array}$$

□

Corollary 2.4. *j_E and k_E are elementary embeddings.*

Proof. This is rather straightforward, using the elementarity of k_{aE}'s and the commutativity properties. Let $\varphi(x)$ be any formula.

- Let $x\in V$. We have:

$$\begin{array}{rcll} \varphi(x) \text{ holds} & \longleftrightarrow & M_a \models \varphi(j_a(x)) & \text{(by elementarity of } j_a) \\ & \longleftrightarrow & M_E \models \varphi(k_{aE}\circ j_a(x)) & \text{(by elementarity of } k_{aE}) \\ & \longleftrightarrow & M_E \models \varphi(j_E(x)) & \text{(by commutativity)} \end{array}$$

- Let $x=[a,[f]]\in M_E$. We have:

$$\begin{array}{rcll} M_E \models \varphi(x) & \longleftrightarrow & M_E \models \varphi(k_{aE}([f])) & \text{(by definition of } k_{aE}) \\ & \longleftrightarrow & M_a \models \varphi([f]) & \text{(by elementarity of } k_{aE}) \\ & \longleftrightarrow & M \models \varphi(k_a([f])) & \text{(by elementarity of } k_a) \\ & \longleftrightarrow & M \models \varphi(k_E(x)) & \text{(by definition of } k_a, k_E) \end{array}$$

□

After all the previous discussion, we are finally in the position to establish the basic properties of the embedding j_E and the structure M_E.

Proposition 2.5. [Properties of j_E, M_E]

(*i*) $\mathrm{cp}(k_E)\geqslant\lambda$. *Thus,* $\mathrm{cp}(j_E)=\kappa$ *and* $j_E(\kappa)\geqslant\lambda$.
(*In particular, if* $\lambda=j(\kappa)$ *then* $\mathrm{cp}(k_E)>\lambda$ *and* $j_E(\kappa)=j(\kappa)=\lambda$*).*

(*ii*) $M_E=\{j_E(f)(a) : a\in[\lambda]^{<\omega}, f:[\kappa]^{|a|}\longrightarrow V, f\in V\}$.

(iii) *If, for some ordinal* γ, $(|V_\gamma| \leqslant \lambda)^M$, *then* $V_\gamma^M = V_\gamma^{M_E} \subseteq range(k_E)$ *and* $k_E \upharpoonright V_\gamma^{M_E} = id$.

Proof. As we have already seen, for every $x \in M_E$ there is some $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \longrightarrow V$ such that $x = k_{aE}([f])$. Thus,

$$k_E(x) = k_E(k_{aE}([f])) = k_a([f]) = j(f)(a)$$

and therefore,

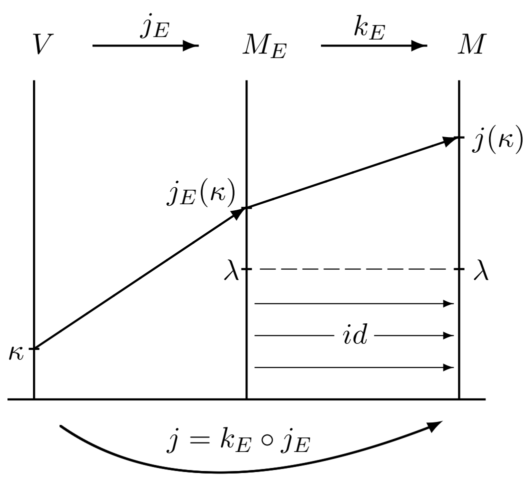
$$(\star) \qquad range(k_E) = \{j(f)(a) : a \in [\lambda]^{<\omega}, f : [\kappa]^{|a|} \longrightarrow V\}.$$

For (i), let $\alpha < \lambda$. Put $a = \{\alpha\} \in [\lambda]^1$ and consider the identity function $f = id^1 : [\kappa]^1 \longrightarrow [\kappa]^1$. Obviously, $a = j(f)(a)$ and it follows by $(\star)$ that $a = \{\alpha\} \in range(k_E)$. It is worth noticing here that the exact same argument shows $[\lambda]^{<\omega} \subseteq range(k_E)$ as well.

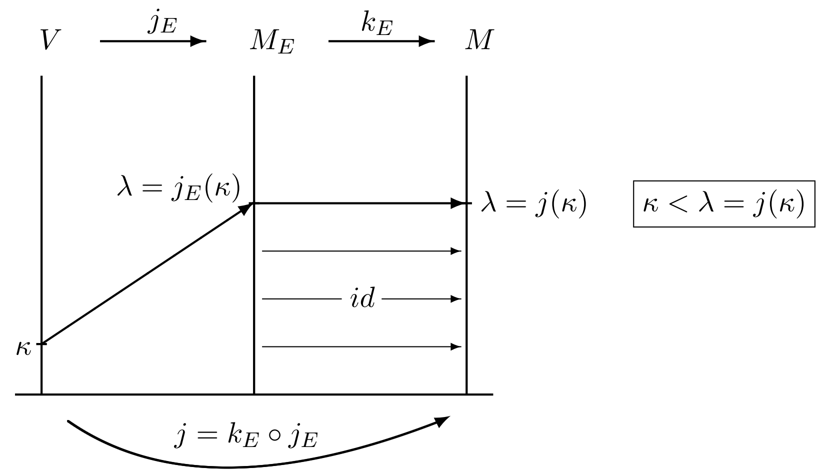
It is now easily seen that, by elementarity, $\{\alpha\} \in range(k_E)$ implies $\alpha \in range(k_E)$ and hence, we have that $\lambda \subseteq range(k_E)$. This shows that $\mathrm{cp}(k_E) \geqslant \lambda$.

For the properties of the critical point of j_E and the "in particular" part, we have the following. On the one hand, it cannot be that $\mathrm{cp}(j_E) = \alpha < \kappa$ because we would then have that $\alpha < j_E(\alpha) \leqslant k_E(j_E(\alpha)) = j(\alpha) = \alpha$. On the other hand, it cannot be that $\mathrm{cp}(j_E) > \kappa$ either, because then, since $k_E(\kappa) = \kappa$, we would have that $j(\kappa) = k_E(j_E(\kappa)) = k_E(\kappa) = \kappa$, contradicting $\mathrm{cp}(j) = \kappa$. Hence, we conclude that $\mathrm{cp}(j_E) = \kappa$.

Finally, $k_E \upharpoonright \lambda = id$, $j(\kappa) = k_E(j_E(\kappa))$ and $\lambda \leqslant j(\kappa)$ imply that $j_E(\kappa) \geqslant \lambda$. The figures below should clarify the situation; the "in particular" part is depicted in the second one:



$\boxed{\kappa < \lambda < j(\kappa)}$



For (ii), as we mentioned above, $[\lambda]^{<\omega} \subseteq range(k_E)$. In fact, $k_E \restriction \lambda = id$ implies that for every $a \in [\lambda]^{<\omega}$, $k_E(a) = a$.

Using $(\star)$ and these observations, we have that for $x = [a, [f]] \in M_E$,

$$k_E(x) = j(f)(a) = k_E \circ j_E(f)(a) = k_E(j_E(f))(k_E(a)) = k_E(j_E(f)(a))$$

and, consequently, since k_E is injective, $x = j_E(f)(a)$ which shows the desired property.

For (iii), let us fix a function $g : [\kappa]^1 \longrightarrow V$ with the property that, for any ordinal α, if $|V_\alpha| \leqslant \kappa$ then $g \restriction [|V_\alpha|]^1 : [|V_\alpha|]^1 \longrightarrow V_\alpha$ is a bijection. This can be done: since κ is inaccessible, $|V_\kappa| = \kappa$ and $|V_\alpha| < \kappa$, for all $\alpha < \kappa$, so one can define g appropriately for the cumulative V_α's, for $\alpha \leqslant \kappa$. Note that by elementarity, $j(g) : [j(\kappa)]^1 \longrightarrow M$ is such that, for any ordinal α, if $(|V_\alpha| \leqslant j(\kappa))^M$ then $j(g) \restriction [|V_\alpha|^M]^1 : [|V_\alpha|^M]^1 \longrightarrow V_\alpha^M$ is a bijection.

Now suppose that, for some ordinal γ, $(|V_\gamma| \leqslant \lambda)^M$. Since $\lambda \leqslant j(\kappa)$, by the bijective property of $j(g) \restriction [|V_\gamma|^M]^1$ we have that for every $x \in V_\gamma^M$, there is some $\xi < \lambda$ such that $j(g)(\{\xi\}) = x$. But then, according to $(\star)$, this means that $x \in range(k_E)$, i.e., we have shown that $V_\gamma^M \subseteq range(k_E)$.

To conclude the proof, note that $k_E^{-1} : range(k_E) \longrightarrow M_E$ is just the collapsing isomorphism and hence, $V_\gamma^M = V_\gamma^{M_E}$ and $k_E \restriction V_\gamma^{M_E} = id$. $\square$

Finally, before we turn to the general definition of an extender (i.e., regardless of a given embedding j), let us point out one important observation. As we have already seen, for any $a \in [\lambda]^{<\omega}$, $k_E(a) = a$. From this, it follows that for any $X \subseteq [\kappa]^{|a|}$,

$$a \in j(X) \longleftrightarrow a \in k_E(j_E(X)) \longleftrightarrow k_E(a) \in k_E(j_E(X)) \longleftrightarrow a \in j_E(X),$$

i.e., if we try to define the *new* (κ, λ)-extender E' derived from j_E, we have that $E' = E$.

3. General theory of extenders

After having discussed extenders derived from an ambient elementary embedding, we now switch to the general definition of an extender which, as we will see, bares all the essential features that came up in our previous study.

Following the formal definition, we will comment on the importance of these features, establish the connection between them and the extenders studied in the previous section and, then, derive a similar elementary embedding $j_E : V \longrightarrow M_E$ and give several of its properties.

Definition 3.1. Let κ be a regular cardinal and fix $\lambda > \kappa$. We say that $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ is a (κ, λ)-**extender** if the following conditions hold:

(1) (*i*) For all $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$.
 (*ii*) There is some $a \in [\lambda]^{<\omega}$, such that E_a is not κ^+-complete.
 (*iii*) For all $\gamma < \kappa$, there is $a \in [\lambda]^{<\omega}$ such that $\{s : \gamma \in s\} \in E_a$.

(2) (Coherence) For all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$,
$$X \in E_a \longleftrightarrow \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b$$
(where $\pi_{ba} : [\kappa]^{|b|} \longrightarrow [\kappa]^{|a|}$ is the "projection" function defined in the previous section).

(3) (Normality) If for some $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \longrightarrow V$ we have that
$$\{s \in [\kappa]^{|a|} : f(s) \in max(s)\} \in E_a,$$
then there is some $b \supseteq a$ such that
$$\{s \in [\kappa]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b.$$

(4) (Well-foundedness) If there are $a_n \in [\lambda]^{<\omega}$ and $X_n \subseteq [\kappa]^{|a_n|}$ with $X_n \in E_{a_n}$, for all $n \in \omega$, then there is an order-preserving function $d : \bigcup_{n\in\omega} a_n \longrightarrow \kappa$ such that $d``a_n \in X_n$, for all $n \in \omega$.

Given such a (κ, λ)-extender E, we follow a similar route to the one we took in the previous section. We summarize the procedure below, with appropriate references to our earlier discussion.

- Initially, we construct, for all $a \in [\lambda]^{<\omega}$, the corresponding ultrapowers $\mathrm{Ult}_{E_a}(V)$. Note that both condition (1)(*i*) and condition (4) of the definition imply that each E_a is *countably complete*, i.e.,
$$\text{if } X_n \in E_a, \text{ for all } n \in \omega, \text{ then } \bigcap_{n\in\omega} X_n \neq \varnothing.$$
To see that this follows from (4), just let $a_n = a$, for all $n \in \omega$. Since countable completeness is sufficient, it follows that all the ultrapowers are well-founded and we can therefore work with their transitive collapses. Thus, we have as usual, $M_a \cong \mathrm{Ult}_{E_a}(V)$ and the standard corresponding elementary embedding $j_a : V \longrightarrow M_a$, given by $j_a(x) = [c^a_x]_{E_a}$, $x \in V$.

- Next, for all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we define –exactly as before– the embeddings $i_{ab} : M_a \longrightarrow M_b$ by:
$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}, \text{ for all } f : [\kappa]^{|a|} \longrightarrow V.$$
By condition (2) of the definition (the coherence property), we notice that the exact same argument which we used in the previous section, shows that these are elementary embeddings and that they commute with the j_a's, i.e., $i_{ab} \circ j_a = j_b$.
- At this point, we can again form the directed system
$$\big\langle \langle M_a : a \in [\lambda]^{<\omega}\rangle ; \langle i_{ab} : a \subseteq b \in [\lambda]^{<\omega}\rangle \big\rangle$$
from which we construct the direct limit $\widetilde{M_E} = \langle D_E, \in_E\rangle$. As before, this construction comes together with the maps $j_E : V \longrightarrow \widetilde{M_E}$ and $k_{aE} : M_a \longrightarrow \widetilde{M_E}$, defined by
$$\begin{aligned} j_E(x) = [a, [c^a_x]] \quad &, \quad \text{for each } x \in V, \\ &\quad\ \text{and for some (any) } a \in [\lambda]^{<\omega} \\ k_{aE}([f]) = [a, [f]] \quad &, \quad \text{for each } f : [\kappa]^{|a|} \longrightarrow V. \end{aligned}$$
Using the exact same arguments as in the previous section (essentially coming from the coherence property of the measures), one shows that k_{aE} is elementary and then, that the same is true for j_E. The only difference here is that there is no reference to an ambient elementary embedding j but nevertheless, the remaining part of the arguments works just fine.
This concludes the description of our construction.

As one should expect, a commutative diagram that encapsulates all the revelant embeddings might be formed at this point. Yet, we refrain ourselves from stating this explicitly since there is one final thing that needs to be checked.

The attentive reader might have already noticed that we have not said anything about the well-foundedness of the direct limit $\widetilde{M_E}$. Recall that, in the previous section, the well-foundedness of $\widetilde{M_E}$ was –essentially– established from the elementarity of the map $k_E : M_E \longrightarrow M$ (and at any rate, with reference to the given embedding j). Since this is no longer the case, we have to modify our arguments. Evidently, condition (4) of the definition has to be exploited.

Proposition 3.2. *Condition* (4) *is equivalent to the well-foundedness of the direct limit* $\widetilde{M_E}$.

Proof. ($\Longrightarrow$) Suppose that $\widetilde{M_E}$ is ill-founded, i.e., there are $x_n = [a_n, [f_n]] \in \widetilde{M_E}$ such that, for all $n \in \omega$, $x_{n+1} \in_E x_n$. First, we claim that the a_n's can be chosen so that $m \leqslant n \longrightarrow a_m \subseteq a_n$.

Recall that, by construction of the direct limit $\widetilde{M_E}$, for every $a \subseteq b$ and each $f : [\kappa]^{|a|} \longrightarrow V$, we have that $[a, [f]] = [b, [f \circ \pi_{ba}]]$. Therefore, given a descending chain $\langle x_n = [a_n, [f_n]] : n \in \omega \rangle$, if we define the sequence $\langle y_n = [b_n, [g_n]] : n \in \omega \rangle$ where, for each $n \in \omega$,

$$b_n = \bigcup_{k \leqslant n} a_k \qquad \text{and} \qquad g_n = f_n \circ \pi_{b_n\, a_n}$$

then, using the observation we just mentioned, it easy to check that the y_n's form a descending chain as well. This proves our claim.

We now define, recursively for $n \in \omega$, the following sets:

$$\begin{aligned} X_0 &= [\kappa]^{|a_0|} \\ X_{n+1} &= \{s \in [\kappa]^{|a_{n+1}|} : f_{n+1}(s) \in f_n \circ \pi_{a_{n+1}\, a_n}(s)\}. \end{aligned}$$

Note that, for every n, $x_{n+1} \in_E x_n$ implies that $X_{n+1} \in E_{a_{n+1}}$. Obviously, $X_0 \in E_{a_0}$ and thus, $X_n \in E_{a_n}$ for all $n \in \omega$. We are about to derive a contradiction from condition (4).

For, suppose that there is an order-preserving function $d : \bigcup_{n \in \omega} a_n \longrightarrow \kappa$ such that $d``a_n \in X_n$, for all $n \in \omega$. This means that for all $n \in \omega$,

$$f_{n+1}(d``a_{n+1}) \in f_n \circ \pi_{a_{n+1}\, a_n}(d``a_{n+1}) = f_n(d``a_n),$$

i.e., the sequence $\langle f_n(d``a_n) : n \in \omega \rangle$ is an infinite descending chain in V, which is absurd.

($\Longleftarrow$) Conversely, suppose that condition (4) fails, i.e., let $a_n \in [\lambda]^{<\omega}$ and $X_n \subseteq [\kappa]^{|a_n|}$ with $X_n \in E_{a_n}$, for all $n \in \omega$, such that there is no order-preserving $d : \bigcup_{n \in \omega} a_n \longrightarrow \kappa$ with the property $d``a_n \in X_n$, for all $n \in \omega$.

Our aim is to show that $\widetilde{M_E}$ is ill-founded. To begin with, we claim that the a_n's and the X_n's can be chosen so that:

(1) $m \leqslant n \longrightarrow a_m \subseteq a_n$ and
(2) $s \in X_n \wedge m \leqslant n \longrightarrow \pi_{a_n\, a_m}(s) \in X_m$.

To show both of them, a similar idea to the one we used in the first part of the proof is employed. For (1), we define for each $n \in \omega$:

$$b_n = \bigcup_{k \leqslant n} a_k \qquad \text{and} \qquad Y_n = \{s \in [\kappa]^{|b_n|} : \pi_{b_n\, a_n}(s) \in X_n\}.$$

Note that, for every $n \in \omega$, $Y_n \in E_{b_n}$ by coherence. Now, if there is an order-preserving $d : \bigcup_{n \in \omega} b_n \longrightarrow \kappa$ such that $d``b_n \in Y_n$, for all $n \in \omega$, then this means that, $\pi_{b_n\, a_n}(d``b_n) \in X_n$, i.e., $d``a_n \in X_n$, for all $n \in \omega$ and this contradicts our assumption.

For (2), since (1) can be assumed by now, fix some $n \in \omega$ and define, for each $m \leqslant n$, the sets

$$A_m = \{s \in [\kappa]^{|a_n|} : \pi_{a_n\, a_m}(s) \in X_m\}$$

all of which belong to E_{a_n} by coherence. By the finite intersection property of the measures, we consequently have that the set

$$Y_n =_{\text{def.}} \bigcap_{m \leqslant n} A_m = \{s \in [\kappa]^{|a_n|} : (\forall\, m \leqslant n)\, \pi_{a_n\, a_m}(s) \in X_m\}$$

belongs to E_{a_n}, for each $n \in \omega$.

Now, it is evident that the Y_n's satisfy (2). Moreover, the exact same computation used for (1) shows that there cannot be an order-preserving function $d : \bigcup_{n\in\omega} a_n \longrightarrow \kappa$ such that $d``a_n \in Y_n$, for all $n \in \omega$. This concludes our two-part claim and we now proceed with the rest of the proof.

We define the following set:

$$T = \left\{ \langle s_i : i < n\rangle : n \in \omega, \exists\, s \in [\kappa]^{|a_{n-1}|} \text{ s.t. } \begin{array}{l} (i)\ s \in X_{n-1} \\ (ii)\ \pi_{a_{n-1}\, a_i}(s) = s_i,\ \text{ all } i < n \end{array} \right\}$$

Thus, a typical element $s^* \in T$ is of the form $s^* = \langle s_0, s_1, \ldots, s_{n-2}, s_{n-1}\rangle$, or equivalently,

$$s^* = \langle \pi_{a_{n-1}\, a_0}(s), \pi_{a_{n-1}\, a_1}(s), \ldots, \pi_{a_{n-1}\, a_{n-2}}(s), s\rangle$$

where $s_{n-1} = s \in X_{n-1}$. Note that by the second part of our claim, the latter implies that $s_i = \pi_{a_{n-1}\, a_i}(s) \in X_i$, all $i < n$.

We moreover define an order relation on T by:

$$s^* \prec t^* \longleftrightarrow s^* \text{ properly extends } t^*.$$

We now show that, under this ordering, T is a well-founded poset. Towards a contradiction, suppose there is an infinite $\prec$-descending chain in T, i.e.,

$$\ldots \prec \langle s_0, s_1, \ldots, s_{n-1}\rangle \prec \ldots \prec \langle s_0, s_1\rangle \prec \langle s_0\rangle$$

(note that we can always assume that the first element of the chain is an one-element sequence). Then, it is readily seen that one can define appropriately an order-preserving $d : \bigcup_{n\in\omega} a_n \longrightarrow \kappa$ such that $d``a_n = s_n$, for all $n \in \omega$. But this contradicts our initial assumption, since $s_n \in X_n$, all $n \in \omega$.

The well-foundedness of the poset T enables us to define the usual rank function on its elements, i.e., $\mathrm{rk}_T : T \longrightarrow \mathbf{ON}$ such that, for every $s^* \in T$,

$$\mathrm{rk}_T(s^*) = \sup\{\mathrm{rk}_T(t^*) + 1 : t^* \in T \wedge t^* \prec s^*\}$$

Using the rank function, we are about to establish the ill-foundedness of $\widetilde{M_E}$. For this, define for each $n \in \omega$, a function $F_n : [\kappa]^{|a_n|} \longrightarrow \mathbf{ON}$ by:

$$F_n(s) = \begin{cases} \mathrm{rk}_T(\underbrace{\langle \pi_{a_n\, a_0}(s), \pi_{a_n\, a_1}(s), \ldots, \pi_{a_n\, a_{n-1}}(s), s\rangle}_{s^*}) & , \quad s \in X_n \\ \varnothing & , \quad \text{otherwise} \end{cases}$$

We should point out that these are well-defined: if $s \in X_n \subseteq [\kappa]^{|a_n|}$, then $s^* = \langle \pi_{a_n\, a_0}(s), \pi_{a_n\, a_1}(s), \ldots, \pi_{a_n\, a_{n-1}}(s), s\rangle$ is uniquely defined by "projecting down" to all the previous indices and moreover, $s^* \in T$ because $s \in X_n$.

But now observe that for every $s \in X_n$, $s^* \prec \left(\pi_{a_n\, a_{n-1}}(s)\right)^*$ and so, by the order-preserving property of the rank function, we get that, for every $n \geqslant 1$, $X_n \subseteq \{s \in [\kappa]^{|a_n|} : F_n(s) \in F_{n-1}(\pi_{a_n\, a_{n-1}}(s))\}$.

Therefore, since $X_n \in E_{a_n}$ we get, by definition of membership in the direct limit, that $[a_n, [F_n]] \in_E [a_{n-1}, [F_{n-1}]]$, for all $n \geqslant 1$, i.e., $\widetilde{M_E}$ is ill-founded. The proof is complete. $\square$

Having established the well-foundedness of $\widetilde{M_E}$, we also remark that the definition of the membership relation $\in_E$ (recall Scott's trick as well) implies its "set-likeness" and hence, we can work with the transitive collapse of the direct limit structure, called M_E. Thus, we get the anticipated commutative diagram of embeddings, as shown below:

$$\begin{tikzcd}
V \arrow[rr, "j_E"] \arrow[d, "j_a"'] \arrow[ddr, "j_b"] & & M_E \\
M_a \arrow[urr, "k_{aE}"] \arrow[dr, "i_{ab}"'] & & \\
& M_b \arrow[uur, "k_{bE}"'] &
\end{tikzcd}$$

$$k_{aE} \circ j_a = k_{bE} \circ j_b = j_E$$

$$k_{bE} \circ i_{ab} = k_{aE}$$

$$i_{ab} \circ j_a = j_b$$

Our goal now is to present several properties of the model M_E, to give the connection between the two kinds of extenders studied so far and to discuss some of their applications. As an indication of what to expect, let us first verify that any extender derived from an embedding satisfies the general definition.

Lemma 3.3. *Suppose that $j : V \longrightarrow M$ is an elementary embedding with $\mathrm{cp}(j) = \kappa$. Let $\kappa < \lambda \leqslant j(\kappa)$ and consider E, the (κ, λ)-extender derived from j. Then, E is a (κ, λ)-extender.*

Proof. Towards verifying all the clauses of Definition 3.1, let us first point out that conditions (1)(i) and (2) have already been checked in the previous section. Moreover, by Proposition 3.2, condition (4) follows as well since as we have seen $\widetilde{M_E}$ is well-founded. For the rest of the argument, we have the following:

(1)(ii) We want some $a \in [\lambda]^{<\omega}$ such that E_a is not κ^+-complete. Let $a = \{\kappa\}$ where $\kappa = \mathrm{cp}(j)$. Now, to see that κ^+-completeness fails, define, for each $\alpha < \kappa$, $X_\alpha = \{\{\xi\} : \xi > \alpha\} \subseteq [\kappa]^1$. The fact that $j(\kappa) > \kappa$ immediately implies that, for all $\alpha < \kappa$, $X_\alpha \in E_{\{\kappa\}}$. On the other hand, we clearly have that $\bigcap\limits_{\alpha<\kappa} X_\alpha = \varnothing$.

By the way, recall that, in general, if U is an ultrafilter on a set S of cardinality κ then:

$$U \text{ is not } \kappa^+\text{-complete } \longleftrightarrow U \text{ is non-principal.}$$

Thus, since in our case the ultrafilters are on the sets $[\kappa]^n$ (for some $n \in \omega$), condition (1)(ii) is actually equivalent to requiring that for some $a \in [\lambda]^{<\omega}$, E_a is non-principal. See also the discussion on long extenders and the related remarks before Corollary 3.7.

(1)(iii) This is immediate since, for each $\gamma < \kappa$, $\{s : \gamma \in s\} \in E_a$ if and only if $j(\gamma) = \gamma \in a$ and so, we may pick $a = \{\gamma\}$.

(3) To check normality, suppose that for some $a \in [\lambda]^{<\omega}$ and some f on $[\kappa]^{|a|}$ we have that $\{s \in [\kappa]^{|a|} : f(s) \in max(s)\} \in E_a$, which means that $j(f)(a) \in max(a)$.

Now, since $j(f)(a) \in max(a) \in \lambda$, we also have $j(f)(a) \in \lambda$ by transitivity. So, let $b = a \cup \{j(f)(a)\}$ and it is now easy to check that $\{s \in [\kappa]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b$.

$\square$

Before stating the properties of the map j_E and the structure M_E, some remarks are in order regarding Definition 3.1 and related points.

Condition (1)(iii) and long extenders

Note that, in the previous proof and as far as condition (1)(iii) is concerned, we may always find an appropriate $a \in [\lambda]^{<\omega}$ as long as $j(\gamma) < \lambda$. In our case where $\kappa < \lambda \leqslant j(\kappa)$, this was true (exactly) for $\gamma < \kappa$.

More generally, we may modify Definition 3.1 of a (κ, λ)-extender to get a broader version, by requiring that *for some* $\zeta \geqslant \kappa$, the same clauses hold with respect to the sets $[\zeta]^{|a|}$ (where $a \in [\lambda]^{<\omega}$), i.e., the ultrafilters E_a and the "projections" π_{ba} are now taken on the sets $[\zeta]^{|a|}$ and also, the order-preserving function in condition (4) is of the form $d : \bigcup_{n\in\omega} a_n \longrightarrow \zeta$.

In this general case, the only essential change is in clause (1)(iii) which now becomes:

(1)(iii)* For all $\gamma < \zeta$, there is $a \in [\lambda]^{<\omega}$ s.t. $\{s \in [\zeta]^{|a|} : \gamma \in s\} \in E_a$.

Then, as we will later see, this new clause specifies the ζ, in the sense that it is the least ordinal such that $\lambda \leqslant j_E(\zeta)$.

Something similar can be done for the case of Definition 2.1 as well, i.e., towards a more general setting and after fixing some $\lambda > \kappa = \mathrm{cp}(j)$, define (as before) the ultrafilters E_a on the sets $[\zeta]^{|a|}$ (with $a \in [\lambda]^{<\omega}$) where ζ is taken to be *the least ordinal such that* $\lambda \leqslant j(\zeta)$. This latter requirement will then ensure that the updated clause (1)(iii)* is satisfied, i.e., we may again conclude that any such extender derived from an embedding is also an extender in the sense of Definition 3.1 (note that the argument for coherence is the same).

These generalized versions are called *long extenders* and are used in the study of rather strong large cardinal assumptions, such as supercompactness and extendibility. Here, "long" means that $j(\kappa) < \lambda$ or $j_E(\kappa) < \lambda$, according to the situation (i.e., depending on whether the extender is derived from an embedding or not). See also Definition 3.6 where the "length" of an extender is introduced. So far we have been discussing *short extenders*, where the rôle of ζ was played by κ. As we will see, short extenders are sufficient for the study of large cardinals up to superstrongs but for stronger hypotheses (e.g., supercompacts) we will need the long versions in an essential way.

Let us also mention that everything we have been discussing in the previous section can be directly adapted to the case of long extenders; in particular, long extenders derived from an embedding satisfy the properties given in Proposition 2.5 (modulo the obvious modifications with respect to ζ). In the case of the general Definition 3.1, we should point out that clause $(1)(ii)$ is actually "tailored" for long rather than short extenders: in fact, it is superfluous for the latter. See also the (overall) discussion after Corollary 3.5 where we give more details on this issue and, at the same time, a unifying view on extenders is established.

At any rate, from now on we will freely use the generalized setting, i.e., assume that $\zeta \geqslant \kappa$, either because it is essential for the argument at hand, or for the sake of generality. Turning now to the opposite direction, let us briefly discuss restrictions of extenders as well.

<u>Restrictions of an extender</u>

Suppose that E is a (κ, λ)-extender and let $\kappa < \beta < \lambda$. If we have that, for every $a \in [\beta]^{<\omega}$, conditon (1) of Definition 3.1 is satisfied, then the restriction $E \restriction [\beta]^{<\omega}$ is a (κ, β)-extender.

To verify this, the only thing that needs to be checked is the normality condition. For that, suppose that for some $a \in [\beta]^{<\omega}$ and some f on $[\zeta]^{|a|}$, we have that $\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a$. We know that there is some $c \in [\lambda]^{<\omega}$ with $a \subseteq c$, such that $\{s \in [\zeta]^{|c|} : f \circ \pi_{ca}(s) \in s\} \in E_c$. We want to find some $b \in [\beta]^{<\omega}$ with $a \subseteq b$ such that the same holds with respect to the set b. We let $b = c \cap \beta$ and we show that this choice works.

First note that $a \subseteq b \subseteq c$. Since $\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a$, by coherence we have that $\{s \in [\zeta]^{|c|} : f \circ \pi_{ca}(s) \in max(\pi_{ca}(s))\} \in E_c$. Intersecting this set with $\{s \in [\zeta]^{|c|} : f \circ \pi_{ca}(s) \in s\} \in E_c$, it follows that $\{s \in [\zeta]^{|c|} : f \circ \pi_{ca}(s) \in s \cap max(\pi_{ca}(s))\} \in E_c$.

Now, note that since $\pi_{ca}(s) \subseteq \pi_{cb}(s)$, we get that $s \cap max(\pi_{ca}(s)) \subseteq \pi_{cb}(s)$ and therefore, we have that $\{s \in [\zeta]^{|c|} : f \circ \pi_{ca}(s) \in \pi_{cb}(s)\} \in E_c$, or equivalently,

$$\{s \in [\zeta]^{|c|} : f \circ \pi_{ba} \circ \pi_{cb}(s) \in \pi_{cb}(s)\} \in E_c.$$

Hence, by coherence, we get that $\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b$ which is the desired conclusion.

After the previous remarks, we are now in position to state the basic properties of the embedding j_E and the structure M_E. As a matter of notation, for $n \in \omega$, let $id^n : [\zeta]^n \longrightarrow [\zeta]^n$ be the identity function. Note that, trivially, for $a \subseteq b \in [\lambda]^{<\omega}$, $i_{ab}([id^{|a|}]_{E_a}) = [\pi_{ba}]_{E_b}$. With these in mind, we have the following basic proposition (compare with Proposition 2.5):

Proposition 3.4. [Properties of j_E, M_E]

(*i*) *For every* $a \in [\lambda]^{<\omega}$, $k_{aE}([id^{|a|}]) = a$.
(*ii*) $\mathrm{cp}(j_E) = \kappa$ *and* ζ *is the least ordinal such that* $\lambda \leqslant j_E(\zeta)$.
(*iii*) $M_E = \{j_E(f)(a) : a \in [\lambda]^{<\omega}, f : [\zeta]^{|a|} \longrightarrow V, f \in V\}$.
(*iv*) *For any set* X *with* $|X| > \zeta$, $j_E``X \notin M_E$.
(*v*) $E \notin M_E$.

Proof. We first deal with (i). This important property comes essentially from the normality condition of Definition 3.1. We first show that it holds for all singletons $a \in [\lambda]^1$ and then generalize the idea to show the full property.

So, let us show by induction that, for every $\alpha < \lambda$, $k_{\{\alpha\}E}([id^1]) = \{\alpha\}$. The base case is $\alpha = \varnothing$. In this case, first recall that by condition $(1)(iii)^*$ there is some $a \in [\lambda]^{<\omega}$ such that $\{s \in [\zeta]^{|a|} : \varnothing \in s\} \in E_a$. It is now easy to see that, by coherence, $\varnothing \in a$ and moreover, $\{s \in [\zeta]^{|b|} : \varnothing \in s\} \in E_b$ for every $b \supseteq a$. We also have that $k_{\{\varnothing\}E}([id^1]) = [\{\varnothing\}, [id^1]]$ (by definition) and that, for any $a \in [\lambda]^{<\omega}$, $j_E(\varnothing) = [a, [c^a_\varnothing]] = \varnothing$ in M_E (by definition and elementarity). So, our goal here is to show that $[\{\varnothing\}, [id^1]] = \{j_E(\varnothing)\}$.

For this, suppose that $[a, [f]] \in [\{\varnothing\}, [id^1]]$, for some a and f. By what we said above, we may assume that (by moving to a superset) $\varnothing \in a$ and that $\{s \in [\zeta]^{|a|} : \varnothing \in s\} \in E_a$. Thus, we have that for almost all $s \in [\zeta]^{|a|}$, $f(s) \in id^1 \circ \pi_{a\{\varnothing\}}(s) = \pi_{a\{\varnothing\}}(s)$. In other words, if s_0 denotes the first (in the standard ordering) element of s, we have that for almost all $s \in [\zeta]^{|a|}$, $s_0 = \varnothing$ and $f(s) = s_0$, i.e., $[f] = [c^a_\varnothing]$. This implies that $[a, [f]] = [a, [c^a_\varnothing]] = j_E(\varnothing)$ and shows the desired property for the base case.

For the inductive step, assume that for some $\gamma < \lambda$, $k_{\{\alpha\}E}([id^1]) = \{\alpha\}$ for every $\alpha < \gamma$. We first show that $k_{\{\gamma\}E}([id^1])$ is a singleton.

For this, assume that for some element $[a, [f]] \in M_E$, we have that $[a, [f]] \in [\{\gamma\}, [id^1]] = k_{\{\gamma\}E}([id^1])$. We may assume that $\gamma \in a$. Notice that, by definition of equality in M_E, $[\{\gamma\}, [id^1]] = [a, [\pi_{a\{\gamma\}}]]$ and hence, we have that

$$\{s \in [\zeta]^{|a|} : f(s) \in \pi_{a\{\gamma\}}(s)\} \in E_a$$

and since $\pi_{a\{\gamma\}}(s)$ is a singleton, we equivalently have

$$\{s \in [\zeta]^{|a|} : \{f(s)\} = \pi_{a\{\gamma\}}(s)\} \in E_a$$

which means that $\{[f]_{E_a}\} = [\pi_{a\{\gamma\}}]_{E_a}$. But now, by moving to membership in the direct limit M_E, we have that $\{[a, [f]]\} = [a, [\pi_{a\{\gamma\}}]]$, i.e., $\{[a, [f]]\} = k_{\{\gamma\}E}([id^1])$, which is what we wanted.

Let us note that, in order to show that $k_{\{\gamma\}E}([id^1])$ is a singleton, a slightly more general argument may be employed: for every $a \in [\lambda]^{<\omega}$,

$$\{s \in [\zeta]^{|a|} : s \text{ is a set of ordinals of cardinality } |a|\} = [\zeta]^{|a|} \in E_a$$

and thus, in M_a, the same holds for $[id^{|a|}]_{E_a}$. Moreover, by elementarity of the maps k_{aE}, the same is true for $k_{aE}([id^{|a|}])$.

In particular, in our case, $k_{\{\gamma\}E}([id^1]) = \{\delta\}$ for some ordinal δ and we will now show that $\delta = \gamma$.

On the one hand, suppose that $\alpha < \gamma$ and let $a = \{\alpha, \gamma\}$. For every $s \in [\zeta]^2$, let us fix some notation and write $s = \{\xi_0^s, \xi_1^s\}$ (where as usual $\xi_0^s < \xi_1^s$). Note that in this case, $\pi_{a\{\alpha\}}(s) = \{\xi_0^s\}$ and $\pi_{a\{\gamma\}}(s) = \{\xi_1^s\}$. This means that, for every $s \in [\zeta]^{|a|} = [\zeta]^2$, the unique element of $\pi_{a\{\alpha\}}(s)$ (i.e., ξ_0^s) belongs to the unique element of $\pi_{a\{\gamma\}}(s)$ (i.e., ξ_1^s). Similarly, by moving to membership in the M_E, we have that every element of $[a, [\pi_{a\{\alpha\}}]]$ belongs to every element of $[a, [\pi_{a\{\gamma\}}]]$. But now note that

$$\{\alpha\} \overset{\text{I.H.}}{=} k_{\{\alpha\}E}([id^1]) = [\{\alpha\}, [id^1]] = [a, [\pi_{a\{\alpha\}}]]$$

and similarly

$$\{\delta\} = k_{\{\gamma\}E}([id^1]) = [\{\gamma\}, [id^1]] = [a, [\pi_{a\{\gamma\}}]]$$

and therefore, it follows that $\alpha < \delta$. This shows that $\gamma \leqslant \delta$.

On the other hand, suppose that $\beta = [a, [f]] < \delta$ where we may assume that $\gamma \in a$. Since $\delta \in k_{\{\gamma\}E}([id^1]) = [a, [\pi_{a\{\gamma\}}]]$ we have that

$$(1) \;\; \{s \in [\zeta]^{|a|} : f(s) \text{ belongs to the unique member of } \pi_{a\{\gamma\}}(s)\} \in E_a$$

which implies (by transitivity of ordinals) that

$$\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a.$$

At this point, we apply the normality condition to get some $b \in [\lambda]^{<\omega}$, with $b \supseteq a$, such that

$$\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b.$$

By the finite intersection property of the measures, there is some $i < \omega$ such that

$$\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) = s_i\} \in E_b$$

where s_i denotes the i^{th} member of s, in the increasing ordinal order. If we now let α be the i^{th} member of b, we can equivalently write

$$(2) \;\; \{s \in [\zeta]^{|b|} : \{f \circ \pi_{ba}(s)\} = \pi_{b\{\alpha\}}(s)\} \in E_b.$$

Next, by applying coherence to (1) above, we also have that

$$\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) \text{ belongs to the unique member of } \pi_{b\{\gamma\}}(s)\} \in E_b.$$

These last two statements say that, for almost all $s \in [\zeta]^{|b|}$, the unique member of $\pi_{b\{\alpha\}}(s)$ belongs to the unique member of $\pi_{b\{\gamma\}}(s)$ and therefore, $\alpha < \gamma$.

Thus, using (2), we have the following in M_E:

$$\{[a,[f]]\} = \{[b,[f\circ\pi_{ba}]]\} = [b,[\pi_{b\{\alpha\}}]] = [\{\alpha\},[id^1]] = k_{\{\alpha\}E}([id^1]).$$

To conclude, we employ the induction hypothesis and we finally get

$$\{\beta\} = \{[a,[f]]\} = k_{\{\alpha\}E}([id^1]) \overset{\text{I.H.}}{=} \{\alpha\} \subseteq \gamma,$$

i.e., $\beta < \gamma$ which gives $\delta \leqslant \gamma$. Thus, $\delta = \gamma$ and so $k_{\{\gamma\}E}([id^1]) = \{\gamma\}$ which concludes the induction.

Having established the desired property for every $a \in [\lambda]^1$, we proceed to the general case by employing similar arguments.

Let $a \in [\lambda]^{<\omega}$ and let $x = [b,[f]] \in M_E$ be any element, where we may assume that $a \subseteq b$. Then,

$$\begin{array}{lcll}
x \in k_{aE}([id^{|a|}]) & \longleftrightarrow & [f]_{E_b} \in [\pi_{ba}]_{E_b} & \\
 & \longleftrightarrow & \{s \in [\zeta]^{|b|} : f(s) \in \pi_{ba}(s)\} \in E_b & \\
 & \longleftrightarrow & \{s \in [\zeta]^{|b|} : \{f(s)\} = \pi_{b\{\alpha\}}(s)\} \in E_b & \text{, some } \alpha \in a \\
 & & \text{(by the finite intersection property)} & \\
 & \longleftrightarrow & \{[f]_{E_b}\} = [\pi_{b\{\alpha\}}]_{E_b} & \text{, some } \alpha \in a \\
 & \longleftrightarrow & \{[b,[f]]\} = [b,[\pi_{b\{\alpha\}}]] & \text{, some } \alpha \in a \\
 & \longleftrightarrow & \{x\} = [\{\alpha\},[id^1]] & \text{, some } \alpha \in a \\
 & \longleftrightarrow & \{x\} = k_{\{\alpha\}E}([id^1]) = \{\alpha\} & \text{, some } \alpha \in a
\end{array}$$

which shows that $k_{aE}([id^{|a|}]) = a$ and concludes the proof of (i).

For (ii), we first check that $\mathrm{cp}(j_E) = \kappa$. This comes essentially from conditions $(1)(i)$ and $(1)(ii)$ of Definition 3.1. We initially proceed by induction to show that, for every $\alpha < \kappa$, $j_E(\alpha) = \alpha$.

Let $\alpha < \kappa$ and suppose that for every $\gamma < \alpha$, $j_E(\gamma) = \gamma$. It is enough to show that $j_E(\alpha) \leqslant \alpha$. So, pick some $x = [a,[f]] \in M_E$ where $x \in j_E(\alpha)$. Recall that $j_E(\alpha) = [a,[c^a_\alpha]]$. Thus, we have that $\{s : f(s) \in \alpha\} \in E_a$ and so, by the κ-completeness of the measures, there exists $\gamma < \alpha$ such that $\{s : f(s) = \gamma\} \in E_a$, i.e., $[f]_{E_a} = [c^a_\gamma]_{E_a}$ or, equivalently, $[a,[f]] = [a,[c^a_\gamma]]$, i.e., $x = j_E(\gamma) \overset{\text{I.H.}}{=} \gamma < \alpha$. This shows that $j_E(\alpha) \leqslant \alpha$ and hence, we can conclude that for every $\alpha < \kappa$, $j_E(\alpha) = \alpha$, i.e., $\mathrm{cp}(j_E) \geqslant \kappa$.

Next, we show that $\kappa < j_E(\kappa)$. For this, we use condition $(1)(ii)$. So, let $a \in [\lambda]^{<\omega}$ be such that E_a is not κ^+-complete. This means that there is a partition of $[\zeta]^{|a|}$ into κ-many sets not in the ultrafilter, i.e., there is, for each $\alpha < \kappa$, some $X_\alpha \subseteq [\zeta]^{|a|}$ so that $X_\alpha \notin E_a$, with the X_α's being pairwise disjoint and $\displaystyle\bigcup_{\alpha<\kappa} X_\alpha = [\zeta]^{|a|}$.

We define the function $f : [\zeta]^{|a|} \longrightarrow \kappa$ by $f(s) =$ the unique $\alpha < \kappa$ such that $s \in X_\alpha$. Our aim now is to show that

$$\kappa \leqslant [a,[f]] < [a,[c^a_\kappa]] = j_E(\kappa)$$

which will give the desired conclusion (i.e., that $\mathrm{cp}(j_E) = \kappa$).

The second inequality is immediate, since $range(f) = \kappa$. For the first inequality, we fix $\xi < \kappa$ and we show that $\xi < [a, [f]]$. For that, first observe that $\xi < f(s)$, for almost all s (mod E_a): otherwise, we would have that $\bigcup_{\alpha \leqslant \xi} X_\alpha \in E_a$, which would contradict the κ-completeness of the measure.

Consequently, we have that $[a, [c^a_\xi]] < [a, [f]]$ or, in other words, $j_E(\xi) < [a, [f]]$ and hence, $\xi \leqslant j_E(\xi) < [a, [f]]$. This shows the first inequality and concludes the proof of $\mathrm{cp}(j_E) = \kappa$.

Next, to show that $j_E(\zeta) \geqslant \lambda$, let $\alpha < \lambda$ and put $a = \{\alpha\}$. We want to see that $a \subseteq j_E(\zeta)$ or equivalently, using (i), to see that $k_{aE}([id^{|a|}]) \subseteq j_E(\zeta)$. Using the commutativity of the embeddings, this is equivalent to $k_{aE}([id^{|a|}]) \subseteq k_{aE} \circ j_a(\zeta)$, i.e., $[id^{|a|}] \subseteq j_a(\zeta)$. But the latter is certainly true, since if $[f] \in [id^{|a|}]$ (in M_a), then, for almost all s, $f(s) \in s \in [\zeta]^{|a|}$, i.e., $f(s) \in \zeta$ almost everywhere, which means that $[f] \in [c^a_\zeta] = j_a(\zeta)$. This shows that $j_E(\zeta) \geqslant \lambda$.

To conlude the proof of this part, we show the minimality of ζ. For this, note that by condition $(1)(iii)^*$, if $\gamma < \zeta$ then for some $a \in [\lambda]^{<\omega}$ we have that $\{s \in [\zeta]^{|a|} : \gamma \in s\} \in E_a$, i.e., $[c^a_\gamma] = j_a(\gamma) \in [id^{|a|}]$ and thus, $k_{aE} \circ j_a(\gamma) \in k_{aE}([id^{|a|}])$ which by (i) gives $j_E(\gamma) \in a \subseteq \lambda$. We conclude that ζ is indeed the least ordinal such that $\lambda \leqslant j_E(\zeta)$.

For (iii), we argue similarly to Proposition 2.5. Let $x = [a, [f]] \in M_E$. In the ultrapower M_a, we have that $[f] = j_a(f)([id^{|a|}])$ and thus,

$$x = k_{aE}([f]) = k_{aE}(j_a(f)([id^{|a|}])) = k_{aE} \circ j_a(f)(k_{aE}([id^{|a|}])) \overset{(i)}{=} j_E(f)(a).$$

For (iv), suppose that $|X| > \zeta$, let $x = [a, [f]] \in M_E$ be any arbitrary element and consider the set $A = \{s \in [\zeta]^{|a|} : |f(s)| \leqslant |\zeta|\}$. We distinguish two cases.

First, suppose that $A \in E_a$. Note that $B =_{\text{def.}} X \setminus \bigcup_{s \in A} f(s) \neq \varnothing$, so let $w \in B$. It now follows that $\{s \in [\zeta]^{|a|} : w \notin f(s)\} \in E_a$, since for every $s \in A$, $w \notin f(s)$. In other words, $[c^a_w] \notin [f]$, i.e., $j_E(w) \notin x$. This means that $x \neq j_E``X$.

Alternatively, suppose that $A \notin E_a$. Then, we have that $C \in E_a$, where $C =_{\text{def.}} \{s \in [\zeta]^{|a|} : |f(s)| > |\zeta|\}$. Hence, we may construct appropriately an injective function h on C such that $h(s) \in f(s)$, for every $s \in C$. But then, it is clear that $[a, [h]] \in [a, [f]] = x$ but $[a, [h]] \notin range(j_E)$, i.e., $x \neq j_E``X$. In either case, since the element $x \in M_E$ was arbitrary, we conclude that $j_E``X \notin M_E$ which is what we wanted.

For (v), suppose, towards a contradiction, that $E \in M_E$. It is easy to see that, for each $a \in [\lambda]^{<\omega}$, $E_a \in M_E$ as well and, clearly, $[\zeta]^{|a|} = \bigcup E_a \in M_E$. Moreover, by observing that $\mathcal{P}([\zeta]^{|a|}) = E_a \cup \{[\zeta]^{|a|} \setminus X : X \in E_a\}$ we also get that $\mathcal{P}([\zeta]^{|a|}) \in M_E$. Thus (using the Gödel pairing), it follows that $\mathcal{P}([\zeta]^{|a|} \times [\zeta]^{|a|}) \in M_E$. But this latter set contains every well-ordering of

$[\zeta]^{|a|}$ (with order-type some ordinal in $|\zeta|^+$) and therefore, we after all have that, for every $a \in [\lambda]^{<\omega}$, ${}^{[\zeta]^{|a|}}\big(|\zeta|^+\big) \in M_E$.

Now note that, for every $\alpha \in |\zeta|^+$, $j_E(\alpha)$ is the order-type of the set $\{[a,[f]] : a \in [\lambda]^{<\omega}, f \in {}^{[\zeta]^{|a|}}\alpha\}$. Therefore, the set of all such order-types, which is definable from the sets ${}^{[\zeta]^{|a|}}\big(|\zeta|^+\big)$ and the E_a's, belongs to M_E, i.e., $j_E``|\zeta|^+ \in M_E$. But this contradicts (iv). $\square$

Let us point out one important corollary to the above proposition which, together with Lemma 3.3, establish the connection of the two kinds of extenders considered so far in this exposition.

Corollary 3.5. *Suppose that $E = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ is a (κ,λ)-extender and let $j_E : V \longrightarrow M_E$ be the extender embedding. If E' is the (κ,λ)-extender derived from j_E, then $E' = E$.*

Proof. Suppose that $E' = \langle E'_a : a \in [\lambda]^{<\omega}\rangle$ is the (κ,λ)-extender derived from j_E. First, note that if each E_a is on $[\zeta]^{|a|}$ then, by Proposition 3.4(ii), the same is true for E'_a. Thus, for every $a \in [\lambda]^{<\omega}$ and every $X \subseteq [\zeta]^{|a|}$, we have that:

$$\begin{array}{rcll} X \in E'_a & \longleftrightarrow & a \in j_E(X) & \text{(definition of } E'_a\text{)} \\ & \longleftrightarrow & k_{aE}([id^{|a|}]) \in k_{aE} \circ j_a(X) & \text{(Proposition 3.4}(i)\text{)} \\ & \longleftrightarrow & [id^{|a|}] \in j_a(X) & \text{(elementarity)} \\ & \longleftrightarrow & X \in E_a & \text{(ultrapower } M_a\text{)} \end{array}$$

and we therefore conclude that $E' = E$. $\square$

Thus, every extender E is derived from an embedding (namely j_E) and vice versa. This means that the two kinds of extenders that we have discussed are essentially the two opposite sides of the same coin, i.e., it all reduces to a matter of perspective. From now on, we can freely refer to extenders without determining whether they are derived from an embedding or not; yet, whenever it seems appropriate, we do indicate which of the two perspectives we adopt.

Let us also give the following definition, which introduces some standard terminology related to any extender.

Definition 3.6. Given a (κ,λ)-extender E, we say that κ is the **critical point** and that λ is the **length** of E. Moreover, if $V_\alpha \subseteq M_E$ we say that the extender is α-**strong**. Finally, we define the **strength** of E to be the largest α such that E is α-strong.

Note that, in the light of our new definitions, an extender is "short" if it has length $\leqslant j_{(E)}(\kappa)$ and it is "long" otherwise. As we will see in the next section (Lemma 4.5), there are certain limitations on the strength of short extenders and this, in turn, implies limitations of such extenders in capturing large cardinal notions beyond superstrongs.

Finally, as the last word on "long" vs. "short" extenders, let us elaborate on a previously made remark regarding condition (1)(ii) of Definition 3.1. We will not give a fully detailed account, but we hope that one can fill in the gaps without too much burden.

In the case of short extenders, it is not hard to show that, for every $\alpha < \kappa$, $E_{\{\alpha\}}$ is principal and in fact, $X \in E_{\{\alpha\}} \longleftrightarrow \{\alpha\} \in X$. Indeed, this can be established without appealing to condition (1)(ii). Then, it easily follows that same is true for every $a \in [\kappa]^{<\omega}$, i.e., E_a is a principal ultrafilter and $X \in E_a \longleftrightarrow a \in X$.

Moreover, still without appealing to condition (1)(ii), we can directly show that $E_{\{\kappa\}}$ is non-principal (and hence, since the underlying set has cardinality κ, it is not κ^+-complete). For this, we argue as follows.

Since the extender is short, the ultrafilter $E_{\{\kappa\}}$ is on $[\kappa]^1$. Towards a contradiction, suppose that it is principal, i.e., there exists some $\alpha < \kappa$ such that $X \in E_{\{\kappa\}} \longleftrightarrow \{\alpha\} \in X$. But then, we know that $E_{\{\alpha\}}$ is principal and thus, we get that $E_{\{\alpha\}} = E_{\{\kappa\}}$; in particular, $\{\{\alpha\}\} \in E_{\{\alpha\}} = E_{\{\kappa\}}$. Now, if we let $a = \{\alpha, \kappa\}$, then it follows by coherence that

$$\{s \in [\kappa]^2 : s_0 = \alpha \wedge s_1 = \alpha\} \in E_a,$$

where s_i denotes the i^{th} element of s. But this is a clear contradiction.

A similar argument shows that, for every $\gamma \geqslant \kappa$, $E_{\{\gamma\}}$ is non-principal as well. Furthermore, it is a simple observation that if E_a is non-principal and $b \supseteq a$, then E_b is non-principal (and this does not depend on the extender being short). Therefore, we may conclude that, if the extender is short, then condition (1)(ii) is superfluous and

$$E_a \text{ is non-principal } \longleftrightarrow E_a \text{ is not } \kappa^+\text{-complete } \longleftrightarrow a \notin [\kappa]^{<\omega}.$$

On the other hand, in the case of long extenders, the argument that we used for the non-principality of $E_{\{\kappa\}}$ does not go through, since now, the ultrafilter is on the set $[\zeta]^1$ with $\zeta > \kappa$. In fact, if $|\zeta| \geqslant \kappa^+$ then non-principality does not necessarily imply failure of κ^+-completeness. Of course, the issue here is that we have no control over the ordinal ζ and its cardinality.

Finally, recall that condition (1)(ii) was essentially used in order to deduce that the critical point of the embedding j_E is exactly κ (Proposition 3.4(ii)). This indicates that, when the extender is long, it is the avoidance of κ^+-completeness which is the desirable feature; not of non-principality. At any rate, even in the case of long extenders one can show that, for every $a \in [\kappa]^{<\omega}$, E_a is principal and $X \in E_a \longleftrightarrow a \in X$.

We conclude this section by giving a corollary following from Propositions 2.5 and 3.4, which shows some other limitations of extenders, this time regarding the closure properties of the structure M_E.

Corollary 3.7. *Suppose that $j : V \longrightarrow M$ is elementary with* $\mathrm{cp}(j) = \kappa$, *and suppose $\lambda > \kappa$ is a beth fixed point of countable cofinality with $V_\lambda \subseteq M$. Let E be the (κ, λ)-extender derived from j. Then, ${}^\omega M_E \nsubseteq M_E$.*

Proof. First of all, note that $|V_\lambda| = \lambda$. Also, by $V_\lambda \subseteq M$ and an easy inductive argument, we get that, for all $\alpha < \lambda$, $(\beth_\alpha)^M < \beth_{\alpha+1} < \lambda$. Thus, ($\lambda$ is a beth fixed point)^M and consequently, $(|V_\lambda| = \lambda)^M$. Therefore, by Proposition 2.5, we get that $V_\lambda = V_\lambda^M = V_\lambda^{M_E}$.

Moreover, recall that the ultrafilters of the extender E are on the sets $[\zeta]^{|a|}$, where $\kappa \leqslant \zeta \leqslant \lambda$ is the least ordinal such that $\lambda \leqslant j(\zeta)$ (and since λ is limit, ζ must be limit as well). We need to consider two cases.

Suppose first that $\zeta = \lambda$, which in turn implies that, for all $n \in \omega$, $\kappa_n = j^n(\kappa)$ is below λ. Now, let $\delta = \sup_{n<\omega} \kappa_n$ and note that $j(\delta) = \delta$.

If $\delta < \lambda$, then since $V_\lambda \subseteq M$, the restriction $j \restriction V_{\delta+2} : V_{\delta+2} \longrightarrow V_{\delta+2}$ gives a contradiction by a very well-known result of Kunen (see [5]). Therefore, it must be the case that $\delta = \lambda = \zeta$ (which, by the way, means that j was actually an I2-embedding).

In this situation, if we consider the usual embeddings j_E and k_E commuting with j, then $j(\lambda) = j_E(\lambda) = k_E(\lambda) = \lambda$. Hence, we may form the following restricted version of the familiar commuting diagram:

$$\begin{tikzcd}
V_\lambda \arrow[r, "j\restriction V_\lambda"] \arrow[d, "j_E\restriction V_\lambda"'] & V_\lambda \\
V_\lambda \arrow[ur, "k_E\restriction V_\lambda"'] &
\end{tikzcd}$$

where, by Proposition 2.5(i), $k_E \restriction \lambda = id$ and then (by a standard inductive argument) we have $k_E \restriction V_\lambda = id$; therefore, $j \restriction V_\lambda = j_E \restriction V_\lambda : V_\lambda \longrightarrow V_\lambda$.

Finally, observe that $j``\kappa_n = j_E``\kappa_n \in V_\lambda \subseteq M_E$, for each $n \in \omega$. On the other hand, the proof of Kunen's theorem shows that $j_E``\lambda \notin M_E$ and thus, ${}^\omega M_E \nsubseteq M_E$ as required.

Alternatively, suppose that $\zeta < \lambda$. In this case, let $\langle \lambda_n : n \in \omega \rangle$ be an increasing sequence with $\sup_{n<\omega} \lambda_n = \lambda$. Since λ is strong limit, note that for all $n \in \omega$, $E \restriction [\lambda_n]^{<\omega} \in M_E$: this follows from the facts that $[\lambda_n]^{<\omega} \in V_\lambda$, $E_a \in \mathcal{PP}([\zeta]^{|a|}) \in V_\lambda$ (for each $a \in [\lambda_n]^{<\omega}$) and $V_\lambda \subseteq M_E$.

Therefore, if we had ${}^\omega M_E \subseteq M_E$, then it would follow that $E \in M_E$. But according to Proposition 3.4(v), the latter is not the case. $\square$

4. Extenders and large cardinals

After having discussed the basic theory of extenders, we now turn to some applications which are related to certain large cardinal notions. In particular, we will deal with (λ-)strong, superstrong and Woodin cardinals, although other well-known notions will come into play as well. We begin by recalling the (informal) definitions of strong and superstrong cardinals.

Definition 4.1. A cardinal κ is λ-**strong** ($\lambda \geqslant \kappa$), if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $\lambda < j(\kappa)$ and $V_\lambda \subseteq M$. Moreover, κ is **strong** if it is λ-strong for every $\lambda \geqslant \kappa$.

Definition 4.2. A cardinal κ is **superstrong**, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

Some remarks are in order, regarding these definitions. First of all, one can "refine" the definition of λ-strongness, in order to consider $\lambda < \kappa$ as well. The only difference is that in this case, we require $V_{\kappa+\lambda} \subseteq M$. Note that then, the two versions coincide for (not very) large λ; more precisely, for all $\lambda \geqslant \kappa \cdot \omega$ since then, $\kappa + \lambda = \lambda$. Hence this distinction will cause no essential change in our discussion, since for the strong hypotheses we consider sufficiently large λ. Nevertheless, the "refined" version gives an immediate reformulation of measurable cardinals:

$$\kappa \text{ is measurable } \longleftrightarrow \kappa \text{ is 0-strong } \longleftrightarrow \kappa \text{ is 1-strong.}$$

Also note that, in either version, if κ is λ-strong and $\beta < \lambda$, then κ is β-strong as well and thus, for full strongness, it sufficient to consider $\lambda \geqslant \kappa$. Finally, it is easy to see that, if κ is supercompact then κ is strong and that, if κ is huge then κ is superstrong.

As one basic application of our discussion in the previous sections, we now show how these notions can be reformulated in terms of extenders.

Proposition 4.3. [Characterization of (super)strongness]

(i) *The cardinal κ is λ-strong ($\lambda \geqslant \kappa$) if and only if there is a $(\kappa, |V_\lambda|^+)$-extender E such that $\lambda < j_E(\kappa)$ and $V_\lambda \subseteq M_E$.*

(ii) *The cardinal κ is superstrong if and only if there is, for some ordinal $\lambda > \kappa$, a (κ, λ)-extender E such that $V_{j_E(\kappa)} \subseteq M_E$.*

Proof. For (i), note that the converse is immediate by the definition of λ-strongness. For the forward implication, let $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$, $\lambda < j(\kappa)$ and $V_\lambda \subseteq M$ witness the fact that κ is λ-strong.

Let E be the $(\kappa, |V_\lambda|^+)$-extender derived from j and $j_E : V \longrightarrow M_E$ with $\mathrm{cp}(j_E) = \kappa$ be the corresponding extender embedding. As we have seen, in this case $k_E \restriction V_\lambda = id$ and thus, since $\lambda < |V_\lambda|^+$, $k_E \circ j_E = j$ and $\lambda < j(\kappa)$, we have that $\lambda < j_E(\kappa)$.

It remains to see that $V_\lambda \subseteq M_E$. For this, let $|V_\lambda|^+ = \delta$ and note that, since $V_\lambda^M \subseteq V_\lambda$, $(|V_\lambda| \leqslant \delta)^M$ and so by Proposition 2.5(iii) we get that $V_\lambda^{M_E} = V_\lambda^M = V_\lambda$, i.e., $V_\lambda \subseteq M_E$.

For (ii), again note that the converse is immediate by the definition of superstrongness. For the forward implication, let $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$ witness the fact that κ is superstrong.

Let E be the $(\kappa, j(\kappa))$-extender derived from j and $j_E : V \longrightarrow M_E$ with $\mathrm{cp}(j_E) = \kappa$ be the corresponding extender embedding. Recall that in this case $j_E(\kappa) = j(\kappa)$. Since $(j(\kappa)$ is inaccessible$)^M$, we have that $(|V_{j(\kappa)}| = j(\kappa))^M$ and therefore $V^M_{j(\kappa)} = V^{M_E}_{j(\kappa)}$, which gives

$$V_{j_E(\kappa)} = V_{j(\kappa)} = V^M_{j(\kappa)} = V^{M_E}_{j(\kappa)} \subseteq M_E$$

and concludes the proof. □

As one can easily see, the same argument works for the "refined" version of λ-strongness as well. Moreover, we should point out that, in that case, the requirement $\lambda < j(\kappa)$ is superfluous for successor λ and may thus be dropped. This is analogous to the case of λ-supercompactness (see 23.15 in [4] for details). Given this fact, we may use similar arguments to the ones above in order to show that κ is $(\lambda + 1)$-strong if and only if there is, for some $\beta > \kappa$, a (κ, β)-extender E such that $V_{\kappa+\lambda+1} \subseteq M_E$.

This observation provides us with (yet) another very useful characterization of strongness:

Lemma 4.4. *The cardinal κ is strong if and only if, for any set X, there is an elementary embedding $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$ and $X \in M$.*

Proof. The forward direction is rather immediate from the definition of strongness: given any set X, pick $\lambda > \mathrm{rk}(X)$ and consider the λ-strong embedding $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$. Then, $X \in V_\lambda \subseteq M$.

Conversely, in order to show that κ is strong, it is enough to show that for every ordinal λ, κ is $(\lambda + 1)$-strong. So, fix some $\lambda + 1 \in \mathbf{ON}$. By assumption, there is $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$ and $V_{\kappa+\lambda+1} \in M$. If we now let E be the $(\kappa, |V_{\kappa+\lambda+1}|^+)$-extender derived from j, then we again have that $V_{\kappa+\lambda+1} \subseteq M_E$ and therefore, by the previous observation, κ is $(\lambda + 1)$-strong. □

The aforementioned characterizations of strongness and superstrongness in terms of extenders lead to direct formalizations of these large cardinal notions in (the language of) ZFC. This is an important point to which we will frequently return in what follows and so, before continuing, it is worth discussing some related details.

Formalization issues

In order to see how one can formalize properly the aforementioned large cardinals, it is sufficient to make a case regarding how the clauses of the sort "E is a (κ, λ)-extender", "$\lambda < j_E(\kappa)$" and "$V_\lambda \subseteq M_E$" can be formalized.

First of all, being a (κ, λ)-extender is something that can be faithfully checked inside some V_α, for sufficiently large α: if V_α contains $[\lambda]^{<\omega}$, all the ultrafilters E_a, the "projection" functions π_{ba} and the functions of the sort $d : \bigcup_{n\in\omega} a_n \longrightarrow \zeta$, then conditions (1)($i$),($ii$),($iii$), (2) and (4) of Definition 3.1 can be immediately checked inside V_α. By straighforward computations, all these sets belong to V_α if $\alpha > \lambda+3$ and moreover, one can easily see that if E is a (κ,λ)-extender, then $\mathrm{rk}(E) \leqslant \lambda+6$. For the normality condition, although it seems to be referring to any function $f : [\zeta]^{|a|} \longrightarrow V$, in fact, it is sufficient to consider only functions $f : [\zeta]^{|a|} \longrightarrow \zeta$. This is because, if for any f, $\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a$ then since $max(s) \in \zeta$, we may as well assume that $f : [\zeta]^{|a|} \longrightarrow \zeta$ modulo E_a. All this shows that,

$$E \text{ is a } (\kappa,\lambda)\text{-extender} \longleftrightarrow V_\alpha \models \text{“}E \text{ is a } (\kappa,\lambda)\text{-extender”}$$

for some (any) $\alpha > \mathrm{rk}(E)$. Hence, $\alpha > \lambda + 6$ will in general do, although sometimes smaller ordinals work as well. In any case, finding the least such α is just a matter of straightforward computations.

Moreover, if for some sufficiently large α, $V_\alpha \models$ “E is a (κ,λ)-extender”, then we can build *inside* V_α the structure $(M_E)^{V_\alpha}$ together with the embedding $(j_E)^{V_\alpha}$, where these are the “direct limit” and the “extender embedding” as computed in V_α. Since all this procedure is completely determined by E, one can easily see that, in fact, these are initial segments of the “whole” structure M_E and embedding j_E (more precisely, they are subsets of M_E and j_E containing initial segments of both). In other words, our standard construction is carried out inside V_α as long as it is defined (this of course depends on the ordinal α, which can be chosen sufficiently large and correct in the universe).

Now, if for some (sufficiently large) α, it happens to be the case that $j_E \restriction V_{\kappa+1} \subseteq (j_E)^{V_\alpha}$ and $V^{M_E}_{j_E(\kappa)+1} \subseteq (M_E)^{V_\alpha}$, then inside V_α we are able to say something regarding, e.g., the size of $j_E(\kappa)$ (and by what we said above, this is going to be correct). These observations lead to a direct formalization of the clauses “$\lambda < j_E(\kappa)$” and “$V_\lambda \subseteq M_E$”: we just require that these hold *inside some* (sufficiently large) V_α, i.e., there exists α such that $\lambda < (j_E)^{V_\alpha}(\kappa)$ and $V_\lambda^{V_\alpha} \subseteq (M_E)^{V_\alpha}$ (where clearly, $V_\lambda = V_\lambda^{V_\alpha}$ for $\alpha > \lambda$).

Finally, let us point out that our preceding discussion applies also to the case of Woodin cardinals; as we will see, they have analogous characterizations in terms of extenders and thus, by arguments similar to the ones given here, these cardinal notions can be formalized in ZFC as well.

At this point, let us open here a brief parenthesis to show something that was mentioned at the end of the previous section (right after Definition 3.6). The following lemma shows that by using short extenders one cannot hope to “capture” large cardinals beyond superstrongs.

Lemma 4.5. *Suppose that E is a short (κ,λ)-extender with λ limit. Then, $\kappa + 1 \leqslant strength(E) \leqslant \lambda$.*

Proof. The first inequality is clear. For the second, by straightforward computations one can easily see that $E \in V_{\lambda+1}$ (note here that $\zeta = \kappa$ since the extender is short). But as we have seen, $E \notin M_E$ and thus, $V_{\lambda+1} \nsubseteq M_E$. $\square$

In particular, if $\lambda = j_{(E)}(\kappa)$ then $V_{j_{(E)}(\kappa)+1} \nsubseteq M_E$ and so, superstrongness is the best we can hope for.

We continue our study by giving the following proposition, which links superstrong to 1-extendible cardinals.

Proposition 4.6. *If κ is 1-extendible, then κ is superstrong and, moreover, there is a normal measure U on κ such that*

$$\{\alpha < \kappa : \alpha \text{ is superstrong}\} \in U.$$

Proof. Suppose that κ is 1-extendible, i.e., there is an elementary embedding $j : V_{\kappa+1} \longrightarrow V_{j(\kappa)+1}$ with $\mathrm{cp}(j) = \kappa$. Let E be the $(\kappa, j(\kappa))$-extender derived from j. Note that this makes sense: for every $n \in \omega$, $\mathcal{P}([\kappa]^n) \subseteq V_{\kappa+1}$ so we can define the ultrafilters E_a, for $a \in [j(\kappa)]^{<\omega}$.

First of all, since our j is between sets and not class models, we have to check that E is indeed a $(\kappa, j(\kappa))$-extender, i.e., that all the relevant sets needed to verify this are actually present (in other words, that the proof of Lemma 3.3 goes through here). First of all note that $[j(\kappa)]^{<\omega} \in V_{j(\kappa)+1}$; next, the ultrafilters E_a (all of which are on some $[\kappa]^n$, for $n \in \omega$) belong to $V_{j(\kappa)}$. Moreover, the "projection" functions $\pi_{ba} : [\kappa]^{|b|} \longrightarrow [\kappa]^{|a|}$ belong to $V_{\kappa+1}$ and finally, any function of the sort $d : \bigcup_{n\in\omega} a_n \longrightarrow \kappa$ belongs to $V_{j(\kappa)+1}$.

It is also easy to see that, in fact, $E \in V_{j(\kappa)+1}$. So, it can be checked *inside* $V_{j(\kappa)+1}$ that conditions (1), (2) and (4) of Definition 3.1 hold for E (note here that for each π_{ba}, $j(\pi_{ba}) \in V_{j(\kappa)+1}$). Also, for normality, as we have mentioned it is sufficient to consider functions $f : [\kappa]^n \longrightarrow \kappa$ (for $n \in \omega$), all of which are in $V_{\kappa+1}$. Thus, exactly as in the proof of Lemma 3.3, normality can be checked inside $V_{j(\kappa)+1}$ as well.

Therefore, we may conclude that $V_{j(\kappa)+1} \models$ "E is a $(\kappa, j(\kappa))$-extender" and, by our previous discussion, this is correct in V.

Of course, our eventual aim is to show that $V_{j(\kappa)+1} \models$ "κ is superstrong" and then consider the usual normal ultrafilter U defined from j, in order to get the second part of the desired conclusion. Before this though, it might seem tempting to try directly to conclude, at this point, that κ *is* superstrong (by Proposition 4.3). The problem again is that, in the proof of Proposition 4.3, we used several facts that were proved for extenders derived from embeddings between *class models* and not sets, as we have here. So, as tedious as it may seem, we do have to verify that analogous properties hold in our case as well, and then conclude that Proposition 4.3 applies in order to get the superstrongness of κ. Note that this was exactly the reason why we had to check all these details in the previous paragraphs, instead of just immediately quoting Lemma 3.3.

So, our essential aim here is to see that something similar to the proof of Proposition 2.5 (clauses (i) and (iii)) goes through. We have already established that E indeed is a $(\kappa, j(\kappa))$-extender and so let $j_E : V \longrightarrow M_E$

be the extender embedding. Note that we may not get a "full" embedding of the sort $k_E : M_E \longrightarrow M$ as we did in Section 2 but, nevertheless, we can define the "restricted" version $k_E^* : V_{j_E(\kappa)}^{M_E} \longrightarrow V_{j(\kappa)}$, by letting:

$$k_E^*([a,[f]]) = j(f)(a)$$

for all $[a,[f]] \in V_{j_E(\kappa)}^{M_E}$ (for $a \in [j(\kappa)]^{<\omega}$ and appropriate $f : [\kappa]^{|a|} \longrightarrow V$).

Let us first see that this is a well-defined map. For this, note that if $[a,[f]] \in V_{j_E(\kappa)}^{M_E}$ (for some a and some f on $[\kappa]^{|a|}$), since (by elementarity and definition) $V_{j_E(\kappa)}^{M_E} = j_E(V_\kappa) = [a, [c^a_{V_\kappa}]]$, we have that $f(s) \in V_\kappa$ for almost all $s \in [\kappa]^{|a|}$, i.e., we may as well assume that $f : [\kappa]^{|a|} \longrightarrow V_\kappa$. Then, $f \in V_{\kappa+1}$ hence, in this case, $j(f)(a)$ makes sense and belongs to $V_{j(\kappa)}$. Also, recall that in M_E,

$$[a,[f]] \ {}^{\in}\!/_{=} \ [\, b,[g]\,] \longleftrightarrow j(f)(a) \ {}^{\in}\!/_{=} \ j(g)(b) \ .$$

and so it follows that k_E^* is well-defined (actually, it is an $\{\in\}$-embedding).

Next, we show that k_E^* is in fact the identity map. Fix some bijection $g : [\kappa]^1 \longrightarrow V_\kappa$ and note that $g \in V_{\kappa+1}$. Then, by elementarity, we have that $j(g) : [j(\kappa)]^1 \longrightarrow V_{j(\kappa)}$ is also a bijection and $j(g) \in V_{j(\kappa)+1}$. Thus, for every $x \in V_{j(\kappa)}$, there is some $\xi < j(\kappa)$ such that $x = j(g)(\{\xi\})$. But this means that for every $x \in V_{j(\kappa)}$, $x = k_E^*([\{\xi\},[g]])$ for some $\xi < j(\kappa)$, i.e., k_E^* is also surjective. Therefore, it must be the identity, since its domain and range are transitive sets. This means that $V_{j_E(\kappa)} \cap M_E = V_{j(\kappa)}$ and so $V_{j(\kappa)} \subseteq M_E$ and $j(\kappa) = j_E(\kappa)$ and therefore, using Proposition 4.3, we have finally established that κ *is* a superstrong cardinal.

We now turn to the second part of the enunciation. As we have pointed out, our aim is to show that $V_{j(\kappa)+1} \models$ "κ is superstrong". For this, it is sufficient to argue that the condition "$V_{j_E(\kappa)} \subseteq M_E$" (which we just showed) can be in fact verified inside $V_{j(\kappa)+1}$.

As we have said, $V_{j(\kappa)+1}$ will compute $(M_E)^{V_{j(\kappa)+1}}$ and $(j_E)^{V_{j(\kappa)+1}}$, which are the "direct limit" and the "extender embedding" in *its* sense. We will now show that, in fact, $(j_E)^{V_{j(\kappa)+1}}(\kappa) = j_E(\kappa)$ and $(j_E)^{V_{j(\kappa)+1}}(V_\kappa) = j_E(V_\kappa)$ from which we will be able to deduce what we are aiming for.

So, for the first, recall that $j_E(\kappa)$ is the order-type of the set

$$\{[a,[f]] : a \in [j(\kappa)]^{<\omega}, f : [\kappa]^{|a|} \longrightarrow \kappa\}$$

and since $[j(\kappa)]^{<\omega} \in V_{j(\kappa)+1}$ and $V_{\kappa+1} \subseteq V_{j(\kappa)+1}$, this is computed correctly, i.e., $(j_E)^{V_{j(\kappa)+1}}(\kappa) = j_E(\kappa) = j(\kappa)$ (as we saw above).

For the second, we have actually already argued –when dealing with the fact that k_E^* is well-defined– that $j_E(V_\kappa) = V_{j_E(\kappa)}^{M_E} = V_{j(\kappa)} \subseteq (M_E)^{V_{j(\kappa)+1}}$ and so $V_{j(\kappa)}^{V_{j(\kappa)+1}} \subseteq (M_E)^{V_{j(\kappa)+1}}$ or, in other words, $V_{j(\kappa)+1} \models V_{j(\kappa)} \subseteq M_E$ which is what we wanted. All the above verify that, indeed,

$$V_{j(\kappa)+1} \models \text{“}\kappa \text{ is superstrong”}.$$

To complete the proof, let U be the usual normal ultrafilter defined from j. Then, by what we just showed and by the definition of U, we (finally) have that $\{\alpha < \kappa : \alpha \text{ is superstrong}\} \in U$. $\square$

Let us remark that, although all these verifications that had to be made in the previous proof may seem rather tedious, in some sense they are "the price to pay" for being able to formalize these large cardinal notions in ZFC. At any rate, from now on we will not always be as detailed and meticulous, most of the time leaving such verifications as straightforward exercises for the interested reader.

Before turning to the important concept of a Woodin cardinal, let us give some easy lemmata showing certain properties of strong and superstrong cardinals. The first lemma shows that, in the presence of strong cardinals, the constructibility axiom fails in a –as their name indicates– strong sense.

Lemma 4.7. *If there is a strong cardinal, then $V \neq L(A)$, for any set A.*

Proof. This is similar to Scott's proof regarding measurable cardinals and $V \neq L$. Let κ be the least strong cardinal and assume, towards a contradiction, that for some set A, $V = L(A)$. Now, by the characterization of Lemma 4.4, let $j : V \longrightarrow M$ be an elementary embedding with $\mathrm{cp}(j) = \kappa$ and $A \in M$.

Note that, since M is a transitive model which contains all ordinals and the set A, our assumption implies that $V = L(A) = M$. But then, by elementarity, $M \models$"$j(\kappa)$ is the least strong cardinal" which is clearly a contradiction, since $j(\kappa) > \kappa$. $\square$

The next lemma shows that strong cardinals have strong reflection properties. So, they continue to keep up with their names.

Lemma 4.8. *If κ is strong, then $V_\kappa \prec_2 V$.*

Proof. Suppose that κ is strong. Let $\exists\, y\, \phi(x, y)$ be a Σ_2 formula, where $\phi(x, y)$ is Π_1 and fix some $a \in V_\kappa$. On the one hand, if $V_\kappa \models \phi[a, b]$, for some $b \in V_\kappa$, then since κ is inaccessible, we know that $V_\kappa \prec_1 V$ and thus $\phi[a, b]$ holds.

On the other hand, if $\phi[a, b]$ holds for some b (in V), let $\lambda > \mathrm{rk}(b)$ and consider $j : V \longrightarrow M$, an elementary embedding witnessing the λ-strongness of κ, i.e., $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. Then, notice that $b \in M \cap V_{j(\kappa)}$.

In this case, since the formula $\phi(x, y)$ is Π_1 (and $j(a) = a$), we have that $M \models \phi[j(a), b]$. Then, $M \models \exists y \in V_{j(\kappa)}\, \phi(j(a), y)$ and thus, by elementarity, $\exists\, y \in V_\kappa\, \phi(a, y)$ holds, i.e., $\phi[a, c]$, for some $c \in V_\kappa$. By the inaccessibility of κ, the latter is equivalent to $V_\kappa \models \phi[a, c]$ and hence, $V_\kappa \models \exists\, y\, \phi(a, y)$. $\square$

Finally, the next two lemmata show that strong cardinals do not imply existence of other large cardinals above them, while superstrong cardinals do. So, strong cardinals do not keep up with the Joneses.

Lemma 4.9. *If κ is superstrong, witnessed by $j : V \longrightarrow M$ with* $\mathrm{cp}(j) = \kappa$ *and $V_{j(\kappa)} \subseteq M$, then $\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is unbounded in $j(\kappa)$.*

Proof. Suppose that κ is superstrong, witnessed by $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. We define the usual (normal) ultrafilter U from the embedding, by $X \in U \longleftrightarrow \kappa \in j(X)$, for $X \subseteq \kappa$.

Now, $U \in \mathcal{P}\mathcal{P}(\kappa) \in V_{\kappa+3} \subseteq V_{j(\kappa)}$ and thus $U \in M$. It follows that $M \models$ "κ is measurable" which means, by definition of the ultrafilter, that $\{\alpha < \kappa : \alpha \text{ is measurable}\} \in U$ and so this set is unbounded in κ. By elementarity, the same holds for the set $\{\alpha < j(\kappa) : (\alpha \text{ is measurable})^M\}$, i.e., it is unbounded in $j(\kappa)$.

Now note that, since $V_{j(\kappa)} \subseteq M$, for every $\alpha < j(\kappa)$, $\mathcal{P}\mathcal{P}(\alpha) \in M$ and thus, α is measurable if and only if $M \models$ "α is measurable". Therefore, we conclude that $\{\alpha < j(\kappa) : \alpha \text{ is measurable}\}$ is unbounded in $j(\kappa)$, as required. $\square$

Lemma 4.10. *If* ZFC + "$\exists\,\kappa\,(\kappa$ *is strong*)" *is consistent, then so is* ZFC + "$\exists\,\kappa\,(\kappa$ *is strong* $\wedge\, \forall\,\lambda > \kappa\,(\lambda$ *is not inaccessible*))".

Proof. Suppose that κ is strong and $\mu > \kappa$ is the least inaccessible above it (if there is one at all). We show that, in this case, $V_\mu \models$ "κ is strong". Equivalently, that for all $\kappa \leqslant \gamma < \mu$, $V_\mu \models$ "κ is γ-strong".

Fix $\kappa \leqslant \gamma < \mu$. Since κ is γ-strong, there exists a $(\kappa, |V_\gamma|^+)$-extender $E = \langle E_a : a \in [|V_\gamma|^+]^{<\omega}\rangle$, such that $\gamma < j_E(\kappa)$ and $V_\gamma \subseteq M_E$. Recall that each ultrafilter E_a is on $[\zeta]^{|a|}$, where $\kappa \leqslant \zeta$ is the least ordinal such that $|V_\gamma|^+ \leqslant j_E(\zeta)$. Obviously, $\zeta \leqslant |V_\gamma|^+$ and since μ is inaccessible and $\gamma < \mu$, we have that $\zeta \leqslant |V_\gamma|^+ < \mu$. Thus, again by inaccessibility, all the relevant sets are in V_μ and it easily follows that, in fact, $E \in V_\mu$. Therefore, we have that $V_\mu \models$ "E is a $(\kappa, |V_\gamma|^+)$-extender".

Moreover, similarly to what we did in Proposition 4.6, one can verify that $j_E(\kappa) = (j_E)^{V_\mu}(\kappa)$ and $j_E(V_\kappa) = (j_E)^{V_\mu}(V_\kappa)$ from which, $\gamma < (j_E)^{V_\mu}(\kappa)$ and $V_\gamma = V_\gamma^{V_\mu} \subseteq (M_E)^{V_\mu}$ follow respectively. This means that the clauses $\gamma < j_E(\kappa)$ and $V_\gamma \subseteq M_E$ can be (correctly) checked inside V_μ and so, we may conclude that, for all $\gamma < \mu$, $V_\mu \models$ "κ is γ-strong" and therefore, $V_\mu \models$ "$\exists\,\kappa\,(\kappa$ is strong $\wedge\, \forall\,\lambda > \kappa\,(\lambda$ is not inaccessible))".

The proof might as well have ended here, but let us be a little bit more careful (i.e., formal) with these issues.

Consider the relevant formal theories $\mathsf{S} = \mathsf{ZFC}$ + "$\exists\,\kappa\,(\kappa$ is strong)" and $\mathsf{T} = \mathsf{ZFC}$ + "$\exists\,\kappa\,(\kappa$ is strong $\wedge\, \forall\,\lambda > \kappa\,(\lambda$ is not inaccessible))". In order to show that $\mathrm{Con}(\mathsf{S}) \Longrightarrow \mathrm{Con}(\mathsf{T})$, we argue as follows.

Define, in S, the (non-empty) class (i.e., predicate)

$$M = \{x : \forall\,\lambda\,(\lambda > \kappa_0 \wedge \text{“}\lambda \text{ is inaccessible”} \longrightarrow x \in V_\lambda)\}$$

where κ_0 throughout denotes the least strong cardinal, provably existing from the theory S. Now note that

$$\mathsf{S} \vdash M = V \vee (\text{“}\mu \text{ is the least inaccessible} > \kappa_0\text{”} \wedge M = V_\mu)$$

where "$M = V$" is just a shorthand for "$\forall x\,(x \in M)$" (and similarly for the second disjunct). Of course, on the basis of S it cannot be decided which of the two is the case (actually, this is what we're trying to show here) but this turns out to be irrelevant. The point is that, all of the argument given above shows that

$$\mathsf{S} \vdash \text{"}\mu \text{ is the least inaccessible} > \kappa_0\text{"} \longrightarrow M = V_\mu \models \text{"}\kappa_0 \text{ is strong"}$$

and therefore, in either case, $\mathsf{S} \vdash$ "M is a model of T", where by this we mean that, for every formula $\varphi \in \mathsf{T}$, $\mathsf{S} \vdash \varphi^M$.

The latter is enough in order to establish $\mathrm{Con}(\mathsf{S}) \Longrightarrow \mathrm{Con}(\mathsf{T})$: given any proof of an inconsistency from T, arguing in S, we get a contradiction in M and this produces a proof of an inconsistency from S. $\square$

We remark that this was a completely finitistic (in the meta-theory) relative consistency result, i.e., without reference to the "true" V or any other "external" model of ZFC. The reader who is interested in these delicate issues will greatly benefit by looking at Kunen's [6], where one can find a very careful treatment. In particular, at the relevant Ch. IV of the book, in the spirit of which the previous proof was carried out.

We now turn to the important concept of a Woodin cardinal. This was introduced by W. Hugh Woodin in 1984 and has become a central notion for inner model theory, closely related to (the variants of) the axiom of determinacy. See Kanamori [4], Martin-Steel [7], and Neeman [9] for more details on these issues.

We begin by giving the "informal" definition of a Woodin cardinal, i.e., in terms of existence of elementary embeddings, and we then establish several of its properties in connection with other large cardinal notions. As we proceed, more characterizations of "Woodinness" will appear, enabling us to eventually give an equivalent reformulation in terms of extenders.

Definition 4.11. A cardinal δ is **Woodin** if, for every $f \in {}^{\delta}\delta$, there is a $\kappa < \delta$ with $f``\kappa \subseteq \kappa$, and an elementary embedding $j : V \longrightarrow M$ into some transitive M, with $\mathrm{cp}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.

The next lemma shows some basic properties of Woodin cardinals, placing them –via a lower bound– in the picture of the large cardinal hierarchy.

Lemma 4.12. *Suppose that δ is Woodin. Then, δ is regular and moreover $\{\alpha < \delta : \alpha \text{ is measurable}\}$ is stationary in δ. In particular, δ is δ-Mahlo.*

Proof. Let δ be Woodin and suppose that it is singular, i.e., $\mathrm{cof}(\delta) = \gamma < \delta$. Let $f : \delta \longrightarrow \delta$ be such that $\langle f(\xi) : \xi < \gamma\rangle$ is increasing and cofinal in δ, with $f(0) > \gamma$. By definition, there is some $\kappa < \delta$ with $f``\kappa \subseteq \kappa$. Now note that it cannot be that $\kappa \leqslant \gamma$, since then $f(0) > \kappa$ and so $f``\kappa \nsubseteq \kappa$. Also, it cannot be that $\gamma < \kappa$ either, because then $f``\gamma \subseteq f``\kappa \subseteq \kappa$, while $f``\gamma$ is supposed to be cofinal in δ. In either case we get a contradiction and thus, we conclude that δ is regular.

For the second part, let $A = \{\alpha < \delta : \alpha \text{ is measurable}\}$. First of all, note that this set is unbounded in δ: for every $\alpha < \delta$, consider any function $f \in {}^{\delta}\delta$ such that $f(0) > \alpha$; then, if $\kappa < \delta$ is such that $f``\kappa \subseteq \kappa$ given by Definition 4.11, it follows that κ is a measurable above α. We will now show that the set A is in fact stationary in δ.

For this, fix some club $C \subseteq \delta$ and we want to see that $A \cap C \neq \varnothing$. Now consider the function $f : \delta \longrightarrow \delta$ that enumerates C, i.e., for every $\xi < \delta$, $f(\xi) =$ the ξ^{th} element of C.

Since δ is Woodin, there is some (measurable) $\kappa < \delta$ with $f``\kappa \subseteq \kappa$. Now, observe that, for every $\xi < \delta$, $\xi \leqslant f(\xi)$ and, moreover, $f$ is strictly increasing. Thus, it follows that $f``\kappa$ is unbounded in κ: if $\alpha < \kappa$, then $\alpha \leqslant f(\alpha) < f(\alpha+1) < \kappa$. Hence, since C is a club (in δ), we get that $\kappa \in C$ and thus $\kappa \in A \cap C$, which is what we wanted.

To complete the proof, for the "in particular" part, we have the following. First, note that δ is a strong limit since there are unboundedly many measurables below it (thus, being also regular, we get that it is inaccessible). In fact, as we just saw, there are *stationarily* many measurable below it, which implies that δ is (strongly) Mahlo. Now recall that if α is measurable, then it is also α-Mahlo. So, for every ordinal $\kappa < \delta$, if we let $A_\kappa = \{\alpha < \delta : \alpha \text{ is } \kappa\text{-Mahlo}\}$, then it is readily seen that $A \cap [\kappa, \delta) \subseteq A_\kappa$ and thus, A_κ is stationary in δ. This shows that δ is $(\kappa+1)$-Mahlo for every $\kappa < \delta$. Consequently, δ is indeed δ-Mahlo. □

As we will later see, "δ is Woodin" is actually a Π^1_1 property of the structure $\langle V_\delta, \in\rangle$ and so the least Woodin is not weakly compact. Thus, the lower bound established in the previous lemma is (essentially) optimal. Next, we turn to the connection between Woodin and superstrong cardinals, this time giving an upper bound for the former.

Proposition 4.13. *If κ is superstrong then it is Woodin and, moreover, there is a normal measure U on κ such that*

$$\{\alpha < \kappa : \alpha \textit{ is Woodin}\} \in U.$$

Proof. Suppose that κ is superstrong and that this is witnessed by the elementary embedding $j : V \longrightarrow M$, i.e., $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. In order to verify that κ is Woodin, let us fix some $f : \kappa \longrightarrow \kappa$. The idea is to show that the desired properties hold for $j(f)$ in M and then, by elementarity, to conclude the same for the function f in V. In this direction, first of all note that since $\mathrm{cp}(j) = \kappa$, for every $\xi < \kappa$, $j(f)(\xi) = f(\xi)$ and so $j(f)``\kappa = f``\kappa \subseteq \kappa$. We distinguish two cases.

If $j(f)(\kappa) \leqslant \kappa$, then let U be the usual normal ultrafilter on κ derived from j and note that $\{\xi < \kappa : f(\xi) = j(f)(\xi) \leqslant \xi\} \in U$ and, in addition, $M \models$ "U is a normal measure on κ". Therefore, we have that:

$$\begin{aligned} M \models \exists\, \alpha < j(\kappa)\, \exists\, U\, (&\text{“}U \text{ is a normal measure on } \alpha\text{”} \wedge j(f)``\alpha \subseteq \alpha \\ &\wedge j(f)(\alpha) \leqslant \alpha \wedge \{\xi < \alpha : j(f)(\xi) \leqslant \xi\} \in U) \end{aligned}$$

and so, by elementarity, we may find some measurable $\alpha < \kappa$ and a D, such that D is a normal measure on α, $f``\alpha \subseteq \alpha$, $f(\alpha) \leqslant \alpha$ and, moreover, $\{\xi < \alpha : f(\xi) \leqslant \xi\} \in D$.

Now, let $j_D : V \longrightarrow M_D$ with $\mathrm{cp}(j_D) = \alpha$ be the ultrapower elementary embedding. Since D is normal, we have that $\alpha = [id]_D$ and therefore, $\{\xi < \alpha : f(\xi) \leqslant \xi\} \in D$ implies that $j_D(f)(\alpha) \leqslant \alpha$. Consequently, $V_{j_D(f)(\alpha)} \subseteq V_\alpha \subseteq M_D$ which is what we wanted.

If, on the other hand, $j(f)(\kappa) > \kappa$, then we argue as follows. Since we have that $(j(\kappa)$ is inaccessible$)^M$ and $j(f)(\kappa) < j(\kappa)$, it follows that $|V_{j(f)(\kappa)}|^M < j(\kappa)$. So, if we let E be the $(\kappa, |V_{j(f)(\kappa)}|^M)$-extender derived from j (notice that in this case $\zeta = \kappa$), then it is easy to see that $E \in V_{j(\kappa)} \subseteq M$ and, in addition, $M \models$ "E is a $(\kappa, |V_{j(f)(\kappa)}|)$-extender".

Next, we show that $V_{j_E(f)(\kappa)} \subseteq M_E$. For this, first observe that by Proposition 2.5, we have that $V_{j(f)(\kappa)} \subseteq M_E$ and so, it is sufficient to show that $j_E(f)(\kappa) = j(f)(\kappa)$. Consider $k_E : M_E \longrightarrow M$, the usual embedding commuting with j_E and j. Again by Proposition 2.5, $k_E \restriction \left(|V_{j(f)(\kappa)}|^M\right) = id$. So, since $\kappa < j(f)(\kappa) \leqslant |V_{j(f)(\kappa)}| \leqslant |V_{j(f)(\kappa)}|^M$, we have that $k_E(\kappa) = \kappa$ and thus,

$$j(f)(\kappa) = k_E \circ j_E(f)(\kappa) = k_E(j_E(f))(k_E(\kappa)) = k_E(j_E(f)(\kappa)) \geqslant j_E(f)(\kappa).$$

But now, observe that it cannot be the case that $j(f)(\kappa) > j_E(f)(\kappa)$ since then, $k_E(j_E(f)(\kappa)) = j_E(f)(\kappa)$, i.e., $j(f)(\kappa) = j_E(f)(\kappa)$. Thus, we anyway conclude that $V_{j_E(f)(\kappa)} \subseteq M_E$.

The final step towards establishing the "Woodinness" of κ, is to show that all the above actually hold in M and then, to use elementarity to "pull back things" in V, as we did before. Now, since $V_{j(\kappa)} \subseteq M$, it is easily verified that (as in the proof of Proposition 4.6) $j_E(V_\kappa) = (j_E)^M(V_\kappa)$ and then, since $j_E(f)(\kappa) < j(\kappa)$ and $V_{j_E(f)(\kappa)} \subseteq M_E$, it follows that $M \models V_{j_E(f)(\kappa)} \subseteq M_E$.

Moreover, $f = j(f) \cap V_\kappa$ and so $(j_E)^M(f) = (j_E)^M(j(f)) \cap V^{M_E}_{j_E(\kappa)}$, from which it follows that $(j_E)^M(f)(\kappa) = (j_E)^M(j(f))(\kappa)$. Consequently,

$$\begin{aligned} M \models \exists\, \alpha < j(\kappa)\, \exists\, \beta\, \exists\, E\ (&\text{“}\alpha \text{ is measurable”} \wedge j(f)``\alpha \subseteq \alpha \wedge E \in V_{j(\kappa)} \wedge \\ &\text{“}E \text{ is an } (\alpha,\beta)\text{-extender”} \wedge V_{j_E(j(f))(\alpha)} \subseteq M_E\,) \end{aligned}$$

and then, by elementarity, we may find a measurable $\alpha < \kappa$ with $f``\alpha \subseteq \alpha$ and an (α, β)-extender $E \in V_\kappa$ (for some β), such that $V_{j_E(f)(\alpha)} \subseteq M_E$. Thus, we may conclude that κ is Woodin.

For the second part of the proposition, if we consider again U, the usual normal ultrafilter derived from the embedding j, then it is sufficient to show that $M \models$ "κ is Woodin" from which, the conclusion follows. For this, fix some $f \in {}^\kappa\kappa \subseteq M$. Observe that our previous arguments (in either case) can be completely done inside M, since $V_{j(\kappa)} \subseteq M$: in the first case, $D \in M$ and then M can construct (enough of) j_D to verify that $V_{j_D(f)(\alpha)} \subseteq M_D$; in the second case, the witnessing extender $E \in V_\kappa$ and then M can certainly verify the fact that $V_{j_E(f)(\alpha)} \subseteq M_E$. This completes the proof. $\square$

Recall that, as we have already remarked (but not yet shown), Woodin cardinals may not even be weakly compact; let alone measurable. But, despite the fact that Woodin cardinals have the strange characteristic of not necessarily being (very) large cardinals themselves, they nevertheless imply the existence of many large cardinals below them. This is the content of the next proposition. Note that in its enunciation, owing to the lack of measurability, the usual "normal measure" clause is –as one should expect– replaced by stationarity.

Proposition 4.14. *If κ is Woodin, then the set*

$$S = \{\alpha < \kappa : \alpha \textit{ is } \gamma\textit{-strong for every } \gamma < \kappa\}$$

is stationary in κ.

Proof. Suppose that κ is Woodin, S is as above and let $C \subseteq \kappa$ be any club in κ. Our aim of course is to show that $S \cap C \neq \varnothing$. Consider the function $g : \kappa \longrightarrow \kappa$ where, for every $\xi < \kappa$,

$$g(\xi) = \begin{cases} 0 \quad , & \text{if } \xi \text{ is } \gamma\text{-strong for every } \gamma < \kappa \\ \\ \gamma \quad , & \gamma > \xi \text{ is the least inaccessible } < \kappa \\ & \text{such that } \xi \text{ is not } \gamma\text{-strong} \end{cases}$$

(note that this function is well-defined since, as we have seen, there are stationarily many measurables below κ).

Now, for every $\xi < \kappa$, define $f : \kappa \longrightarrow \kappa$ by:

$$f(\xi) = max\left(\{g(\xi) + 5, min(C \backslash \xi)\}\right).$$

Before continuing any further with the proof, let us describe the idea behind these definitions and the strategy that we will follow. Initially, using the "Woodinness" of κ (with respect to the function f), we will find an $\alpha < \kappa$ and a $j : V \longrightarrow M$ as in Definition 4.11. Our aim then is to show that $M \models \alpha \in j(S) \wedge \alpha \in j(C)$ which, by elementarity, will imply the desired conclusion. In order to show this fact, we will proceed as follows: for the second conjunct, the clause "$min(C \backslash \xi)$" of the definition of f will be used; for the first conjunct, the clause "$g(\xi) + 5$" and the definition of the function g, will ensure that the extender witnessing the γ-strongness of α (for any $\gamma < j(\kappa)$) belongs to M and that this γ-strongness can be faithfully verified inside the latter.

So, in this direction, let $\alpha < \kappa$ and $j : V \longrightarrow M$ be such that $f``\alpha \subseteq \alpha$, $\mathrm{cp}(j) = \alpha$ and $V_{j(f)(\alpha)} \subseteq M$.

First of all, note that $j(C)$ is club in $j(\kappa)$ and that $j(C) \cap \alpha = C \cap \alpha$. We claim that the latter is club in α: the fact that it is closed is immediate; if $C \cap \alpha \subseteq \beta < \alpha$ (i.e., bounded in α), then consider any $\beta < \xi < \alpha$ and observe that, since $min(C \backslash \xi) \geqslant \alpha$, it follows that $f(\xi) \geqslant \alpha$ which contradicts $f``\alpha \subseteq \alpha$. Therefore, we may conclude that $\alpha \in j(C)$ which is the second conjunct of what we wanted.

For the first conjunct now, note that if $j(g)(\alpha) = 0$, then by elementarity and the definition of g we have that $\alpha \in j(S)$, i.e., we are done. So assume otherwise and let E be the $(\alpha, j(g)(\alpha)+1)$-extender derived from j. We will actually show that, in this case, $M \models$ "α is $j(g)(\alpha)$-strong", a contradiction from which the first conjunct will follow.

Let us verify that $E \in M$. Obviously, for every $\xi < \kappa$, $g(\xi) < f(\xi)$ and so, by elementarity, $j(g)(\alpha) < j(f)(\alpha)$. Moreover, for every $\beta < \alpha$, $f(\beta) < \alpha$ and so, by elementarity, $j(f)(\alpha) < j(\alpha)$. Thus, we have the following order:

$$\alpha < j(g)(\alpha) < j(g)(\alpha) + 5 \leqslant j(f)(\alpha) < j(\alpha)$$

(where the third inequality comes again from the definitions of g and f). In particular, each ultrafilter E_a of the extender, is on the set $[\alpha]^{|a|}$ (i.e., $\zeta = \alpha$) and therefore, it follows that $E \in V_{j(g)(\alpha)+5} \subseteq V_{j(f)(\alpha)} \subseteq M$ and in addition, $M \models$ "E is an $(\alpha, j(g)(\alpha)+1)$-extender".

Now, if $j_E : V \longrightarrow M_E$ is the extender embedding as usual, then by Proposition 2.5, we get that $\mathrm{cp}(j_E) = \alpha$ and $j(g)(\alpha) < j_E(\alpha)$.

Moreover, $(j(g)(\alpha)$ is inaccessible$)^M$ and so, again by Proposition 2.5, we have that $V^M_{j(g)(\alpha)} = V^{M_E}_{j(g)(\alpha)}$. But since $j(g)(\alpha) < j(f)(\alpha)$ and $V_{j(f)(\alpha)} \subseteq M$, it follows that $V_{j(g)(\alpha)} \subseteq M_E$.

The last two paragraphs showed that α *is* $j(g)(\alpha)$-strong. It remains to verify that this is also the case *in the sense of* M. For this, recall that $j_E(\alpha)$ is the order-type of the set

$$\{[a, [f]] : a \in [j(g)(\alpha) + 1]^{<\omega}, f : [\alpha]^{|a|} \longrightarrow \alpha\}$$

and since $[j(g)(\alpha) + 1]^{<\omega} \in M$ and $V_{\alpha+1} \subseteq M$, by absoluteness of order-types it follows that the embedding $(j_E)^M$ constructed inside M agrees with j_E on α, i.e., $(j_E)^M(\alpha) = j_E(\alpha)$ and thus, $M \models j(g)(\alpha) < j_E(\alpha)$.

Similarly, recalling that $j_E(V_\alpha) = [a, [c^a_{V_\alpha}]]$, it is easy to see that any element $x = [a, [f]] \in [a, [c^a_{V_\alpha}]]$ must be such that f belongs to $V_{\alpha+1} \subseteq M$ and so, $(j_E)^M(V_\alpha) = j_E(V_\alpha)$, i.e., $V^{M_E}_{j_E(\alpha)}$ is the same whether it is computed in V or in M, by the extender E. Thus, and since as we have seen $j(g)(\alpha) < j_E(\alpha)$, $V_{j(g)(\alpha)} \subseteq M$ and $V_{j(g)(\alpha)} \subseteq M_E$, we get $M \models V_{j(g)(\alpha)} \subseteq M_E$ which is what we wanted.

We may conclude at this point that $M \models$ "α is $j(g)(\alpha)$-strong", which contradicts our assumption that $j(g)(\alpha) > 0$. Therefore, $j(g)(\alpha) = 0$ which, by definition and elementarity, means that

$$M \models \text{“}\alpha \text{ is } \gamma\text{-strong for every } \gamma < j(\kappa)\text{”},$$

i.e., $M \models \alpha \in j(S)$. Thus, $M \models \exists \alpha < j(\kappa)(\alpha \in j(S) \cap j(C))$ and so, by elementarity, we may find an $\alpha < \kappa$ such that $\alpha \in S \cap C$. This shows that $S \cap C \neq \varnothing$ and completes the proof. $\square$

Note that since Woodin cardinals are inaccessible, by the previous proposition and the argument given in Lemma 4.10, we get that "Woodinness" is a consistency-wise stronger assumption than strongness.

Next, we turn to some equivalent reformulations of Woodin cardinals. In particular, we give a characterization in terms of extenders and thus, the formalizability of "Woodinness" is established as well. Before this though, we need one more definition.

Definition 4.15. Let κ be a cardinal, $\lambda \geqslant \kappa$ and let A be any set. We say that κ is **λ-strong for** A, if there exists an elementary embedding $j : V \longrightarrow M$ into some transitive M, such that $\mathrm{cp}(j) = \kappa$, $\lambda < j(\kappa)$, $V_\lambda \subseteq M$ and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Of course, as in the case of λ-strongness, one can "refine" this definition so as to include the cases where $\lambda < \kappa$ as well. But again, this causes no essential change in our study. Note that a cardinal κ is λ-strong if and only if it is λ-strong for every $A \in V_\kappa$. We can finally give the following theorem, due to Woodin himself, characterizing such cardinals.

Theorem 4.16. (Woodin) *The following are equivalent:*

(i) *δ is a Woodin cardinal.*

(ii) *For every $A \subseteq V_\delta$, the set*

$$S_A = \{\alpha < \delta : \alpha \textit{ is } \gamma\textit{-strong for } A, \textit{ for every } \gamma < \delta\}$$

is stationary in δ.

(iii) *For every $A \subseteq V_\delta$, there exists a $\kappa < \delta$ which is γ-strong for A, for every $\gamma < \delta$.*

(iv) *For every function $f \in {}^\delta\delta$, there is a $\kappa < \delta$ with $f``\kappa \subseteq \kappa$ and an extender $E \in V_\delta$, such that $\mathrm{cp}(j_E) = \kappa$, $j_E(f)(\kappa) = f(\kappa)$ and $V_{j_E(f)(\kappa)} \subseteq M_E$.*

Proof. The implications $(ii) \Longrightarrow (iii)$ and $(iv) \Longrightarrow (i)$ are obvious. So, let us first show that (i) implies (ii).

For this, suppose that δ is Woodin, fix some $A \subseteq V_\delta$ and consider the set S_A as stated in (ii). Now let $C \subseteq \delta$ be any club in δ. We argue *exactly* as in the proof of Proposition 4.14 in order to show that $S_A \cap C \neq \varnothing$. The only difference here is that we have to make an additional case, regarding the strongness with respect to A. We sketch the argument below; the missing details follow from arguments along the lines of Proposition 4.14.

We first consider the function $g : \delta \longrightarrow \delta$, where for every $\xi < \delta$,

$$g(\xi) = \begin{cases} 0 & , \text{ if } \xi \text{ is } \gamma\text{-strong } \underline{\text{for } A}, \text{ for every } \gamma < \delta \\ \\ \gamma & , \gamma > \xi \text{ is the least inaccessible } < \delta \\ & \quad \text{such that } \xi \text{ is not } \gamma\text{-strong } \underline{\text{for } A} \end{cases}$$

where again, g is well-defined we can use it to define, as before, the function $f : \delta \longrightarrow \delta$, such that for every $\xi < \delta$,

$$f(\xi) = max\left(\{g(\xi) + 5, min(C \backslash \xi)\}\right).$$

Now let $\alpha < \delta$ and $j : V \longrightarrow M$ be such that $f``\alpha \subseteq \alpha$, $\mathrm{cp}(j) = \alpha$ and $V_{j(f)(\alpha)} \subseteq M$. The idea remains the same, i.e., we want to show that the desired properties hold for α in M and then, using elementarity, to conclude the same in V.

As before, it follows that $\alpha \in j(C)$. Moreover, if $j(g)(\alpha) = 0$, then by elementarity and the definition of g, we are done. So assume otherwise and let E be the $(\alpha, j(g)(\alpha) + 1)$-extender derived from j. As one can easily verify, the same arguments that were used in Proposition 4.14 show that, in our current situation as well, $M \models$ "α is $j(g)(\alpha)$-strong". Hence, in order to get that α is $j(g)(\alpha)$-strong *for* $j(A)$ *in* M, it remains to see that

$$M \models j(A) \cap V_{j(g)(\alpha)} = j_E(j(A)) \cap V_{j(g)(\alpha)}$$

or, in other words, that

$$j(A) \cap V^M_{j(g)(\alpha)} = (j_E)^M(j(A)) \cap V^M_{j(g)(\alpha)}.$$

Recall that in Proposition 4.14, we showed $j(g)(\alpha) < min(\{j(\alpha), j_E(\alpha)\})$, $j_E(\alpha) = (j_E)^M(\alpha)$, $j_E(V_\alpha) = (j_E)^M(V_\alpha)$, $V^M_{j(g)(\alpha)} = V^{M_E}_{j(g)(\alpha)} = V_{j(g)(\alpha)}$. Hence, it will be sufficient to show that

$$j(A) \cap V^M_{j(\alpha)} = (j_E)^M(j(A)) \cap (j_E)^M(V_\alpha),$$

since then, we get the desired equality by "cutting down" both sides to $V_{j(g)(\alpha)}$. For this, first observe that, similarly to $j_E(V_\alpha) = (j_E)^M(V_\alpha)$, we may conclude that $j_E(A \cap V_\alpha) = (j_E)^M(A \cap V_\alpha)$; moreover, $j(A \cap V_\alpha) = j_E(A \cap V_\alpha)$ because we also have that $k_E \restriction V_{j(g)(\alpha)} = id$. Therefore, we can now write:

$$j(A) \cap V^M_{j(\alpha)} = j(A \cap V_\alpha) = j_E(A \cap V_\alpha) = (j_E)^M(A \cap V_\alpha).$$

But then, since evidently $A \cap V_\alpha = j(A) \cap V_\alpha$ (given that $\mathrm{cp}(j) = \alpha$), the previous string of equalities gives:

$$j(A) \cap V^M_{j(\alpha)} = (j_E)^M(j(A) \cap V_\alpha) = (j_E)^M(j(A)) \cap (j_E)^M(V_\alpha),$$

as desired. Thus, we may conclude that $M \models$ "α is $j(g)(\alpha)$-strong for $j(A)$", which contradicts our assumption that $j(g)(\alpha) > 0$. Therefore, $j(g)(\alpha) = 0$ which, by definition and elementarity, means that

$$M \models \text{"}\alpha \text{ is } \gamma\text{-strong for } j(A), \text{ for every } \gamma < j(\delta)\text{"},$$

i.e., $M \models \alpha \in j(S_A)$. Thus, $M \models \exists\, \alpha < j(\delta)\, (\alpha \in j(S_A) \cap j(C))$ and so, by elementarity, we may find an $\alpha < \delta$ such that $\alpha \in S_A \cap C$. This shows that $S_A \cap C \neq \varnothing$ and completes the proof of this part.

We now turn to the remaining implication, i.e., that (*iii*) implies (*iv*). So, suppose that (*iii*) holds. We first argue that in this case, for every $f \in {}^\delta\delta$, there is a $\kappa < \delta$ with $f``\kappa \subseteq \kappa$.

Fix a function $f \in {}^\delta\delta$. Since $f \subseteq V_\delta$, by assumption, there is a $\kappa < \delta$ which is γ-strong for f, for every $\gamma < \delta$. We show that this κ works, i.e., we fix $\xi < \kappa$ and we want to see that $f(\xi) < \kappa$ as well.

Let $\beta = max(\{\xi, f(\xi)\}) + 3$. Since $\beta < \delta$, we have that κ is β-strong for f, i.e., there is an embedding $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$, $\beta < j(\kappa)$, $V_\beta \subseteq M$ and $f \cap V_\beta = j(f) \cap V_\beta$. But now note that, by choice of β, the pair $\langle \xi, f(\xi) \rangle$ belongs to $f \cap V_\beta$ and hence, it follows that $j(f)(\xi) = f(\xi)$. Now, using that $f(\xi) < \beta < j(\kappa)$ and $j(f(\xi)) = j(f)(\xi)$, we have that $f(\xi) < \kappa$ which is what we wanted.

By the way, note that a similar argument to the one given in Lemma 4.12 shows that δ must be regular and limit of measurables so, in particular, inaccessible. As we are aiming towards a characterization in terms of extenders, let us also point out the following. Looking at the argument given in Proposition 4.3(i), note that it may be easily adapted in the case of λ-strongness for a set A: given an embedding $j : V \longrightarrow M$ which is λ-strong for A, if we derive again the $(\kappa, |V_\lambda|^+)$-extender E then, as we know, $\lambda < j_E(\kappa)$, $V_\lambda = V_\lambda^M = V_\lambda^{M_E}$, and $k_E \restriction V_\lambda = id$. Hence, given the fact that $A \cap V_\lambda = j(A) \cap V_\lambda$, we can easily see that $A \cap V_\lambda = j_E(A) \cap V_\lambda$ as well.

Therefore, putting together these ideas, we finally argue as follows. Given a function $f \in {}^\delta\delta$, there is some $\kappa < \delta$ which is γ-strong for f, for every $\gamma < \delta$. In particular, $f``\kappa \subseteq \kappa$. Now, if we let $\beta = max(\{\kappa, f(\kappa)\}) + 3$, then since $\beta < \delta$, by the extender characterization we get a $(\kappa, |V_\beta|^+)$-extender E such that $\mathrm{cp}(j_E) = \kappa$, $\beta < j_E(\kappa)$, $V_\beta \subseteq M_E$ and $f \cap V_\beta = j_E(f) \cap V_\beta$.

But as before, it follows that $j_E(f)(\kappa) = f(\kappa) < \beta$ and $V_{j_E(f)(\kappa)} \subseteq M_E$. The only thing that remains to see is that, in fact, $E \in V_\delta$. But the latter follows easily from the inaccessibility of δ and completes the proof. $\square$

The previous theorem has several important consequences to which we now turn our attention.

First of all, it gives an equivalent reformulation of the notion of "Woodinness" in terms of extenders and thus, it shows that Woodin cardinals can be formalized in ZFC, by a procedure analogous to the one we discussed for the case of strong and superstrong cardinals.

In addition, it is readily seen that characterization (iv) is a Π^1_1 property of the structure $\langle V_\delta, \in \rangle$: there is only one second-order universal quantifier, ranging over all functions $f \in {}^\delta\delta$. Consequently, the least Woodin cardinal is Π^1_1-describable and so it is not weakly compact.

Finally, it is worth mentioning that the extender characterization (iv) shows that the "Woodinness" of δ can be faithfully checked inside $V_{\delta+1}$; thus, "δ is Woodin" is absolute between transitive models that contain the initial segment $V_{\delta+1}$. In fact, this last observation has the following consequence, which relates Woodin to supercompact cardinals and concludes this section.

Corollary 4.17. *If κ is 2^κ-supercompact then κ is Woodin and, moreover, there is a normal measure U on κ such that*

$$\{\alpha < \kappa : \alpha \textit{ is Woodin}\} \in U.$$

Proof. Suppose that $j : V \longrightarrow M$ witnesses the 2^κ-supercompactness of κ, i.e., $\mathrm{cp}(j) = \kappa$, $2^\kappa < j(\kappa)$ and ${}^{2^\kappa}M \subseteq M$. Note that, by the closure of M, we have that the restricted embedding $j \restriction V_{\kappa+1} : V_{\kappa+1} \longrightarrow V^M_{j(\kappa)+1}$ actually belongs to M and, moreover, it witnesses there the fact that κ is 1-extendible, i.e., $M \models$ "κ is 1-extendible". Consequently, it now follows from Propositions 4.6 and 4.13 that $M \models$ "κ is Woodin".

Now, on the one hand, the latter statement is computed correctly in M, since $V_{\kappa+1} \subseteq M$, i.e., κ is indeed a Woodin cardinal. Furthermore, and on the other hand, if we consider the usual normal ultrafilter U on κ derived from j then it immediately follows that $\{\alpha < \kappa : \alpha \text{ is Woodin}\} \in U$. □

5. Martin-Steel extenders and Supercompactness

As the final part of our exposition to the machinery of extenders, we briefly present the theory of generalized Martin-Steel extenders. These are extenders which are allowed to have as their "support" sets of the form $[Y]^{<\omega}$, where Y is any transitive set (and not necessarily an ordinal).

We shall also see how such Martin-Steel extenders can be used in order to "capture" large cardinal embeddings at the level of supercompactness. Although the new feature of these generalized extenders comes together with various modifications which need to be done, the underlying intuition is very similar to the one of the ordinary extenders that he had been discussing so far. At any rate, the missing details can be found in the classical Martin-Steel article [7], where such generalized extenders were introduced.

Since we have (hopefully) gained by now some insight into the concept of an extender, we now take the opposite route to the one we took at the very beginning of this exposition and we define our new extender notion by first giving its general properties. Afterwards, we will relate it to a corresponding extender notion derived from an elementary embedding. Thus, the following is very much in the spirit of Definition 3.1.

Definition 5.1. (Martin-Steel)
Let κ be a cardinal and let Y be some transitive set. We say that the sequence $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ is a (κ, Y)-**extender** if, for some $\zeta \geqslant \kappa$, the following conditions are satisfied:

(1) Each E_a is a κ-complete ultrafilter on ${}^a(V_\zeta)$, and at least one E_a is not κ^+-complete.
(2) For every $a \in [Y]^{<\omega}$, $\{s \in {}^a(V_\zeta) : \langle a, \in \rangle \overset{s}{\cong} \langle range(s), \in \rangle\} \in E_a$.
(3) (Coherence) For all $a, b \in [Y]^{<\omega}$ with $a \subseteq b$,
$$X \in E_a \longleftrightarrow \{s \in {}^b(V_\zeta) : s \restriction a \in X\} \in E_b.$$
(4) (Normality) If for some $a \in [Y]^{<\omega}$ and some $f : {}^a(V_\zeta) \longrightarrow V_\zeta$
$$\{s \in {}^a(V_\zeta) : f(s) \in \bigcup range(s)\} \in E_a,$$
then there is some $y \in Y$ such that
$$\{s \in {}^{a\cup\{y\}}(V_\zeta) : f(s \restriction a) = s(y)\} \in E_{a\cup\{y\}}.$$

(5) The direct limit $\widetilde{M_E}$ constructed from E is well-founded.

Several remarks are in order, regarding the previous definition. First of all, our new notion of an extender consists of ultrafilters which are on sets of (finite) functions of the form $s : a \longrightarrow V_\zeta$, instead of just (finite) subsets of ζ. This has some advantages as, for example, the fact that we do not have to deal with "projection" functions any more; we just restrict the finite function on the relevant subset (note this fact in the coherence property).

On the other hand, the absence of a "canonical" well-ordering of the support set Y dictates several changes in order to express the desired properties in the definition. Note that this is the case in condition (2), which should be included if we want to avoid sets of "degenerate" or non order-preserving functions in the ultrafilters.

Regarding the well-foundedness condition (5), let us mention that we are referring to a direct limit structure $\widetilde{M_E} = \langle D_E, \in_E \rangle$ constructed in a totally analogous way to the one in Section 3. The obvious changes which need to be made are along the lines of the (notational) modifications in the coherence property. Thus, for example, after defining the (Scott) equivalence classes of the sort $[a, [f]]$ (where $a \in [Y]^{<\omega}$ and $f : {}^a(V_\zeta) \longrightarrow V$), we stipulate that

$$[a, [f]] \in_E [b, [g]] \longleftrightarrow \exists\, c \supseteq a \cup b \text{ s.t. } \{s \in {}^c(V_\zeta) : f(s \restriction a) \in g(s \restriction b)\} \in E_c.$$

Obviously, after constructing $\widetilde{M_E}$ we may directly consider its transitive collapse M_E and then, as one should expect, we have the usual elementary embedding $j_E : V \longrightarrow M_E$ where, for each $x \in V$, we let $j_E(x) = [a, [c_x^a]]$ for some (any) $a \in [Y]^{<\omega}$.

Let us also remark that there is an equivalent "combinatorial" characterization of the well-foundedness condition, something along the lines of condition (4) in Definition 3.1 (for details, the interested reader is referred directly to the source, i.e., [7]).

This finishes (the sketch of) our description of the new situation, hoping that one can fill in the missing details most of which are straightforward adaptations of previously discussed versions. Let us now give one basic lemma, which establishes two important features of our new extender notion.

Lemma 5.2. *Let $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ be a (κ, Y)-extender (for some transitive Y) with the extender embedding $j_E : V \longrightarrow M_E$. Then,*

(i) $j_E \restriction V_\kappa = id$ *and* $\mathrm{cp}(j_E) = \kappa$.
(ii) $Y \subseteq M_E$.

Proof. For the first part of (i), one employs a standard inductive argument in order to show that, for every $\alpha < \kappa$, $j_E \restriction V_\alpha = id$. In particular, it follows that $\mathrm{cp}(j_E) \geqslant \kappa$. To see that $\mathrm{cp}(j_E) = \kappa$, one then argues similarly to Proposition 3.4(ii).

For (ii), which is a quite important property, condition (2) and normality are employed. Initially, we define for every $y \in Y$ and every $a \in [Y]^{<\omega}$ with

$y \in a$, the function $f_{a,y} : {}^{a}(V_\zeta) \longrightarrow V_\zeta$ by $f_{a,y}(s) = s(y)$, for every $s \in {}^{a}(V_\zeta)$ (these are essentially "projection" functions). Then, we will show that, in fact, $y = [a, [f_{a,y}]] \in M_E$ from which the conclusion follows.

First of all, observe that if $y \in a \cap b$ with $b \in [Y]^{<\omega}$, we then have that $[a, [f_{a,y}]] = [b, [f_{b,y}]]$ in M_E. Now, to show that $y = [a, [f_{a,y}]]$ we proceed inductively on the rank of y (taking care of all $a \in [Y]^{<\omega}$ with $y \in a$ at the same time).

The base case is $y = \varnothing$ (note that by condition (1), $Y \neq \varnothing$ and so $\varnothing \in Y$ by transitivity). Consider any $a \in [Y]^{<\omega}$ such that $\varnothing \in a$. In order to show that $[a, [f_{a,\varnothing}]] = \varnothing$ we argue as follows. Suppose, towards a contradiction, that $[a, [g]] \in [a, [f_{a,\varnothing}]]$ (where, by the above observation we may assume that g is also on ${}^{a}(V_\zeta)$) which means that

$$\{s \in {}^{a}(V_\zeta) : g(s) \in s(\varnothing)\} \in E_a.$$

In particular, $g(s) \in \bigcup range(s)$ for E_a-almost all $s \in {}^{a}(V_\zeta)$. Now, by applying normality, we get some $z \in Y$ such that

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) : g(s \restriction a) = s(z)\} \in E_{a\cup\{z\}},$$

where note that $[a, [g]] = [a \cup \{z\}, [f_{a\cup\{z\},z}]]$. Moreover, by applying coherence, we get

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) : g(s \restriction a) \in s(\varnothing)\} \in E_{a\cup\{z\}}$$

and thus, $\{s \in {}^{a\cup\{z\}}(V_\zeta) : s(z) \in s(\varnothing)\} \in E_{a\cup\{z\}}$. But, by condition (2), it follows that $\{s \in {}^{a\cup\{z\}}(V_\zeta) : z \in \varnothing\} \in E_{a\cup\{z\}}$, which is a contradiction. This shows the base case.

Next, assume that the desired property holds inductively, for all $y' \in Y$ with $\mathrm{rk}(y') < \mathrm{rk}(y) \neq 0$.

On the one hand, let $z \in y$ and $a \in [Y]^{<\omega}$ such that $\{z, y\} \subseteq a$. By condition (2), we have that $\{s \in {}^{a}(V_\zeta) : s(z) \in s(y)\} \in E_a$ which, by our definition means that $\{s \in {}^{a}(V_\zeta) : f_{a,z}(s) \in f_{a,y}(s)\} \in E_a$ and thus, $[a, [f_{a,z}]] \in [a, [f_{a,y}]]$. But now, the inductive hypothesis gives $z \in [a, [f_{a,y}]]$. This shows that $y \subseteq [a, [f_{a,y}]]$ and by our observation, the same holds for any $b \in [Y]^{<\omega}$ with $y \in b$.

On the other hand, if for some element $[a, [g]] \in M_E$ we have that $[a, [g]] \in [a, [f_{a,y}]]$ (where $y \in a$), we argue exactly as in the base case to show that there is some $z \in Y$ such that

$$[a, [g]] = [a \cup \{z\}, [f_{a\cup\{z\},z}]]$$

and

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) : s(z) \in s(y)\} \in E_{a\cup\{z\}}.$$

But now the latter, by condition (2), gives $z \in y$ and then, the inductive hypothesis implies that $z = [a \cup \{z\}, [f_{a\cup\{z\},z}]]$ (since by our observation, the particular finite set is not important, as long as it contains the element in question; in this case z). Thus, $z = [a, [g]] \in y$ which implies the inclusion $[a, [f_{a,y}]] \subseteq y$ and completes the proof. $\square$

The importance of the property "$Y \subseteq M_E$" should be apparent: we are free to choose any transitive set as the support of our extender and then, this set will be included inside the structure M_E. The analogy with $[\lambda]^{<\omega}$ in the case of (κ, λ)-extenders is obvious, just by recalling Proposition 3.4(i). We now turn to a brief discussion of (κ, Y)-extenders derived from ambient elementary embeddings; comparisons with Section 2 are inevitable.

Let $j : V \longrightarrow M$ be an elementary embedding into a transitive model M with $\mathrm{cp}(j) = \kappa$. Let us pick some transitive $Y \subseteq M$ with $\kappa \in Y$, and let $\zeta \geqslant \kappa$ be the least ordinal for which $Y \subseteq V^M_{j(\zeta)} = j(V_\zeta)$. For each $a \in [Y]^{<\omega}$, we define an ultrafilter E_a on ${}^a(V_\zeta)$ by letting:

$$X \in E_a \longleftrightarrow j^{-1} \restriction j(a) \in j(X).$$

Note that $j^{-1} \restriction j(a) : j(a) \longrightarrow a$ is an isomorphism (a is finite) and that if $X \subseteq {}^a(V_\zeta)$, then $j(X) \subseteq {}^{j(a)}(V^M_{j(\zeta)})$; that is, this definition not only makes sense but, arguably, it is the obvious modification of Definition 2.1 which we have to consider. It is again easy to check that for every $a \in [Y]^{<\omega}$, E_a is in fact a κ-complete ultrafilter and that the coherence property is satisfied.

As one should expect, $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ is called *the* (κ, Y)-*extender derived from* j and moreover:

Lemma 5.3. *If $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ is the (κ, Y)-extender derived from $j : V \longrightarrow M$ as above, then E is a (κ, Y)-extender.*

Proof. We argue similarly to Lemma 3.3, but in our new context. First of all, we show that $E_{\{\kappa\}}$ is not κ^+-complete, which follows from $\kappa = \mathrm{cp}(j)$.

For each $\alpha < \kappa$, let $X_\alpha = \{s \in {}^{\{\kappa\}}(V_\zeta) : \alpha < s(\kappa) < \kappa\}$ and we note that $X_\alpha \in E_{\{\kappa\}}$ because $\big(j^{-1} \restriction j(\{\kappa\})\big)(j(\kappa)) = \big(j^{-1} \restriction \{j(\kappa)\}\big)(j(\kappa)) = \kappa$. On the other hand though, it is clear that $\bigcap_{\alpha<\kappa} X_\alpha = \varnothing$.

Also, $j^{-1} \restriction j(a)$ being an isomorphism implies that condition (2) of Definition 5.1 holds as well. For normality now, suppose that for some $a \in [Y]^{<\omega}$ and some function $f : {}^a(V_\zeta) \longrightarrow V_\zeta$, we have that

$$\{s \in {}^a(V_\zeta) : f(s) \in \bigcup range(s)\} \in E_a.$$

By definition of E_a, this means $j(f)(j^{-1} \restriction j(a)) \in \bigcup a$ and so, by transitivity of Y we have that $y = j(f)(j^{-1} \restriction j(a)) \in Y$. It is now easy to check that, for this particular $y \in Y$, the desired conclusion follows.

Finally, we let $j_E : V \longrightarrow \widetilde{M_E}$ and $k_E : \widetilde{M_E} \longrightarrow M$ be the usual embeddings, as depicted in the following diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_E"'] & M \\ \widetilde{M_E} \arrow[ur, "k_E"'] & \end{tikzcd}$$

$$j_E(x) = [a, [c_x^a]], \text{ for each } x \in V \text{ (and for any } a)$$

$$k_E([a, [f]]) = j(f)(j^{-1} \restriction j(a)), \text{ for } f \in V \cap {}^{a(V_\zeta)}V$$

One checks, along the lines of Section 2, that these are elementary embedding commuting with j and so, in particular, $\widetilde{M_E}$ is well-founded. $\square$

Having seen that such a derived E is actually a (κ, Y)-extender, we may work as usual with the transitive collapse M_E of the direct limit structure. Now, recalling Lemma 5.2, we have that $Y \subseteq M_E$. Knowing this, we may try define the *new* (κ, Y)-extender E' derived from j_E. As anticipated, we then have that $E' = E$. This, together with some other related properties, are summarized (and proved) below.

Proposition 5.4. *Let $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ be the (κ, Y)-extender derived from $j : V \longrightarrow M$ and let $j_E : V \longrightarrow M_E$ and $k_E : M_E \longrightarrow M$ be the elementary embeddings associated with E. Then:*

(*i*) $k_E \restriction Y = id$.

(*ii*) $M_E = \{j_E(f)(j_E^{-1} \restriction j_E(a)) : a \in [Y]^{<\omega}, f : {}^a(V_\zeta) \longrightarrow V, f \in V\}$.

(*iii*) *If E' is the (κ, Y)-extender derived from j_E, then $E' = E$.*

Proof. For (i), we recall that by the proof of Lemma 5.2 (ii), for every $y \in Y$ and for any $a \in [Y]^{<\omega}$ with $y \in a$, we have that

$$k_E(y) = k_E([a, [f_{a,y}]]) = j(f_{a,y})(j^{-1} \restriction j(a)) = (j^{-1} \restriction j(a))(j(y)) = y.$$

For (ii), first note that, as a direct corollary of the previous part, for every (*finite*) $a \in [Y]^{<\omega}$, $k_E(a) = a$ and also, using the commutativity of the embeddings, it readily follows that $k_E(j_E^{-1} \restriction j_E(a)) = j^{-1} \restriction j(a)$. So, let $x = [a, [f]] \in M_E$ be any element. Then:

$$k_E(x) = j(f)(j^{-1} \restriction j(a)) = k_E(j_E(f)(j_E^{-1} \restriction j_E(a))),$$

and since k_E is injective, the conclusion follows.

For (iii), let $E' = \langle E'_a : a \in [Y]^{<\omega} \rangle$ be the (κ, Y)-extender derived from j_E and recall that, for every $a \in [Y]^{<\omega}$ and every $X \subseteq {}^a(V_\zeta)$, we have $X \in E'_a \longleftrightarrow j_E^{-1} \restriction j_E(a) \in j_E(X)$. If $a = \varnothing$, then it is obvious that $E_\varnothing = E'_\varnothing = \{\{\varnothing\}\}$.

If $a \neq \varnothing$, then for any $y \in a$, $j_E^{-1}(j_E(y)) = y = [a, [f_{a,y}]]$. Thus, if we consider the function F_a on ${}^a(V_\zeta)$ such that, for every s, $F_a(s)$ is a function on a with $F_a(s)(y) = s(y) = f_{a,y}(s)$, for all $y \in a$, then we have that

$$k_E(j_E^{-1} \upharpoonright j_E(a)) = j^{-1} \upharpoonright j(a) = k_E([a, [F_a]])$$

and, therefore, $j_E^{-1} \upharpoonright j_E(a) = [a, [F_a]]$ by injectivity of k_E. But now note that, by definition of F_a,

$$[a, [F_a]] = [a, [\langle s : s \in {}^a(V_\zeta) \rangle]] = [a, [id^a]]$$

where $id^a : {}^a(V_\zeta) \longrightarrow {}^a(V_\zeta)$ is the identity function. Therefore, we after all have that

$$X \in E'_a \longleftrightarrow j_E^{-1} \upharpoonright j_E(a) = [a, [id^a]] \in [a, [c^a_X]] = j_E(X)$$

and so, $X \in E'_a \longleftrightarrow \{s \in {}^a(V_\zeta) : s \in X\} \in E_a \longleftrightarrow X \in E_a$. □

At this point, we are ready for our basic application which will be to "encode" a given λ-supercompact embedding via an appropriately derived Martin-Steel extender. Having benefitted from the material of [7], we now diverge from this source. The results which are presented below can be found in § 5 of [1]. Let us first describe the ideas and motivation behind the several details with which we will then proceed.

Given a λ-supercompact embedding $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$, the main issue is to pick the right transitive set $Y \subseteq M$ as the support of the derived extender E. Of course, our goal is to pick this set in a way that the usual extender embedding $j_E : V \longrightarrow M_E$ is also λ-supercompact; i.e., such that ${}^\lambda M_E \subseteq M_E$. The dominant idea for showing the latter is to include $j``\lambda$ in $Y$ (and thus in $M_E$), and use it as a "prototype" $\lambda$-sequence in order to encode any other such sequence from $M_E$. Of course, $j``\lambda$ is *essentially* but not literally a λ-sequence, but we are only trying to pinpoint the idea.

To be a little bit more specific, by the representation of M_E given in Proposition 5.4, suppose that we are given $\{x_i : i < \lambda\} \subseteq M_E$ where for each $i < \lambda$, x_i is of the form $x_i = j_E(f_i)(j_E^{-1} \upharpoonright j_E(b_i))$ for some $b_i \in [Y]^{<\omega}$ and some function f_i on ${}^{b_i}(V_\zeta)$. Our aim of course is to find an $A \in [Y]^{<\omega}$ and a function F on ${}^A(V_\zeta)$, such that the element $X = j_E(F)(j_E^{-1} \upharpoonright j_E(A))$ encodes this particular λ-sequence in M_E, i.e., $X(i) = x_i$, for all $i < \lambda$.

As we shall show below, if apart from including $j``\lambda$, we also choose the support set Y in such a way that it is closed under finite subsets, closed under λ-sequences and closed under j, then $j \upharpoonright Y = j_E \upharpoonright Y$ and we may encode the entire λ-sequence of the $j_E^{-1} \upharpoonright j_E(b_i)$'s as a single element of Y. In fact, these conditions on Y, together with the requirement of it being transitive, are sufficient in order to define the A and the F that work. Let us now see how to do it, provided we are given such a Y. After that, we will briefly describe how to obtain such a $Y \subseteq M$; this will conclude the construction and accomplish our goal.

Proposition 5.5. *Suppose that κ is λ-supercompact, witnessed by the embedding $j : V \longrightarrow M$. Suppose that $Y \subseteq M$ is transitive, $[Y]^{<\omega} \subseteq Y$, ${}^\lambda Y \subseteq Y$, $j``Y \subseteq Y$ and $j``\lambda \in Y$. Let E be the (κ, Y)-extender derived* *from j and let $j_E : V \longrightarrow M_E$ be the extender embedding. Then, j_E is λ-supercompact for κ.*

Proof. As we have already seen, $Y \subseteq M_E$ and $k_E \restriction Y = id$, where k_E is the usual embedding commuting with j and j_E. We first note that, in fact, $j \restriction Y = j_E \restriction Y$; this follows easily from commutativity, $k_E \restriction Y = id$ and $j``Y \subseteq Y$. Consequently, for every $a \in [Y]^{<\omega}$, we have that $j_E^{-1} \restriction j_E(a) = j^{-1} \restriction j(a) \in Y$ and so, in particular, in the representation of M_E given in Proposition 5.4(ii), we may replace j_E by j as shown below:

$$M_E = \{j_E(f)(j^{-1} \restriction j(a)) : a \in [Y]^{<\omega}, f : {}^{a}(V_\zeta) \longrightarrow V, f \in V\}.$$

Recall here that ζ is the least ordinal such that $Y \subseteq j(V_\zeta)$. Moreover, if $\mu = \sup(\mathbf{ON} \cap Y)$, then $\lambda < \mu$ and $\mu \subseteq Y$ (since $\lambda < j(\kappa)$, $j``\lambda \in Y$ and $Y$ is transitive) and thus, since $j_E$ and $j$ agree on $Y$, we have that $j_E``\lambda = j``\lambda$. In particular, $\lambda < j_E(\kappa) = j(\kappa)$ where, of course, $\mathrm{cp}(j_E) = \kappa$. So, the crux of the matter towards establishing the λ-supercompactness of the embedding j_E, is to check that ${}^{\lambda}M_E \subseteq M_E$ holds. For this, we make extensive use of the several closure properties of the given Y.

Fix throughout some collection $\{x_i : i < \lambda\} \subseteq M_E$ where, for each $i < \lambda$, $x_i = j_E(f_i)(j^{-1} \restriction j(b_i))$ for some $b_i \in [Y]^{<\omega}$ and some function f_i on ${}^{b_i}(V_\zeta)$. As we have remarked, we want to find some $A \in [Y]^{<\omega}$ and some function F on ${}^{A}(V_\zeta)$, such that the element $X = j_E(F)(j^{-1} \restriction j(A)) \in M_E$ is the λ-sequence of the x_i's, i.e., $X(i) = x_i$, for all $i < \lambda$.

Let $b = \langle j^{-1} \restriction j(b_i) : i < \lambda \rangle$ and observe that, by the closure of Y, both b and the function $j \restriction \lambda : \lambda \longrightarrow j``\lambda$ belong to Y. We now let $A = \{j \restriction \lambda, b\} \in [Y]^{<\omega}$. Before dealing with the definition of the function F on ${}^{A}(V_\zeta)$, let us first point out that

$$j_E(A) = j(A) = \{j(j \restriction \lambda), j(b)\},$$

and then,

$$j^{-1} \restriction j(A) : \{j(j \restriction \lambda), j(b)\} \longrightarrow \{j \restriction \lambda, b\}$$

is the function with values

$$j(j \restriction \lambda) \mapsto j \restriction \lambda \text{ and } j(b) \mapsto b.$$

In addition,

$$j_E^{-1} \restriction j_E(A) = j^{-1} \restriction j(A) \in {}^{j_E(A)}j_E(V_\zeta) \cap {}^{j(A)}j(V_\zeta).$$

We also note that any element $s \in {}^{A}(V_\zeta)$ is of the form

$$\{\langle j \restriction \lambda, s(j \restriction \lambda)\rangle, \langle b, s(b)\rangle\}.$$

These were easy observations which will be needed in the final computations. Finally, let $f = \langle f_i : i < \lambda \rangle$. We now turn to the definition of F on ${}^{A}(V_\zeta)$.

Given $s \in {}^{A}(V_\zeta)$, we first define an auxiliary function g_s as follows:

- If both $s(j \restriction \lambda)$ and $s(b)$ are functions with domain the same ordinal, say α, then g_s is a function on α such that, for every $i < \alpha$,

$$g_s(i) = \begin{cases} f(s(j \restriction \lambda)(i))(s(b)(i)) & , \text{ if } s(j \restriction \lambda)(i) \in dom(f) \text{ and} \\ & \quad s(b)(i) \in dom(f(s(j \restriction \lambda)(i))) \\ \varnothing & , \text{ otherwise.} \end{cases}$$

- Otherwise, $g_s = \varnothing$.

We then define the function F by letting, for every $s \in {}^{A}(V_\zeta)$,

$$F(s) = g_s.$$

Now, by elementarity, the function $j_E(F)$ is on ${}^{j_E(A)}j_E(V_\zeta)$ and so it follows that $j_E(F)(j^{-1} \restriction j(A))$ makes sense, since for the particular element $s = j^{-1} \restriction j(A)$ we have that $s \in {}^{j_E(A)}j_E(V_\zeta)$ as we remarked above. In this situation, $s(j(j \restriction \lambda)) = j \restriction \lambda$ and $s(j(b)) = b$ are certainly functions with domain the same ordinal, namely, λ.

Thus, by the explicit definition of F and elementarity, $j_E(F)(s)$ is going to be the (non-empty) auxiliary function g_s on λ, as described above. Moreover, notice that the second alternative in the definition of g_s does not occur: for every $i < \lambda$,

$$s(j(j \restriction \lambda))(i) = (j \restriction \lambda)(i) = j(i) = j_E(i) \in dom(j_E(f))$$

and, then, $j_E(f)(s(j(j \restriction \lambda))(i)) = j_E(f)(j_E(i)) = j_E(f(i)) = j_E(f_i)$. Hence,

$$s(j(b))(i) = b(i) = j^{-1} \restriction j(b_i) \in dom(j_E(f_i)),$$

since f_i was a function on ${}^{b_i}(V_\zeta)$ and $j^{-1} \restriction j(b_i) = j_E^{-1} \restriction j_E(b_i)$. Therefore, we after all have that, for every $i < \lambda$,

$$X(i) = j_E(F)(j^{-1} \restriction j(A))(i) = j_E(f_i)(j^{-1} \restriction j(b_i)),$$

i.e., $X(i) = x_i$ as desired. This completes the proof. $\square$

Towards concluding, we now briefly describe a way in which, given a λ-supercompact embedding $j : V \longrightarrow M$, one may construct a $Y \subseteq M$ that meets all the requirements stated in the previous proposition.

The idea is simple: we start with $j``\lambda$ (which belongs to M) and we recursively close under everything that we care about. We repeat λ^+-many times (taking unions at limit stages) and the resulting set is our Y. Formally, we define by transfinite recursion on λ^+:

$$\begin{aligned} Y_0 &= \operatorname{trcl}(\{j``\lambda\}) \\ Y_{\alpha+1} &= \operatorname{trcl}(Y_\alpha \cup [Y_\alpha]^{<\omega} \cup {}^{\lambda}Y_\alpha \cup j``Y_\alpha) \\ Y_\alpha &= \bigcup_{\xi<\alpha} Y_\xi \,, \text{ if } \alpha \text{ is limit} \end{aligned}$$

and we then let $Y = Y_{\lambda^+}$. It straightforward to check (by induction) that $Y \subseteq M$, Y is transitive, $[Y]^{<\omega} \subseteq Y$, $j``Y \subseteq Y$ and –of course– $j``\lambda \in Y$. In addition, ${}^{\lambda}Y \subseteq Y$ because given any set of elements $\{a_i : i < \lambda\} \subseteq Y$, there exists some $\alpha < \lambda^+$ such that $\{a_i : i < \lambda\} \subseteq Y_\alpha$ and then, by definition of $Y_{\alpha+1}$, $\langle a_i : i < \lambda \rangle \in Y_{\alpha+1} \subseteq Y$.

This finishes the construction which, together with the previous proposition, jointly accomplish our goal of "encoding" a λ-supercompact embedding by Martin-Steel extenders. In fact, they provide us with the following characterization.

Theorem 5.6. *A cardinal κ is λ-supercompact, for some $\lambda > \kappa$, if and only if there exists a (κ, Y)-extender E such that Y is transitive, $[Y]^{<\omega} \subseteq Y$, ${}^{\lambda}Y \subseteq Y$, $j_E``Y \subseteq Y$, $j_E``\lambda \in Y$ and $\lambda < j_E(\kappa)$.*

Proof. The forward direction follows immediately from the previous proposition and the construction which we just described. For the converse, consider $j_E : V \longrightarrow M_E$. In order to get the additional ${}^{\lambda}M_E \subseteq M_E$ condition, repeat the arguments of the previous proposition replacing j by j_E everywhere. $\square$

At this point, the (remaining) reader might be wondering why we even bothered going through all this analysis, only to conclude something known, i.e., that the property "κ is λ-supercompact" has an equivalent (formalizable) "combinatorial" characterization; it is well-known that this can be expressed using normal fine measures on $\mathcal{P}_\kappa(\lambda)$. Nevertheless, our characterization has at least one advantage over the one in terms of $\mathcal{P}_\kappa(\lambda)$-measures.

If $j : V \longrightarrow M$ is a λ-supercompact embedding which is constructed from a normal fine measure U on $\mathcal{P}_\kappa(\lambda)$, then j has the limitating property $2^{\lambda^{<\kappa}} < j(\kappa) < (2^{\lambda^{<\kappa}})^+$ (see, e.g., Proposition 22.11 in [4]). In such a case, $j(\kappa)$ is not even a cardinal (in V). Consequently, if we want to describe a λ-supercompact embedding j such that $j(\kappa)$ is, in addition, a cardinal, then normal fine measures are of no help.

On the other hand, the use of Martin-Steel extenders circumvents these limitations. Indeed, via the use of such extenders we can describe a variety of λ-supercompact embeddings, requiring each time anything (consistent) for their image $j_E(\kappa)$ (e.g., being a cardinal, being inaccessible, being a Σ_n-correct ordinal, etc.). This idea is repeatedly used in the context of the $C^{(n)}$-cardinals, which are hierarchies of large cardinals that were introduced recently by Bagaria (cf. [1]; see also [10] for a further study of such notions).

For the time being, we shall not tire the reader with more details.

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Department of Logic, History & Philosophy of Science, University of Barcelona, 08001 Barcelona, Spain

E-mail address: kostas.tsap@gmail.com