ON RESURRECTION AXIOMS

KONSTANTINOS TSAPROUNIS

ABSTRACT. The resurrection axioms are forms of forcing axioms that were introduced recently by Hamkins and Johnstone, who developed on earlier ideas of Chalons and Veličković. In this note, we introduce a stronger form of resurrection (which we call *unbounded* resurrection) and show that it gives rise to families of axioms which are consistent relative to extendible cardinals, and which imply the strongest known instances of forcing axioms, such as Martin's Maximum^{++}. In addition, we study the unbounded resurrection postulates in terms of consistency lower bounds, obtaining, for example, failures of the weak square principle.

1. INTRODUCTION

The study of forcing axioms has a long tradition in set theory, emerging shortly after Cohen's introduction of the outstanding method of forcing in 1963 (cf. [6]). Initially, Martin formulated a sort of closure principle for c.c.c. posets, isolating it from the proof of Solovay and Tennenbaum on the consistency of *Suslin's Hypothesis* (cf. [24]). This principle, which is now known as *Martin's Axiom* (MA), generalizes the familiar *Baire category theorem* and has had many applications in various mathematical contexts.

Progressively, other similar postulates were considered, mainly by expanding the class of posets to which such a closure principle applies; two famous strengthenings of MA (in fact of $\mathsf{MA}_{\aleph_1}$) are the *Proper Forcing Axiom* (PFA) and *Martin's Maximum* (MM), both introduced in the 1980's. Strong forms of forcing axioms have dramatic implications for the set-theoretic universe and, in particular, for the continuum and its structure. For instance, PFA (and, a fortiori, MM as well) settles many problems originating from a wide spectrum of mathematical areas, while at the same time it answers important set-theoretic questions, among which lies the *Continuum Hypothesis* (CH): it is known that PFA implies $2^{\aleph_0} = \aleph_2$ (thus $\neg$ CH).

Forcing axioms tend to annihilate the effect of forcing, in the sense that they render certain existential statements absolute between the universe and its relevant generic extensions. Indeed, forcing axioms are often regarded as instances of *generic absoluteness*, an aspect which is well-known;[1] for example, MA is equivalent to asserting that, for any c.c.c. poset $\mathbb{P}$, $H_{\mathfrak{c}}$ is a Σ_1-elementary substructure

Date: November 7, 2013. This is a preliminary version. The finalized published version of this work appears in **The Journal of Symbolic Logic** 80(2), 587–608 (2015); doi: 10.1017/jsl.2013.39.

2010 *Mathematics Subject Classification.* 03E35, 03E55, 03E57.

Key words and phrases. Forcing axioms, Resurrection axioms, Extendible cardinals.

Final parts of this research were carried out during the author's visit to the University of East Anglia (UEA); his visit was supported by the European Science Foundation (ESF), under the INFTY exchange grant No. 4143. The author expresses his gratitude to the members of the School of Mathematics of UEA, and in particular to Prof. Mirna Džamonja, for their hospitality.

[1] And a rather important one, since forcing is essentially our only effective tool for obtaining independence results in set theory.

of the $H_{\mathfrak{c}}$ of the forcing extension $V^{\mathbb{P}}$, written as $H_{\mathfrak{c}} \prec_1 H_{\mathfrak{c}}^{V^{\mathbb{P}}}$ (cf. [1] and, independently, [25]). In addition, the *Bounded Proper Forcing Axiom* (BPFA) is equivalent to asserting that, for any proper poset $\mathbb{P}$, $H_{\aleph_2} \prec_1 H_{\aleph_2}^{V^{\mathbb{P}}}$. Likewise, we have an analogous characterization for *Bounded Martin's Maximum* (BMM), by replacing proper posets with stationary preserving ones in the previous statement (these two latter results are due to Bagaria; see [2]).

Unfortunately, such absoluteness results quickly run into inconsistency if we allow too much liberty of choice regarding the class of posets, the cardinal κ (specifying the structure H_κ), the complexity of the formulas considered, or combinations thereof. These limitations motivate the idea of resurrection from this perspective;[2] namely, we require the existence of an appropriate (name for a) poset $\dot{\mathbb{R}}$ such that, by further forcing with it, we "resurrect" the full elementarity of $H_{\mathfrak{c}}$ into that of the whole forcing extension; that is, we have $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$. With these ideas in mind, we now give the formal definition, as introduced by Hamkins and Johnstone.

Definition 1.1 ([15]). For any (definable) class Γ of posets, the **Resurrection Axiom** for Γ, denoted by $\mathsf{RA}(\Gamma)$, is the assertion that for any $\mathbb{Q} \in \Gamma$, there exists a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$, with $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", such that $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$.

In [15], such axioms are mainly motivated by the model-theoretic concept of *existential closure*, which is exactly the situation where a submodel $\mathcal{N} \subseteq \mathcal{M}$ is a Σ_1-elementary substructure of $\mathcal{M}$. Indeed, Hamkins and Johnstone show that the concept of resurrection generalizes that of existential closure, and they then argue that: [...]*resurrection may allow us to formulate more robust forcing axioms than existential closure or than combinatorial assertions about filters and dense sets*.[3]

They subsequently study several axioms of this form, by either varying the class Γ or by considering *weak resurrection*, where no restriction is imposed on the "resurrecting" poset $\dot{\mathbb{R}}$. Among other results, it is shown in [15] that, in many cases, the axiom $\mathsf{RA}(\Gamma)$ has consistency strength below that of a Mahlo cardinal.

There are many natural and intriguing open questions surrounding the area of resurrection axioms and, with this note, we would like to call further attention to this fruitful subject. The structure of the note is as follows. The rest of this section is devoted to the necessary preliminaries, as well as to a brief review of some relevant issues regarding Laver functions and extendible cardinals.

In Section 2, we motivate and introduce the principles of *unbounded resurrection* (denoted by UR). Subsequently, we show that several UR axioms are consistent relative to (the consistency of) an extendible cardinal, giving also some results on the relation between such principles and various well-known forcing axioms.

In Section 3, we briefly diverge from the context of extendible cardinals; we present an argument due to Asperó which, using Woodin's stationary tower forcing, gives an improved consistency upper bound for the axiom $\mathsf{UR}(\Gamma)$, when Γ is the class of stationary preserving posets.

[2] Notwithstanding, the phenomenon of resurrection has been around for a longer while. For instance, the so-called *Maximality Principle* was initially introduced by Stavi and Väänänen in [25]; following an idea of Chalons, this principle was then rediscovered and further studied by Hamkins (cf. [14]). Moreover, Veličković has also considered similar principles, formulating resurrection particularly in the context of rank-into-rank embeddings. Finally, Woodin's work related to the *stationary tower* forcing provides more background and early considerations of this phenomenon in set theory (see [19] and [30]).

[3] See [15] for more details and further discussion.

In Section 4, we deal with obtaining consistency lower bounds for some UR axioms, mainly via failures of (weak) squares and other related principles. We also introduce and give a brief account of the notion of an *indestructibly generically extendible* cardinal. Finally, in Section 5, we close this note with some concluding thoughts and a few open questions. Let us begin.

1.1. **Preliminaries.** Our notation and terminology are mostly standard.[4] ZFC stands for the usual first-order axiomatization of Zermelo–Fraenkel set theory, together with the Axiom of Choice. The class of ordinals will be denoted by **ON**. If $A \subseteq \mathbf{ON}$, $\sup A$ is the supremum of A (if A is a set) and $\mathrm{Lim}(A)$ is the collection of its limit points, i.e., $\{\xi : \sup(A \cap \xi) = \xi\}$. For any limit ordinal α, $\mathrm{cof}(\alpha)$ is its cofinality. $\mathfrak{c}$ stands for the cardinality of the set of real numbers, i.e., $\mathfrak{c} = 2^{\aleph_0}$. For any cardinal λ, H_λ is the collection of sets whose transitive closure has size less than λ. Following [3], for every natural number n, we let $C^{(n)}$ denote the closed and unbounded proper class of ordinals α which are Σ_n-correct in V, that is, such that V_α is a Σ_n-elementary substructure of V.

Given any function f and any $S \subseteq dom(f)$, we write $f \restriction S$ for the restriction of f to S; moreover, we write $f``S$ for the pointwise image of $S$ under $f$, i.e., $f``S = \{f(x) : x \in S\}$. We use the three dots in order to indicate partial functions, i.e., $f \vdots X \longrightarrow Y$ means that $dom(f) \subseteq X$, with the inclusion possibly being proper.

In the context of forcing posets, we write $p < q$ to mean than p is stronger than q. We denote the greatest element of a poset by $\mathbb{1}$. If $\mathbb{P}$ is a partial ordering on sequences indexed by ordinals and $s, t \in \mathbb{P}$, we say that s is an initial segment of t, denoted by $s \sqsubseteq t$, if $t \restriction dom(s) = s$ and $dom(s) = dom(t) \cap \sup\{\xi + 1 : \xi \in dom(s)\}$. For the closure of a given $\mathbb{P}$ we write, for example, "$\leqslant \kappa$-directed closed" to mean that we may find lower bounds of directed subsets whose cardinality is at most κ. When $\kappa = \aleph_1$, we write "σ-closed" instead of "$< \aleph_1$-closed".

Regarding relativized notions in forcing extensions, we write things like $H_{\aleph_2}^{V[G]}$ and $H_{\mathfrak{c}}^{V^{\mathbb{P}}}$, instead of the more precise forms $(H_{\aleph_2})^{V[G]}$ and $(H_{\mathfrak{c}})^{V^{\mathbb{P}}}$ respectively, which would stress the fact that "$\aleph_2$" and "$\mathfrak{c}$" are computed in the corresponding models. Throughout, in order to avoid ambiguities, we understand such notation by assuming that every defined notion is computed in the superscript model; if the superscript is missing, it is understood that the computations take place in V, the fixed initial (ground) model of the argument at hand, whatever that is. In addition, whenever we consider some definable class Γ of posets, we silently fix and work with a background defining formula φ_Γ for it. Therefore, if $\mathbb{Q}$ is a poset and $\dot{\mathbb{R}}$ is a $\mathbb{Q}$-name for a poset, when we write $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$" we mean that $\mathbb{Q} \Vdash \varphi_\Gamma(\dot{\mathbb{R}})$.

A poset $\mathbb{P}$ is called *stationary preserving* (for subsets of ω_1) if every stationary $S \subseteq \omega_1$ remains stationary in $V^{\mathbb{P}}$. We denote by SSP the class of stationary preserving posets. Every c.c.c., σ-closed, or proper poset, is indeed stationary preserving. An important weakening of properness is the notion of $\aleph_1$-*semi properness*, introduced by Shelah (cf. Chapter X in [23]). Shelah has shown that $\aleph_1$-semi properness is preserved under *revised countable support* (RCS).[5]

---

[4] See [16] or [17] for an account of all undefined set-theoretic notions.

[5] For more details on RCS iterations and further development of the theory of proper (and improper) forcing, see [23]. Although the property of being $\aleph_1$-semi proper is stronger than that of being stationary preserving, there are cases in which the two notions coincide. This coincidence is itself a principle of large cardinal strength and is traditionally denoted by (†). For our purposes, we recall Shelah' s result

If j is a non-trivial elementary embedding we write $\mathrm{cp}(j)$ for its critical point. Whenever we lift embeddings to forcing extensions we follow the common convention and use the same letter j for the lifted embedding.

Finally, recall the following natural strengthenings of the usual forcing axioms.

Definition 1.2. Given a class Γ of posets, the **Forcing Axiom**$^+$ for Γ, denoted by $\mathsf{FA}^+(\Gamma)$, asserts that for every $\mathbb{Q} \in \Gamma$, for every collection $\{A_\alpha : \alpha < \omega_1\}$ of maximal antichains of $\mathbb{Q}$ and given a $\mathbb{Q}$-name τ for a stationary subset of ω_1 (i.e., $\mathbb{Q} \Vdash$ "$\tau \subseteq \omega_1$ is stationary"), there is a filter $G \subseteq \mathbb{Q}$ such that $G \cap A_\alpha \neq \varnothing$, for all $\alpha < \omega_1$, and so that $\tau^G = \{\alpha < \omega_1 : \exists p \in G\,(p \Vdash \check{\alpha} \in \tau)\}$ is stationary in ω_1.

$\mathsf{FA}^{++}(\Gamma)$ is a similar axiom, where instead of a single $\mathbb{Q}$-name we are given ω_1-many names $\{\tau_\xi : \xi < \omega_1\}$ for stationary subsets of ω_1.

1.2. **Laver functions and extendible cardinals.** The following result is due to Laver, who used it to show that any supercompact cardinal can be made indestructible under $<\kappa$-directed closed forcing.

Theorem 1.3 (Laver [20]). *If κ is supercompact then there is $\ell \colon \kappa \longrightarrow V_\kappa$ such that, for any cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$, there is a λ-supercompact embedding $j : V \longrightarrow M$ for κ (i.e., $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$), with $j(\ell)(\kappa) = x$.*

In what follows, we shall be interested in similar functions but for extendible cardinals. Recall that κ is called λ-*extendible*, for some $\lambda > \kappa$, if there is some θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ such that $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$; κ is called *extendible* if it is λ-extendible for all $\lambda > \kappa$. Extendibility, which is a stronger notion than that of supercompactness, is witnessed locally by set embeddings. Corazza has shown that extendible cardinals carry appropriate Laver functions (cf. [7]). Nevertheless, since it will be convenient to work with class embeddings, we now give an alternative characterization of extendibility.

Definition 1.4 ([26]). A cardinal κ is called **jointly λ-supercompact and θ-superstrong**, for some $\lambda, \theta \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $V_{j(\theta)} \subseteq M$. In such a case, we say that j *is jointly λ-supercompact and θ-superstrong for κ.*

For the global version(s) of this notion, we say that κ is jointly supercompact and θ-superstrong, for some fixed $\theta \geqslant \kappa$, if it is jointly λ-supercompact and θ-superstrong, for every $\lambda \geqslant \kappa$. Moreover, we say that κ is jointly supercompact and superstrong if it is jointly λ-supercompact and λ-superstrong, for every $\lambda \geqslant \kappa$.

The above notion transcends supercompactness: if κ is the least supercompact, then it is not jointly λ-supercompact and κ-superstrong, for any λ. In fact, global joint supercompactness and κ-superstrongness is equivalent to extendibility:

Theorem 1.5 ([26]). *A cardinal κ is extendible if and only if it is jointly supercompact and κ-superstrong if and only if it is jointly supercompact and superstrong.*

For the proof of the previous theorem, see Corollary 2.31 in [26] and its subsequent remarks.[6] In what follows, we shall be mainly using the last characterization of extendibility, that is, in terms of joint supercompactness and superstrongness.

---

showing that the $\aleph_1$-*Semi Proper Forcing Axiom* (SPFA) implies (†), from which it then follows that SPFA is equivalent to MM (see Theorem 37.10 in [16]).

[6] Indeed, as shown in [26], these characterizations also work for the notions of $C^{(n)}$-extendible cardinals; the latter were introduced by Bagaria in [3].

Let us now redefine the notion of an extendibility Laver function:

Definition 1.6 ([27])**.** Let κ be an extendible cardinal. A function $\ell \vdots \kappa \longrightarrow V_\kappa$ is an **extendibility Laver function** for κ if for every cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$ there is an (extender) elementary embedding $j : V \longrightarrow M$ which is jointly λ-supercompact and λ-superstrong for κ, and such that $j(\ell)(\kappa) = x$.

Theorem 1.7 ([27])**.** *Every extendible cardinal carries an extendibility Laver function as above.*

Proof. (Sketch) Fix an extendible cardinal κ and some well-ordering $\lhd_\kappa$ of V_κ. Towards a contradiction, assume that there is no extendibility Laver function for κ. We recursively construct a (partial) function $\ell \vdots \kappa \longrightarrow V_\kappa$ as follows. For any $\alpha < \kappa$, given $\ell \restriction \alpha$, we define $\ell(\alpha)$ only if $\ell``\alpha \subseteq V_\alpha$ and there exists $\lambda \geqslant \alpha$ and $x \in H_{\lambda^+}$ such that, for every (extender) embedding $j : V \longrightarrow M$ which is jointly λ-supercompact and λ-superstrong for α, $j(\ell \restriction \alpha)(\alpha) \neq x$. In such a case we fix $\lambda_\alpha < \kappa$, the least such cardinal $\lambda \geqslant \alpha$, and we let $\ell(\alpha)$ be the $\lhd_\kappa$-minimal witness $x \in H_{\lambda_\alpha^+}$. Otherwise, we leave ℓ undefined at α. This concludes the recursive definition of the function $\ell \vdots \kappa \longrightarrow V_\kappa$.

By our assumption, there must be some least $\lambda^* \geqslant \kappa$ and some $x^* \in H_{\lambda^{*+}}$, such that every jointly λ^*-supercompact and λ^*-superstrong (extender) embedding j fails to "anticipate" the set x^* (i.e., $j(\ell)(\kappa) \neq x^*$). Let $\psi(\lambda^*, x^*)$ be a fixed Π_2-statement asserting this fact (using κ, ℓ as parameters). Now fix some $\theta \in C^{(2)}$ with $\theta > \lambda^*$, some inaccessible $\overline{\theta} > \theta$, and an elementary embedding $j : V \longrightarrow M$ witnessing the joint $\overline{\theta}$-supercompactness and $\overline{\theta}$-superstrongness of κ, with $j(\overline{\theta})$ inaccessible. It is easy to see that, in M, λ^* is the least cardinal μ for which ψ holds for some $x \in H_{\mu^+}$; that is, in M, $\lambda^* = \lambda_\kappa$ in the above notation. Therefore, by elementarity, there is $y \in H_{\lambda^{*+}}$ such that $j(\ell)(\kappa) = y$, where y is chosen using $j(\lhd_\kappa)$ in M. Observe that, essentially by definition, $M \models \psi(\lambda^*, y)$. This will give the desired contradiction, once we extract an appropriate factor embedding of j which anticipates the set y and which, moreover, is witnessed by some extender in M (the latter is not the case for j itself).

We now use an elementary chain construction, aiming at obtaining a jointly λ^*-supercompact and $\overline{\theta}$-superstrong factor embedding of j.[7] We pick some initial limit ordinal $\beta_0 \in (j(\kappa), j(\overline{\theta}))$, and some $\gamma \in (\beta_0, j(\overline{\theta}))$ with $\mathrm{cof}(\gamma) > \lambda^*$. Let

$$X_0 = \{j(f)(j``\lambda^*, x) : \ f \in V, \ f : \mathcal{P}_\kappa\lambda^* \times V_{\overline{\theta}} \longrightarrow V, \ x \in V_{\beta_0}\} \prec M.$$

For any $\xi + 1 < \gamma$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\overline{\theta})) + \omega$ and

$$X_{\xi+1} = \{j(f)(j``\lambda^*, x) : \ f \in V, \ f : \mathcal{P}_\kappa\lambda^* \times V_{\overline{\theta}} \longrightarrow V, \ x \in V_{\beta_{\xi+1}}\}.$$

If $\xi \leqslant \gamma$ is limit, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha$. Let us consider

$$X_\gamma = \{j(f)(j``\lambda^*, x) : \ f \in V, \ f : \mathcal{P}_\kappa\lambda^* \times V_{\overline{\theta}} \longrightarrow V, \ x \in V_{\beta_\gamma}\} \prec M.$$

Note that $\beta_\gamma < j(\overline{\theta})$, with $\mathrm{cof}(\beta_\gamma) = \mathrm{cof}(\gamma) > \lambda^*$. We let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and define the map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $\mathrm{cp}(j_\gamma) = \kappa$ and $j_\gamma(\kappa) = j(\kappa)$, producing a commutative diagram (with $k_\gamma = \pi_\gamma^{-1}$). One now checks that j_γ is a jointly λ^*-supercompact and $\overline{\theta}$-superstrong factor of j, with $\mathrm{cp}(k_\gamma) = j_\gamma(\overline{\theta}) = \beta_\gamma$. Moreover, by inaccessibility of $j(\overline{\theta})$, for every $\alpha < j(\overline{\theta})$ we have that $j_\gamma(\alpha) < j(\overline{\theta})$. Then, we may derive from j_γ some (Martin–Steel)

[7] For various related examples and more details on such constructions, see Section 2 in [26].

extender $E \in M$ such that $j_\gamma(\ell) = j_E(\ell)$, $M \models$"E is jointly λ^*-supercompact and λ^*-superstrong for κ", and, moreover, such that M correctly computes $j_E(\ell)(\kappa)$.

Now note that κ, λ^*, $H_{\lambda^{*+}}$ and y all belong to V_{β_γ} and are, thus, fixed by π_γ. Hence, $j_\gamma(\ell)(\kappa) = j_E(\ell)(\kappa) = y$, which contradicts the fact that $M \models \psi(\lambda^*, y)$. □

Without loss of generality, we may assume that extendibility Laver functions have the following additional property: given λ and x as above, there is a jointly λ-supercompact and λ-superstrong embedding j for κ such that, apart from satisfying $j(\ell)(\kappa) = x$, it also satisfies $dom(j(\ell)) \cap (\kappa, \lambda] = \varnothing$.[8]

2. Unbounded resurrection axioms

In order to motivate unbounded resurrection, we first consider the axiom RA(SSP) and we briefly describe how it can be forced from an extendible cardinal.

It should be stressed at the outset that, as far as the consistency strength of RA(SSP) (with $\neg$CH) is concerned, Hamkins and Johnstone indeed have a better upper bound (cf. Theorem 29 in [15]). Notwithstanding, and as it will hopefully become clear below, employing the assumption of extendibility leads us naturally to the formulation of the unbounded resurrection principles.

2.1. **Forcing resurrection from an extendible.** We use the techniques of Foreman, Magidor, and Shelah (cf. [13]), replacing the supercompactness assumption by extendibility. Fix κ extendible, and let $\ell\colon \kappa \longrightarrow V_\kappa$ be an extendibility Laver function. Exactly as in [13], but using now the extendibility Laver function instead, we define $\mathbb{P}$, an RCS forcing iteration of length κ guided by ℓ. Thus, at stage $\alpha < \kappa$ and given $\mathbb{P}_\alpha$, if $\mathbb{P}_\alpha$ forces $\ell(\alpha)$ to be an $\aleph_1$-semi proper poset, we then force with the $\mathbb{P}_\alpha$-name $\ell(\alpha)$ followed by the collapse (as computed in $V^{\mathbb{P}_\alpha * \ell(\alpha)}$, and via countable conditions) of the cardinal $2^{|\mathbb{P}_\alpha * \ell(\alpha)|}$ to $\aleph_1$.[9] For more details on the definition of $\mathbb{P}$, as well as for the fact that $\mathbb{P} \Vdash \mathsf{SPFA}^{++} \wedge \kappa = \aleph_2$, see the proof of Theorem 5 in [13]. Let us now check that, in addition, $\mathbb{P} \Vdash$ RA(SSP).

Fix $G \subseteq \mathbb{P}$-generic over V and suppose that $\mathbb{Q} \in V[G]$ is a stationary preserving poset (equivalently, $\aleph_1$-semi proper since (†) holds). Fix some $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$ and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-semi proper". Now let $\lambda > \text{rank}(\dot{\mathbb{Q}})$ with $\lambda \in C^{(2)}$ and let $j : V \longrightarrow M$ be a jointly λ-supercompact and λ-superstrong elementary embedding for κ, with $j(\ell)(\kappa) = \dot{\mathbb{Q}}$. Note that $M \models \lambda \in C^{(2)}$ and, consequently, $M \models \mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-semi proper". Thus, in M, we may factor $j(\mathbb{P})$ as $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$. We now force to add (any) appropriate generics for the factors of $j(\mathbb{P})$ as displayed above, in order to lift j through $\mathbb{P}$. First, let $g \subseteq \mathbb{Q}$ be $\mathbb{Q}$-generic over $V[G]$ and then, fix any $h \subseteq (\dot{\mathbb{P}}_{\text{tail}})_{G*g}$-generic over $V[G][g]$; let $\widetilde{G} = G * g * h$ be the whole generic filter for $j(\mathbb{P})$, over V. It follows that the ground model embedding lifts to

$$j : V[G] \longrightarrow M[\widetilde{G}],$$

[8] At this point, one may naturally (or naïvely) hope for indestructibility results. However, recent work shows that extendible cardinals (among many other large cardinals) are never Laver indestructible; see [5].

[9] The other case in which non-trivial forcing is done is when $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset but such that $\mathbb{P}_\alpha \nVdash$ "$\ell(\alpha)$ is $\aleph_1$-semi proper". In this situation, we only force to collapse a sufficiently large δ to $\aleph_1$, as explained in Case 2 on page 13 of [13].

a lift which takes place in the enlarged universe $V[\widetilde{G}]$. Observe that $H_\kappa^{V[G]} \prec H_{j(\kappa)}^{M[\widetilde{G}]}$ and that $V[G] \models \mathbb{Q} \Vdash$ "$\dot{\mathbb{P}}_{\text{tail}}$ is $\aleph_1$-semi proper". It is thus enough to show that $H_{j(\kappa)}^{M[\widetilde{G}]} = H_{j(\kappa)}^{V[\widetilde{G}]}$, in order to conclude that $V[G] \models \mathsf{RA}(\text{SSP})$.

For this, we use the fact that the ground model embedding was λ-superstrong, i.e., that $V_{j(\lambda)} \subseteq M$. First, since M and V have the same (maximal) antichains of $j(\mathbb{P})$ and the latter is $j(\kappa)$-c.c. in M, it follows that $j(\mathbb{P})$ is $j(\kappa)$-c.c. in V as well. In particular, $j(\kappa)$ remains regular in both $M[\widetilde{G}]$ and $V[\widetilde{G}]$. Consequently, any given $X \in H_{j(\kappa)}^{V[\widetilde{G}]}$ can be coded in $V[\widetilde{G}]$ by a subset of $\alpha \times \alpha$, for some $\alpha < j(\kappa)$, so that X can be then retrieved by (the transitive collapse of) its code. But any nice name for such a code belongs to M, and so $X \in H_{j(\kappa)}^{M[\widetilde{G}]}$, concluding the argument.

Remark 2.1. Evidently, the above shows that $V[G] \models \mathsf{RA}(\aleph_1\text{-SEMI PROPER})$ as well. In fact, since (†) holds in $V[G]$, RA($\aleph_1$-SEMI PROPER) implies RA(SSP) in this case. However, there is a substantial difference between these axioms in terms of consistency strength: we shall see in Section 4 that RA(SSP) implies that every set has a sharp, whereas, by Theorem 26 in [15], RA($\aleph_1$-SEMI PROPER) has consistency strength below that of a Mahlo cardinal.

We now ask ourselves how much more resurrection can we get in the just obtained model of RA(SSP). This question is based on the intuitive fact that the full power of the various ground model extendibility embeddings has not been entirely exploited. As it turns out, this will lead us to a significantly stronger form of resurrection.

2.2. **Unbounded resurrection.** In the previous subsection, we argued that the ground model elementarity $H_\kappa \prec H_{j(\kappa)}$ lifts to the elementarity $H_\kappa^{V[G]} \prec H_{j(\kappa)}^{V[\widetilde{G}]}$ in the generic extension, witnessing the resurrection axiom in $V[G]$.

Now, using the ground model extendibility of κ, one is tempted to apply similar reasoning for the corresponding H_β and $H_{j(\beta)}$, for various $\beta > \kappa$. Of course, in such a case, we do not get a fully elementary substructure, but an elementary embedding between $H_\beta^{V[G]}$ and $H_{j(\beta)}^{V[\widetilde{G}]}$. It thus seems appropriate to introduce the following principle of *unbounded* resurrection.

Definition 2.2. For any (definable) class Γ of posets, the **Unbounded Resurrection Axiom** for Γ, denoted by UR(Γ), is the assertion that for every cardinal $\beta > \max\{\omega_2, \mathfrak{c}\}$ and every poset $\mathbb{Q} \in H_\beta$ with $\mathbb{Q} \in \Gamma$, there exists a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $\text{cp}(j) = \max\{\omega_2, \mathfrak{c}\}$ and $j(\text{cp}(j)) > \beta$.

In what follows, we focus on the classes of c.c.c., σ-closed, proper, and of stationary preserving posets; we shall occasionally comment on $\aleph_1$-semi properness as well. The apparent ambiguity regarding the value of $\text{cp}(j)$ is included in order to account for the general setting; as we shall see, for c.c.c. posets $\text{cp}(j) = \mathfrak{c}$, whereas, for the other classes of posets just mentioned we have that $\text{cp}(j) = \omega_2$.

The climax of the discussion in §2.1 and the remarks preceding Definition 2.2 is:

Theorem 2.3. *If the theory ZFC + "$\exists\,\kappa\,(\kappa$ is extendible$)$" is consistent, then so are the theories ZFC + UR($\aleph_1$-SEMI PROPER) and ZFC + UR(SSP).*

Proof. We show that in the model $V[G]$ obtained in §2.1, UR($\aleph_1$-SEMI PROPER) holds (and thus UR(SSP) as well, since (†) holds in $V[G]$).[10]

Fix a cardinal $\beta > \kappa = \omega_2^{V[G]}$, some $\aleph_1$-semi proper poset $\mathbb{Q} \in H_\beta^{V[G]}$, and repeat the arguments in §2.1; in particular, we fix some $\lambda \in C^{(2)}$ (in V) above β and a jointly λ-supercompact and λ-superstrong ground model embedding $j : V \longrightarrow M$ anticipating a name for $\mathbb{Q}$. The rest of the argument now proceeds as in §2.1, with the main point being that, for any inaccessible $\theta \in (\kappa, \lambda)$, we have that $H_{j(\theta)}^{M[\widetilde{G}]} = H_{j(\theta)}[\widetilde{G}] = H_{j(\theta)}^{V[\widetilde{G}]}$; this follows from the λ-superstrongness of j, by the usual coding arguments. Therefore, in the fully enlarged universe $V[\widetilde{G}]$, we have available the restricted map $j \restriction H_\beta^{V[G]} : H_\beta^{V[G]} \longrightarrow H_{j(\beta)}^{V[\widetilde{G}]}$, which has the desired properties. □

We now look at the rest of the UR axioms. We state the next theorem again in terms of relative consistency of theories, although we actually argue that, given an extendible cardinal, the forcing iterations that we define work as intended.

Theorem 2.4. *If the theory ZFC + "$\exists\,\kappa\,(\kappa$ is extendible$)$" is consistent, then so is each one of the theories:*

(i) ZFC + UR(C.C.C.).
(ii) ZFC + UR(σ-CLOSED).
(iii) ZFC + UR(PROPER).

Proof. Our treatment of all cases follows a unified pattern: starting from an extendible cardinal, we define the appropriate forcing iteration guided by an extendibility Laver function, while taking into account only the posets which are relevant to the axiom at hand. Note that if Γ is any of these three classes of posets, then Γ has two useful features: first, Γ is closed under two-step iterations; second, there is a way (i.e., choice of support) to handle iterations of posets in Γ so that, at limit stages, the defined limit poset is still in Γ.

Given these remarks, we now fix an extendible cardinal κ and some extendibility Laver function $\ell \vdots \kappa \longrightarrow V_\kappa$. In all three cases, the definition of the iteration follows the same template. We start with $\mathbb{P}_0 = \{\mathbb{1}\}$. Given $\alpha < \kappa$ and $\mathbb{P}_\alpha$, if $\alpha \in dom(\ell)$ and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset with $\mathbb{P}_\alpha \Vdash$ "$\ell(\alpha) \in \Gamma$", we let $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$ and define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Otherwise, trivial forcing is done at stage $\alpha + 1$.

At limits, we use finite support for (*i*); we use Easton support for (*ii*); and we use countable support for (*iii*). In each case, the final forcing $\mathbb{P}$ is defined as the κ-iteration $\mathbb{P}_\kappa$. By standard forcing facts, $\mathbb{P}$ has the c.c.c. in case (*i*); it is σ-closed and has the κ-c.c. in (*ii*); and it is proper and has the κ-c.c. in (*iii*).

It is now straightforward to adapt our "prototype" arguments (as described above, in §2.1 and in the proof of Theorem 2.3) in order to conclude that, in each case, the defined forcing produces a model of the UR axiom at hand. □

Given the (relative) consistency of the UR axioms, we now proceed with some of their consequences, as well as with their relation to other forcing axioms.

[10] Indeed, UR(SSP) directly follows from UR($\aleph_1$-SEMI PROPER); see Corollary 2.7 below.

2.3. **Consequences and relation to forcing axioms.** We start with the following important connection.

Proposition 2.5. *UR(SSP) implies MM^{++}. Moreover, if there is a model in which there exists a supercompact cardinal with a unique inaccessible above it, then there is a model of MM^{++} in which UR(SSP) fails.*

Proof. Suppose that UR(SSP) holds in V. We verify that MM follows; we leave it to the reader to check that a mild modification of the argument produces MM^{++} as well. For this, we fix some poset $\mathbb{Q} \in$ SSP, and let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a collection of dense subsets of $\mathbb{Q}$.

We fix a regular $\beta > \max\{\omega_2, \mathfrak{c}\}$, with $\mathbb{Q} \in H_\beta$ and $\langle D_\alpha : \alpha < \omega_1 \rangle \in H_\beta$. Then, by UR(SSP), there is some $\mathbb{Q}$-name $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in$ SSP", and an elementary embedding $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ of the form

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $\text{cp}(j) = \max\{\omega_2, \mathfrak{c}\}$ and $j(\text{cp}(j)) > \beta$. Finally, we fix any filter $g \subseteq \mathbb{Q}$-generic over V and any filter $H \subseteq \dot{\mathbb{R}}_g$-generic over $V[g]$. Therefore, $j : H_\beta \longrightarrow H_{j(\beta)}^{V[g][H]}$ is elementary with $\text{cp}(j) = \max\{\omega_2, \mathfrak{c}\}$ and $j(\text{cp}(j)) > \beta$. Now, since ω_1 is fixed by the embedding,

$$j(\langle D_\alpha : \alpha < \omega_1 \rangle) = \langle j(D_\alpha) : \alpha < \omega_1 \rangle \in H_{j(\beta)}^{V[g][H]}$$

and, also, the pointwise image $j``g$ belongs to $H_{j(\beta)}^{V[g][H]}$ as well, since it is constructible in $V[g][H]$ from $j$ and $g$, with the latter having size less than $\beta$. But then, as $g$ is $\mathbb{Q}$-generic over $V$, it follows that, in $H_{j(\beta)}^{V[g][H]}$, $j``g$ generates a filter of $j(\mathbb{Q})$ which intersects every $j(D_\alpha)$, for $\alpha < \omega_1$. Hence, by elementarity, there is, in H_β, a filter $G \subseteq \mathbb{Q}$ such that $G \cap D_\alpha \neq \varnothing$, for all $\alpha < \omega_1$. MM now follows.

To separate the axioms MM^{++} and UR(SSP), we fix a model in which there is a supercompact κ and a unique inaccessible $\lambda > \kappa$, and we let $\mathbb{P}$ be the standard κ-iteration which forces MM^{++}, as in [13]. Let us fix a forcing extension V in which MM^{++} holds and λ remains inaccessible. Towards a contradiction, suppose that UR(SSP) holds in V and let $\mathbb{Q} \in H_{\lambda^+}$ be the canonical (σ-closed) poset which collapses λ to ω_1. Then, there is a $\mathbb{Q}$-name $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in$ SSP", and an embedding $j : H_{\lambda^+} \longrightarrow H_{j(\lambda^+)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$, with $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $\text{cp}(j) = \kappa$ and $j(\kappa) > \lambda^+$. By elementarity, $j(\lambda)$ is inaccessible in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$; but this is impossible, since in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$ there are no inaccessibles at all. □

Similarly, if UR(SSP) holds and there is some inaccessible, then there are proper class many inaccessibles. Observe that, in place of inaccessibles, one could also consider other objects which cannot be created (but can be destroyed) by stationary preserving forcing. Clearly, this remark and the previous proof can be also adapted for the case of proper posets. Indeed, a moment's inspection shows that:

Corollary 2.6. *For any (definable) class $\Gamma \subseteq$ SSP, UR(Γ) implies the forcing axiom FA^{++}(Γ).*

Recalling that SPFA implies (†), we immediately get that:

Corollary 2.7. *UR($\aleph_1$-SEMI PROPER) implies UR(SSP).*

The following argument due to Asperó shows that, given enough large cardinals, the converse holds as well.

Proposition 2.8 (Asperó)**.** *If UR(SSP) holds and there is a proper class of supercompact cardinals, then UR($\aleph_1$-SEMI PROPER) holds.*

Proof. We use the fact that, if κ is supercompact and $\mathbb{P} = \mathrm{Coll}(\omega_1, {<}\kappa)$ is the Lévy collapse to make $\kappa = \aleph_2$, then by results in [13] we have that (†) holds in $V^{\mathbb{P}}$. Towards verifying UR($\aleph_1$-SEMI PROPER), suppose that $\mathbb{Q}$ is $\aleph_1$-semi proper and let $\beta > \omega_2$ be a given cardinal with $\mathbb{Q} \in H_\beta$.

Let κ be supercompact with $\kappa > \beta$; clearly, κ remains supercompact in $V^{\mathbb{Q}}$. Consider the ($\mathbb{Q}$-name for the) poset $\dot{\mathbb{Q}}_0$ which is the Lévy collapse $\mathrm{Coll}(\omega_1, {<}\kappa)$ as computed in $V^{\mathbb{Q}}$. Now, $\mathbb{Q} * \dot{\mathbb{Q}}_0 \in$ SSP in V and hence there is a $\mathbb{Q} * \dot{\mathbb{Q}}_0$-name $\dot{\mathbb{R}}$ such that $\mathbb{Q} * \dot{\mathbb{Q}}_0 \Vdash$ "$\dot{\mathbb{R}} \in$ SSP", and an elementary embedding

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}}}$$

with $j \in V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}}$, $\mathrm{cp}(j) = \omega_2$ and $j(\omega_2) > \beta$. Now recall that (†) holds in $V^{\mathbb{Q}*\dot{\mathbb{Q}}_0}$ and so we actually have that $\mathbb{Q} * \dot{\mathbb{Q}}_0 \Vdash$ "$\dot{\mathbb{R}}$ is $\aleph_1$-semi proper". Observe that $\dot{\mathbb{Q}}_0 * \dot{\mathbb{R}}$ is of the form "σ-closed$*\aleph_1$-semi proper" and thus it is $\aleph_1$-semi proper in $V^{\mathbb{Q}}$. Therefore, since $V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}} = V^{\mathbb{Q}*(\dot{\mathbb{Q}}_0*\dot{\mathbb{R}})}$, the conclusion follows. □

Given this result, the undermentioned question suggests itself.

Question 2.9. Are the axioms UR(SSP) and UR($\aleph_1$-SEMI PROPER) equivalent in general? In case of a negative answer, can we improve the extra assumption of a proper class of supercompact cardinals?

It should have been clear by now that, for the cases of proper, of $\aleph_1$-semi proper, and of stationary preserving posets, the critical point of the generic embeddings given by the corresponding UR axiom is ω_2, since both PFA and MM imply that the continuum is equal to $\aleph_2$. On the other hand, the generic embeddings given by the axiom UR(c.c.c.) will have, in general, $\mathrm{cp}(j) = \mathfrak{c}$ since UR(c.c.c.) implies that the continuum is weakly inaccessible. In the other extreme, UR(σ-CLOSED) implies CH. These are direct corollaries to results in [15], as displayed below.

Fact 2.10. *For any definable class Γ, UR(Γ) implies RA(Γ). Consequently:*

(i) UR(c.c.c.) $\Longrightarrow$ MA $+$ "$\mathfrak{c}$ is weakly inaccessible".
(ii) UR(σ-CLOSED) $\Longrightarrow$ $2^{\aleph_0} = \aleph_1$.

Proof. The implication UR(Γ) $\Longrightarrow$ RA(Γ) is immediate. Consequently, (*i*) follows from Theorems 4 and 8 in [15], whereas (*ii*) follows from Theorem 10 in [15]. □

Apropos, we may also deduce that the unbounded resurrection axioms, just like the resurrection axioms, are not monotonous; i.e., if $\Gamma \subseteq \Gamma'$ are given classes of posets, then UR(Γ') does not necessarily imply UR(Γ).

Furthermore, note that one may separate UR(c.c.c.) (indeed, RA(c.c.c.)) from MA easily: either go to a model of CH (e.g., to L), or force MA *and* $\mathfrak{c} = \kappa^+$, for some cardinal κ; in either case, RA(c.c.c.) fails, and so does UR(c.c.c.). Towards concluding the current section, we now give some more easy separation results.

Corollary 2.11. *If there is a model with a supercompact cardinal, then there is a model satisfying MA$^+$(σ-CLOSED) $+ \neg$ UR(σ-CLOSED).*

Proof. By a result of Shelah, MM implies $\mathsf{MA}^+(\sigma\text{-CLOSED})$ (see Theorem 37.26 in [16]). Hence, starting from a supercompact, if we force MM+¬ CH in the usual way, then, in the resulting model, UR(σ-CLOSED) cannot possibly hold, by Fact 2.10. □

In addition, if Γ is the class of proper, of $\aleph_1$-semi proper, or of stationary preserving posets, then UR(Γ) can be easily separated from RA(Γ).

Corollary 2.12. *For any of the three aforementioned classes Γ, RA(Γ)+ ¬ UR(Γ) is relatively consistent.*

Proof. By Theorem 7 in [15], we may force over any model of RA(Γ) in order to get a model of RA(Γ)+CH. On the other hand, in all three cases, $\mathsf{UR}(\Gamma) \Longrightarrow \mathfrak{c} = \aleph_2$. □

For σ-closed posets we may not argue likewise, in light of Fact 2.10. Nevertheless, RA(σ-CLOSED) can indeed be separated from UR(σ-CLOSED), and the same holds for the case of c.c.c. posets. Both of these results will follow from considerations regarding consistency lower bounds, which we shall take up in Section 4.

3. Resurrection from the stationary tower

We now briefly diverge from the context of extendible cardinals and give the following theorem, due to Asperó, which substantially improves the consistency upper bound for the axiom UR(SSP). The reader is warned that, in the current section, we assume great familiarity with the techniques related to Woodin's stationary tower forcing; for more details, [19] gives an excellent account of such techniques.

Theorem 3.1 (Asperó)**.** *Assume MM^{++} and that there is a proper class of Woodin cardinals. Then, UR(SSP) holds.*

Before giving the proof of the theorem, let us first fix some terminology: given a (possibly non-transitive) model N of (a sufficient fragment of) ZFC with $\omega_1 \subseteq N$, some stationary preserving poset $\mathbb{P} \in N$, and some $G \subseteq \mathbb{P} \cap X$-generic filter over N, we say that G is *correct* if for every $\tau \in N$ which is $\mathbb{P}$-name for a stationary subset of ω_1, we have that $\tau^G = \{\alpha < \omega_1 : \exists\, p \in G\, (p \Vdash \check{\alpha} \in \tau)\}$ is stationary in ω_1. The following is a well-known characterization of MM^{++} (see, e.g., Lemma 3 in [8]):

Proposition 3.2. *MM^{++} holds if and only if for every $\mathbb{P} \in$ SSP and every sufficiently large regular θ the set $S_{\mathbb{P}}$ is stationary, where $X \in S_{\mathbb{P}}$ if and only if $X \subseteq H_\theta$, $X \prec H_\theta$, $\omega_1 \subseteq X$, $|X| = \aleph_1$, $\mathbb{P} \in X$, and there exists some $G \subseteq \mathbb{P} \cap X$ which is a correct $\mathbb{P} \cap X$-generic filter over X.*

In fact, we may also require that, for every particular $p \in \mathbb{P}$, the set $S^p_{\mathbb{P}}$ is stationary, where $S^p_{\mathbb{P}}$ is defined just as $S_{\mathbb{P}}$, but with the extra clauses that $p \in X$ and $p \in G$.

Proof of Theorem 3.1. Let $\mathbb{P} \in$ SSP and fix a cardinal $\beta > \omega_2$ with $\mathbb{P} \in H_\beta$. Further fix some inaccessible $\theta > \beta$. By Proposition 3.2, we have that the set $S_{\mathbb{P}}$ consisting of $X \subseteq H_\theta$ such that $X \prec H_\theta$, $\omega_1 \subseteq X$, $|X| = \aleph_1$, $\mathbb{P} \in X$, and there exists some correct $G \subseteq \mathbb{P} \cap X$-generic over X, is stationary.

Now let $\delta > \theta$ be a Woodin cardinal and let $\mathbb{Q}$ be the (full) stationary tower up to δ, restricted to conditions below $S_{\mathbb{P}}$. Note that if H is $\mathbb{Q}$-generic over V then, in $V[H]$, standard facts about the tower give that there is an elementary embedding $j : V \longrightarrow M \subseteq V[H]$, with $\mathrm{cp}(j) = \omega_2$, M closed under $< \delta$-sequences in $V[H]$, and $j(\delta) = \delta$. Also, since $j``H_\theta \in j(S_{\mathbb{P}})$, we have that $|\theta| = \aleph_1$ and $j(\omega_2) > \theta > \beta$. Moreover, since $j(\beta) < \delta$, the $< \delta$-closure of M gives that $H^M_{j(\beta)} = H^{V[H]}_{j(\beta)}$.

Hence, in order to conclude the theorem, it suffices to show that $\mathbb{P}$ embeds completely in $\mathbb{Q}$ and that the quotient $\mathbb{Q}/\dot{G}$ is stationary preserving in $V^{\mathbb{P}}$.[11] For the first part, let H be $\mathbb{Q}$-generic over V and consider $j : V \longrightarrow M \subseteq V[H]$, the tower embedding as above. Then, by elementarity, there exists in M a correct filter G which is $j``H_\theta \cap j(\mathbb{P})$-generic over $j``H_\theta$, because $j``H_\theta \in j(S_{\mathbb{P}})$. But now, if G' is the pre-image of G under j, then $G' \in V[H]$ and is $\mathbb{P}$-generic over V. This shows that $\mathbb{P}$ completely embeds in $\mathbb{Q}$. It remains to see that $\mathbb{Q}/\dot{G} \in$ SSP in $V^{\mathbb{P}}$.

Suppose, towards a contradiction, that there is some condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\mathbb{Q}/\dot{G}$ is not stationary preserving". By the remark following Proposition 3.2, let H be $\mathbb{Q}$-generic over V such that $S^p_{\mathbb{P}} \in H$. Arguing as above, there are, in $V[H]$, an embedding $j : V \longrightarrow M \subseteq V[H]$ and a correct filter $G' \subseteq \mathbb{P}$-generic over V with $p \in G'$ and with the property that, any stationary subset of ω_1 which lies in $H_\theta[G'] = H_\theta^{V[G']}$ remains stationary in M, and thus in $V[H]$.

Hence, arguing in $V[H]$, any further $\widetilde{G} \subseteq \mathbb{Q}/G'$-generic over $V[G']$ preserves the stationary subsets of ω_1 which lie in $V[G']$. This contradicts the choice of $p \in \mathbb{P}$ and completes the proof. □

Observe that the previous technique cannot be used for the other classes of posets mentioned so far, since we cannot expect the quotient forcings to be, for example, proper or $\aleph_1$-semi proper. We now immediately have that:

Corollary 3.3. *If ZFC + "$\exists\,\kappa\,(\kappa$ is supercompact)" + "there is a proper class of Woodin cardinals" is consistent, then so is ZFC + UR(SSP).*

4. On consistency lower bounds

For the current section, the reader is advised to review the definitions of (weak) squares, (good) scales, and the approachability property; a comprehensive account is given in [9] (in particular: Sections 2, 3, and 6).

By results in [15], the RA axioms for the classes of c.c.c., σ-closed, proper, and $\aleph_1$-semi proper posets, all have consistency strength below that of a Mahlo cardinal. However, for the unbounded versions, the assumption of extendibility that we used is outrageously stronger. Thus, enquiries regarding the consistency strength of these axioms cannot be avoided.

Here, we focus on the cases of c.c.c. and of σ-closed posets, providing consistency lower bounds for the corresponding UR axioms by deriving failures of (weak forms of) square principles. Finally, and via a different method, we also give a lower bound for the case of RA(SSP).

To begin with, we may already observe the failure of squares for the class of σ-closed posets, as an immediate consequence of the fact that UR(σ-CLOSED) implies MA$^+$(σ-CLOSED); the latter implies the *Singular Cardinal Hypothesis* (SCH) and other principles (see § 37 in [16]). Therefore:

Corollary 4.1. *UR(σ-CLOSED) $\Longrightarrow$ SCH + "$\square_\lambda$ fails, for $\lambda \geqslant \omega_1$".*

Recall that by Theorem 10 in [15], RA(σ-CLOSED) is actually equivalent to CH; thus, the previous corollary implies that L is a model of RA(σ-CLOSED) + ¬ UR(σ-CLOSED), separating the two axioms. We will return to σ-closed posets later on in this section, obtaining failures of weak squares as well.

---

[11] A similar argument, alas not in the context of resurrection axioms, is given by Viale (see Theorem 2.12 in [28]).

Let us now concentrate on c.c.c. posets and show that UR(c.c.c.) implies the non-existence of good scales. For this, we use an argument due to Bagaria and Magidor (cf. [4]), which they apply in the context of ω_1-strongly compact cardinals.

Theorem 4.2. *Assume UR(c.c.c.). Then, for every cardinal $\lambda > \mathfrak{c}$ such that $cof(\lambda) = \omega$, there is no good λ^+-scale.*

Proof. Fix some $\lambda > \mathfrak{c}$ with $\text{cof}(\lambda) = \omega$ and fix a sequence $\langle \lambda_n : n \in \omega \rangle$ of regular cardinals with $\sup_n \lambda_n = \lambda$. Towards a contradiction, assume that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good λ^+-scale with respect to this sequence. Moreover, fix some regular $\beta > \lambda^+$ with $\langle f_\alpha : \alpha < \lambda^+ \rangle \in H_\beta$. If $\mathbb{Q} = \{\mathbb{1}\}$ is the trivial poset then, by unbounded resurrection, there is a c.c.c. poset $\mathbb{R}$ and an elementary embedding

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{R}}},$$

such that $j \in V^{\mathbb{R}}$, $\text{cp}(j) = \mathfrak{c}$ and $j(\mathfrak{c}) > \beta$. By elementarity, $j(\langle f_\alpha : \alpha < \lambda^+ \rangle) = \langle f^*_\alpha : \alpha < j(\lambda^+) \rangle$ is a good $j(\lambda^+)$-scale, with respect to the sequence $\langle j(\lambda_n) : n \in \omega \rangle$.

Let $\delta = \sup(j``\lambda^+)$ and note that $\delta < j(\lambda^+)$ and $\text{cof}(\delta)^{V^{\mathbb{R}}} = \lambda^+$, with the latter being regular in $V^{\mathbb{R}}$. Hence, by definition of a good scale, there exists, in $H_{j(\beta)}^{V^{\mathbb{R}}}$, some $D \subseteq \delta$ cofinal in δ and some $n \in \omega$ so that, for every $\gamma < \gamma'$ in D and every $m > n$, we have the inequality:

$$f^*_\gamma(m) < f^*_{\gamma'}(m).$$

We now define, recursively for $\xi < \lambda^+$, an increasing sequence of ordinals of the form $D^* = \{\gamma_\xi : \xi < \lambda^+\} \subseteq D$, together with a sequence $\{\alpha_\xi : \xi < \lambda^+\} \subseteq \lambda^+$. Initially, we let $\gamma_0 = \min D$. Given γ_ξ for some $\xi < \lambda^+$, we let $\alpha_\xi < \lambda^+$ be least such that $\gamma_\xi < j(\alpha_\xi)$ and define $\gamma_{\xi+1}$ as the least ordinal in the set D with $j(\alpha_\xi) < \gamma_{\xi+1}$. At limits $\xi < \lambda^+$, we let γ_ξ be the least ordinal in D above the supremum of all the γ_ζ's defined so far. Clearly, for every $\xi < \lambda^+$, $\alpha_\xi < \lambda^+$.

Furthermore, for each $\xi < \lambda^+$, there exists an $n_\xi \in \omega$ so that for all $m \geqslant n_\xi$, the following inequalities hold:

$$f^*_{\gamma_\xi}(m) < f^*_{j(\alpha_\xi)}(m) < f^*_{\gamma_{\xi+1}}(m).$$

Now let $E \subseteq \lambda^+$ be of cardinality λ^+ and such that, for all $\xi \in E$, the corresponding n_ξ is the same; say equal to some fixed $k \in \omega$. Then, for every $\xi < \zeta$ in E, we have the inequalities which are shown below:

$$f^*_{\gamma_\xi}(k) < f^*_{j(\alpha_\xi)}(k) < f^*_{\gamma_{\xi+1}}(k) \leqslant f^*_{\gamma_\zeta}(k) < f^*_{j(\alpha_\zeta)}(k) < f^*_{\gamma_{\zeta+1}}(k).$$

At this point observe that, for any $\xi < \lambda^+$, $f^*_{j(\alpha_\xi)}(k) = j(f_{\alpha_\xi}(k))$ where, by definition of a scale, $j(f_{\alpha_\xi}(k)) \in j``\lambda_k$. But this is impossible, since $\langle f^*_{j(\alpha_\xi)}(k) : \xi \in E \rangle$ has order type λ^+, whereas λ_k is below λ. □

We remark that in the last proof, we made heavy use of the fact that the forcing $\mathbb{R}$, due to its countable chain condition, preserved cofinalities and cardinals. We also relied on the fact that the critical point of the generic embeddings given by UR(c.c.c.) is $\mathfrak{c}$. A moment's reflection shows that we may easily modify the previous argument in order to account for any cofinality below $\mathfrak{c}$.

Corollary 4.3. *Assume UR(c.c.c.). Then, for every cardinal $\lambda > \mathfrak{c}$ such that $cof(\lambda) < \mathfrak{c}$, there is no good λ^+-scale.*

We now draw more conclusions regarding the effect of UR(c.c.c.) on the universe.

Corollary 4.4. *UR(c.c.c.) implies the following:*

(i) The Singular Cardinal Hypothesis.
(ii) $\square^*_\lambda$ fails, for every $\lambda > \mathfrak{c}$ with $cof(\lambda) < \mathfrak{c}$.
(iii) Indeed, the Approachablity Property AP_λ fails, for all $\lambda > \mathfrak{c}$ with $cof(\lambda) < \mathfrak{c}$.

Proof. For (*i*), notice that the SCH holds (vacuously) at every singular $\lambda < \mathfrak{c}$. Shelah has shown that, for singular λ, if the SCH fails at λ then there is a good λ^+-scale (see Section 4.7 in [10]). Thus, by Theorem 4.2, for any singular $\lambda > \mathfrak{c}$ with $\mathrm{cof}(\lambda) = \omega$, the SCH holds at λ and, finally, the SCH holds everywhere by a classical result of Silver (see Theorem 8.13 in [16]).

For (*ii*) and (*iii*), we recall that $\square^*_\lambda$ implies AP_λ, and that, for singular λ, AP_λ implies that every λ^+-scale is good (see Proposition 4.52 in [10], or [12]). The desired results now follow from Corollary 6.3. $\square$

Evidently, there is a substantial gap in consistency strength between RA(c.c.c.) and UR(c.c.c.); the former can be forced from an *uplifting* cardinal (see [15]), while the latter implies (consistency-wise) Woodin cardinals. In fact, we may separate the two axioms as follows: we start from an uplifting cardinal in a model of $V = L$ (where global square holds) and force, as in [15], the axiom RA(c.c.c.) using a c.c.c. iteration. By Corollary 4.4, this produces a model of $\neg$ UR(c.c.c.).

We now return to the case of σ-closed posets. To begin with, we introduce the notion of a *generically extendible* cardinal; this is in accordance with other notions of generic large cardinals appearing in the literature, such as generically supercompact and generically huge cardinals (see, for example, [11]).

Definition 4.5. Let $\kappa = \mu^+$, where μ is regular, and fix some (definable) class Γ of posets. We say that κ is **generically extendible** by Γ if for every cardinal $\lambda > \kappa$, there exists a poset $\mathbb{P} \in \Gamma$ and there is an elementary embedding $j : H_\lambda \longrightarrow H^{V^{\mathbb{P}}}_{j(\lambda)}$, with $j \in V^{\mathbb{P}}$, $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$.

As a consequence of Definition 2.2, UR(Γ) implies that ω_2 is generically extendible by Γ, for σ-closed, proper, or stationary preserving posets. Indeed, more is true of ω_2, in a way parallel to the very definition of unbounded resurrection.

Definition 4.6. Let $\kappa = \mu^+$, where μ is regular, and fix some (definable) class Γ of posets. We say that κ is **indestructibly generically extendible** by Γ if for every cardinal $\lambda > \kappa$ and every $\mathbb{Q} \in \Gamma$ with $\mathbb{Q} \in H_\lambda$, there exists a (name for a) poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding $j : H_\lambda \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\lambda)}$, with $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$.

We also consider the (known) notion of indestructibly generically supercompact cardinals (see [9], or Definition 11.4 in [11]; the following definition is a modification of the latter).

Definition 4.7. Let $\kappa = \mu^+$, where μ is regular, and fix some (definable) class Γ of posets which preserve cofinalities $<\mu$. We say that κ is **indestructibly generically supercompact** by Γ if for every regular $\lambda > \kappa$ and every $\mathbb{Q} \in \Gamma$ with $\mathbb{Q} \in H_\lambda$, there is a (name for a) poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding $j : V \longrightarrow M \subseteq V^{\mathbb{Q}*\dot{\mathbb{R}}}$, where M is transitive, j is a definable subclass of $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $j``\lambda \in M$, $\sup(j``\lambda) < j(\lambda)$ and $\mathrm{cof}(\lambda)^M = \mu$.

We now establish a connection which might already be expected.

Proposition 4.8. *Let $\kappa = \mu^+$, where μ is regular, and fix a (definable) class Γ of posets which preserve cofinalities $< \mu$. If κ is indestructibly generically extendible by Γ, then it is indestructibly generically supercompact by Γ.*

Proof. Fix $\kappa = \mu^+$, where μ is regular, let Γ be a class of posets which preserve cofinalities $< \mu$ and suppose that κ is indestructibly generically extendible by Γ. Fix a regular $\lambda > \kappa$ and some $\mathbb{Q} \in H_\lambda$ with $\mathbb{Q} \in \Gamma$, and fix some $\beta = \beth_\beta > \lambda$. Let $\dot{\mathbb{R}}$ and j witness the indestructible generic extendibility of κ with respect to β^+, i.e., $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$" and $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ is an elementary embedding of the form

$$j : H_{\beta^+} \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta^+)},$$

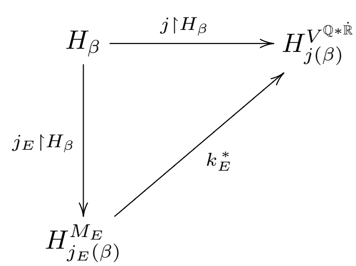
with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \beta^+$. We now extract, in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, a long extender from j, measuring sets in V, and we then argue that the extender ultrapower witnesses the indestructible generic supercompactness of κ.

So, let $E = \langle E_a : a \in [j(\beta)]^{<\omega} \rangle$ where, each E_a is a V-ultrafilter on $[\beta]^{|a|}$ defined as usual: for $X \in \mathcal{P}([\beta]^{|a|}) \cap V$, $X \in E_a \iff a \in j(X)$.[12] Given E, we may now consider the extender embedding

$$j_E : V \longrightarrow M_E \subseteq V^{\mathbb{Q}*\dot{\mathbb{R}}},$$

which is definable in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, M_E is transitive and $\mathrm{cp}(j_E) = \kappa$. We check that $j_E(\kappa) > \lambda$, $j_E``\lambda \in M_E$, $\sup(j_E``\lambda) < j_E(\lambda)$ and $\mathrm{cof}(\lambda)^{M_E} = \mu$.

Consider a restricted version of the usual commutative diagram, defining $k^*_E : H^{M_E}_{j_E(\beta)} \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$ by letting $k^*_E([a,[f]]) = j(f)(a)$, for all $[a,[f]] \in H^{M_E}_{j_E(\beta)}$, where $a \in [j(\beta)]^{<\omega}$ and $f : [\beta]^{|a|} \longrightarrow H_\beta$ with $f \in V$. Then, k^*_E is a well-defined $\{\in\}$-embedding and so, in particular, injective. We thus get the commutative diagram



where $j \restriction H_\beta = k^*_E \circ (j_E \restriction H_\beta)$ and, by standard arguments, k^*_E is also surjective (and thus equal to the identity). Hence, $H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)} \subseteq M_E$ and, so $j_E(\kappa) = j(\kappa) > \lambda$, $j_E(\lambda) = j(\lambda)$ and $j_E``\lambda = j``\lambda \in H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$; therefore, $j_E``\lambda \in M_E$ and $\sup(j_E``\lambda) = \sup(j``\lambda) < j(\lambda) = j_E(\lambda)$ as well.

To conclude, since $V^{\mathbb{Q}*\dot{\mathbb{R}}} \models |\lambda| = \mu$, it follows that $\mathrm{cof}(\lambda)^{M_E} = \mathrm{cof}(\lambda)^{V^{\mathbb{Q}*\dot{\mathbb{R}}}} \leqslant \mu$. But notice that the latter inequality cannot be strict, because the posets in Γ are supposed to preserve cofinalities $< \mu$. □

[12] Note that, in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, κ has cardinality μ and so it is no longer a cardinal; in fact, since $V^{\mathbb{Q}*\dot{\mathbb{R}}} \models j(\kappa) = \mu^+$, the same is true for every ordinal in the interval $[\kappa, j(\kappa))$. For this reason, we avoid using the term "κ-complete" for the E_a's, or the term "$(\kappa, j(\beta))$-extender" for E. Still, E indeed has extender properties and a corresponding ultrapower may be formed; for this, one patiently verifies the defining clauses of an extender, with respect to the $\kappa = \mu^+$ of V.

With the previous proposition in mind, we now depart from full generality and look at the particular axiom UR(σ-CLOSED), from which we obtain the failure of weak squares using an argument due to Foreman and Magidor (cf. § 5 in [12]). This is a result already quoted in [9], of which we now provide a proof.

Theorem 4.9. *Suppose that ω_2 is indestructibly generically supercompact by the class of σ-closed posets. Then, for every (uncountable) strong limit λ with cof$(\lambda) = \omega$, we have that $\square^*_\lambda$ fails.*

Proof. Assume that ω_2 is indestructibly generically supercompact by the class of σ-closed posets. Fix a strong limit $\lambda > \omega_2$ with $\text{cof}(\lambda) = \omega$ and suppose, towards a contradiction, that

$$\mathscr{C} = \langle \mathcal{C}_\alpha : \alpha \in \text{Lim}(\lambda^+) \rangle$$

is a $\square^*_\lambda$-sequence, i.e., $\mathscr{C}$ satisfies the following conditions for every $\alpha \in \text{Lim}(\lambda^+)$:

(*i*) $\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$ and $1 \leqslant |\mathcal{C}_\alpha| \leqslant \lambda$.
(*ii*) Every $C \in \mathcal{C}_\alpha$ is a club in α, with $\text{ot}(C) < \lambda$.
(*iii*) For every $C \in \mathcal{C}_\alpha$ and every $\beta \in \text{Lim}(C)$, $C \cap \beta \in \mathcal{C}_\beta$.
(*iv*) There is some $C \in \mathcal{C}_\alpha$ with $\text{ot}(C) = \text{cof}(\alpha)$.[13]
(*v*) For every $C \in \mathcal{C}_\alpha$ and every club $D \subseteq C$, $D \in \mathcal{C}_\alpha$ as well.

Observe that condition (*v*) can be assumed in the light of the fact that λ is a (singular) strong limit. Now let $\mathbb{Q} = \{\mathbb{1}\}$ be the trivial poset. By the assumption on ω_2, there is some σ-closed poset $\mathbb{R}$ and an elementary embedding

$$j : V \longrightarrow M \subseteq V^{\mathbb{R}},$$

where M is transitive, j is a definable subclass of $V^{\mathbb{R}}$, $\text{cp}(j) = \omega_2$, $j(\omega_2) > \lambda^+$, $j``\lambda^+ \in M$, $\sup(j``\lambda^+) < j(\lambda^+)$ and $\text{cof}(\lambda^+)^M = \omega_1$. Let $\gamma = \sup(j``\lambda^+)$ and let us denote by $\langle \mathcal{C}^*_\alpha : \alpha \in \text{Lim}(j(\lambda^+)) \rangle$ the image $j(\mathscr{C}) \in M$ of the weak square sequence. Then, working temporarily in M, there is a club $D_\gamma \subseteq \gamma$ with $D_\gamma \in \mathcal{C}^*_\gamma$. Using the fact that $\text{cof}(\gamma) = \text{cof}(\lambda^+) = \omega_1$, we may assume by condition (*iv*) that $\text{ot}(D_\gamma) = \omega_1$. Moreover, since $j``\lambda^+$ is an $\omega$-club in $\gamma$, we may further assume by condition (*v*) that $D_\gamma \subseteq j``\lambda^+$. By condition (*iii*), for every $\delta \in \text{Lim}(D_\gamma)$, we have that $D_\delta = D_\gamma \cap \delta \in \mathcal{C}^*_\delta$. But then, from the perspective of $V^{\mathbb{R}}$ now, D_δ is a countable set of ordinals, subset of the range of j. Thus, by the σ-closure of $\mathbb{R}$, there is some (countable) $x \in V$ with $j(x) = D_\delta$. Hence, if $\alpha_\delta < \lambda^+$ is chosen so that $\delta = j(\alpha_\delta)$, then $x \in \mathcal{C}_{\alpha_\delta}$. This shows that $\mathbb{R}$ has added a *thread* E through the $\square^*_\lambda$-sequence $\mathscr{C}$.[14] We now use the closure of $\mathbb{R}$ in order to derive a contradiction.

Claim. *For every $p \in \mathbb{R}$ there exists some $\alpha < \lambda^+$ such that*

$$|\{z \in V \cap [\lambda^+]^\omega : \textit{there is } r \leqslant p \textit{ s.t. } r \Vdash z = E \cap \alpha\}| \geqslant \lambda.$$

Proof of claim. Towards a contradiction, fix some $p \in \mathbb{R}$ which is a counterexample. Fix $\langle \lambda_n : n \in \omega \rangle$ a sequence of regular cardinals cofinal in λ and then, for every $\alpha < \lambda^+$, consider the (non-empty) set

$$T_\alpha = \{z \in V \cap [\lambda^+]^\omega : \text{ there is } r \leqslant p \text{ s.t. } r \Vdash z = E \cap \alpha\}.$$

[13] This clause is not usually included in the definition of a $\square^*_\lambda$-sequence; nevertheless, and as explained in § 1 of [12], it may be assumed without loss of generality.

[14] That is, $E \in V^{\mathbb{R}}$ has order type ω_1, is cofinal in λ^+ and has the property that, for every $\alpha \in \text{Lim}(E)$, $E \cap \alpha \in \mathcal{C}_\alpha$. Namely, E is the pre-image of D_γ under j. Note that, clearly, E does not exist in V, although all of its initial segments do.

By assumption, there is $n_\alpha \in \omega$ with $|T_\alpha| < \lambda_{n_\alpha}$. Then, for any $\alpha < \alpha' < \lambda^+$, $|T_\alpha| \leqslant |T_{\alpha'}|$ and so, there is a fixed $n \in \omega$ such that, for every $\alpha < \lambda^+$, $|T_\alpha| < \lambda_n$.

By regularity of λ_n, for every $\alpha < \lambda^+$ with $\mathrm{cof}(\alpha) = \lambda_n$, there is some $\beta < \alpha$ such that, for every pair $z, z' \in T_\alpha$, if $z \neq z'$ then $z \cap \beta \neq z' \cap \beta$. This produces a regressive function on a stationary set of ordinals below λ^+; thus, there is a stationary $S \subseteq \lambda^+$ and a fixed $\beta < \lambda^+$ so that

$$(\forall\, \alpha \in S)\,(\forall\, z, z' \in T_\alpha)\,(z \neq z' \longrightarrow z \cap \beta \neq z' \cap \beta).$$

Now fix some $z \in T_\beta$; Then, for every $\alpha \in S$ (most interestingly for $\alpha > \beta$), there must be exactly one $z_\alpha \in T_\alpha$ such that $z = z_\alpha \cap \beta$ and $r_\alpha \Vdash z_\alpha = E \cap \alpha$, for some r_α below r, where $r \leqslant p$ and $r \Vdash z = E \cap \beta$. Hence, $E \in V$. This is a clear contradiction which proves the claim. □

Given the claim, we build a tree of conditions in $\mathbb{R}$, indexed by sequences $s \in {}^{<\omega}\lambda$. The construction is recursive based on the length of s, aiming at producing, for each $n \in \omega$, a set $A_n = \{q_s : s \in {}^{n}\lambda\} \subseteq \mathbb{R}$ and some $\delta_n < \lambda^+$ such that, for every $s \in {}^{n}\lambda$, $q_s \in A_n$ determines the segment $E \cap \delta_n$ of the thread. Let $A_0 = \{\mathbb{1}_{\mathbb{R}}\}$ and $\delta_0 = \varnothing$.

Now, suppose that A_n and δ_n are given, for some $n \in \omega$. For any fixed $s \in {}^{n}\lambda$ and $q_s \in A_n$, we show how to extend q_s to q_t, for every $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$. By the claim, there is some $\alpha_s < \lambda^+$ so that the set $T_{\alpha_s} = \{z \in V \cap [\lambda^+]^\omega : \exists\, r \leqslant q_s \text{ s.t. } r \Vdash z = E \cap \alpha_s\}$ has cardinality at least λ. Hence, by choosing for each $z \in T_{\alpha_s}$ some witnessing $r \leqslant q_s$, we produce an antichain D_s of size λ, consisting of conditions below q_s forcing incompatible information about $E \cap \alpha_s$. We index these conditions by $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$, writing $D_s = \{r_t : t \in {}^{n+1}\lambda, s \sqsubseteq t\}$.

We now let $\delta_{n+1} = \sup\{\alpha_s : s \in {}^{n}\lambda\}$ (notice that $\delta_{n+1} < \lambda^+$). In order to define A_{n+1}, for every $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$, we choose some q_t which is an extension of $r_t \in D_s$ and such that q_t decides $E \cap \delta_{n+1}$, i.e., for some $z_t \in V$, we have $q_t \Vdash z_t = E \cap \delta_{n+1}$. Then, we let A_{n+1} be the collection of the chosen extensions q_t's. It is now immediate that $A_{n+1} = \{q_t : t \in {}^{n+1}\lambda\}$ along with δ_{n+1} satisfy the construction requirement. Furthermore, by the σ-closure of $\mathbb{R}$, we may assume, enlarging δ_{n+1} and extending each q_t if necessary, that every q_t forces that $z_t = E \cap \delta_{n+1}$ is unbounded in δ_{n+1}. Finally, let $\delta = \sup_n \delta_n < \lambda^+$.

For each function $f : \omega \longrightarrow \lambda$, let $q_f \in \mathbb{R}$ be a lower bound of the descending chain $\{q_{f\restriction n} : n \in \omega\}$. For every such function, q_f determines $E \cap \delta$; namely, if $z_f = \bigcup_{n\in\omega} z_{f\restriction n}$, then z_f is countable, $z_f \in V$ and $q_f \Vdash z_f = E \cap \delta$. Moreover, q_f forces that z_f is unbounded in δ. In particular, as E is supposed to be a thread, we have that $z_f \in \mathcal{C}_\delta$. Consequently, if $f \neq g$ are distinct functions from ${}^{\omega}\lambda$, then $z_f \neq z_g$ and so $|\mathcal{C}_\delta| \geqslant \lambda^\omega > \lambda$. But this is a contradiction since, by condition (*i*) of the weak square sequence, $|\mathcal{C}_\delta| \leqslant \lambda$. This completes the proof. □

A direct generalization of the previous proof gives:

Corollary 4.10. *Let $n \in \omega$ and suppose that ω_{n+2} is indestructibly generically supercompact by the class of $<\omega_{n+1}$-closed posets. Then, for every (uncountable) strong limit λ with cof$(\lambda) = \omega_n$, $\square^*_\lambda$ fails.*

Recalling that UR(σ-CLOSED) implies that ω_2 is indestructibly generically extendible by the class of σ-closed posets, Theorem 4.9 combined with Proposition 4.8 immediately give the following (adding to Corollary 4.1).

Corollary 4.11. *UR(σ-CLOSED) implies that, for every (uncountable) strong limit cardinal λ with cof$(\lambda) = \omega$, $\square^*_\lambda$ fails.*

Observe that, in the proof of Theorem 4.9, the assumption $\mathrm{cof}(\lambda) = \omega$ was only used at the final step (König's theorem). At any rate, we may ask:

Question 4.12. Can we dispense with the "strong limit" assumption in the previous result(s)? Moreover, does UR(σ-CLOSED) imply failure of even weaker principles, such as the approachability property AP?

As a concluding result of this note, we consider anew the axiom RA(SSP) and show that, unlike the other RA axioms, it has consistency strength beyond the realm of large cardinals compatible with $V = L$; namely, it already implies that every set has a sharp. By a result of Schindler (cf. Theorem 1.3 in [21]), this is certainly the case for RA(SSP)$+\neg$CH, since the latter implies BMM (by Theorem 5 in [15]). Hence, it is the case in which CH holds that is of interest. We shall use some techniques due to Schindler; let us briefly recall some material from [21].[15]

Let $r \subseteq \omega$. We describe a construction of length at most ω_1, producing an ordinal $\xi_r \leqslant \omega_1$, a function $f_r : \xi_r \longrightarrow \omega_1$, a sequence $d^{(r)} = \langle d^{(r)}_i : i < \xi_r \rangle$, and some $A_r \subseteq \xi_r$. Suppose that, for some $\nu \leqslant \omega_1$, we have defined $f_r \restriction \nu$, $\langle d^{(r)}_i : i < \nu \rangle$, and $A_r \cap \nu$. If $\nu = \omega_1$ or if $\nu < \omega_1$ and ν is uncountable in $L[A_r \cap \nu]$, we then set $\xi_r = \nu$ and finish the construction. Otherwise, we define $f_r(\nu)$ to be the least $\beta < \omega_1$ such that $L_{\beta+1}[A_r \cap \nu] \models$ "ν is countable", and we let $d^{(r)}_\nu$ be the $L[A_r \cap \nu]$-least $d \subseteq \omega$ which is almost-disjoint from all the $d^{(r)}_i$'s, for $i < \nu$. Finally, $\nu \in A_r$ if and only if $d^{(r)}_\nu \cap r$ is finite. Following [21], we say that r *codes a reshaped subset of* ω_1 if this construction can be carried out all the way up to ω_1, i.e., if $\xi_r = \omega_1$.

Obviously, if $r \subseteq \omega$ codes a reshaped subset of ω_1, then $\omega_1^{L[r]} = \omega_1^V$. Moreover, given $r \subseteq \omega$, r codes a reshaped subset of ω_1 if and only if this is witnessed in $H_{\aleph_1}$; i.e., $H_{\aleph_1}$ can faithfully verify that the previous construction goes through for all ordinals. In addition, by absoluteness of the computations, the triple $\langle f_r, d^{(r)}, A_r \rangle$ is the same, whether it is computed in $H_{\aleph_1}$ or in V.

Theorem 4.13. *RA(SSP) implies that, for all $X \in V$, $X^\#$ exists.*

Proof. By our previous comments, it is sufficient to consider the case in which CH holds. Towards a contradiction, assume that for some X, $X^\#$ does not exist. In such a case, Schindler has shown (see the proof of Theorem 1.3 in [21]) that there is a poset $\mathbb{P} \in$ SSP which adds a real $r \subseteq \omega$ coding a reshaped subset of ω_1. The latter fact is witnessed in $H^{V^{\mathbb{P}}}_{\aleph_1}$. But then, if $\dot{\mathbb{R}}$ is a further stationary preserving poset achieving resurrection (i.e., $H_{\aleph_1} \prec H^{V^{\mathbb{P}*\dot{\mathbb{R}}}}_{\aleph_1}$), and since ω_1 is preserved, we have that r codes a reshaped subset of ω_1 in $V^{\mathbb{P}*\dot{\mathbb{R}}}$ and thus, the same is true in $H^{V^{\mathbb{P}*\dot{\mathbb{R}}}}_{\aleph_1}$. Hence, by elementarity, there must exist reals $r \in V$ which code reshaped subsets of ω_1. Now, let $<^*$ be the ordering relation on functions in ${}^{\omega_1}\omega_1$ defined by:

$$f <^* g \Longleftrightarrow \exists\, C \subseteq \omega_1 \; (\text{"}C \text{ is a club"} \wedge \forall\, \alpha \in C\, (f(\alpha) < g(\alpha)))\,.$$

By the well-foundedness of $<^*$, fix $r \in V$ coding a reshaped subset of ω_1, with its associated f_r being $<^*$-minimal among functions f_x associated with $x \in V$ coding

[15] The author is grateful to Ralf Schindler for his suggestions, and for kindly explaining his methods that are involved in the proof of Theorem 4.13.

reshaped subsets of ω_1. Let $d^{(r)} = \langle d^{(r)}_\alpha : \alpha < \omega_1 \rangle \in L[r]$ be the sequence of almost-disjoint subsets associated with this r.

Then, by Lemma 3.3 in [21], there is a $\mathbb{Q}_1 \in$ SSP forcing the existence of some r' and of some club $C \subseteq \omega_1$, so that r' codes a reshaped subset of ω_1 and C witnesses that $f_{r'} <^* f_r$. In $V^{\mathbb{Q}_1}$, let $\dot{\mathbb{Q}}_2$ be the (name for the) c.c.c. poset which codes C by a real z, relative to the sequence $d^{(r)}$. That is, $\dot{\mathbb{Q}}_2$ is the Jensen–Solovay almost-disjoint coding, producing a $z \subseteq \omega$ such that, in $V^{\mathbb{Q}_1 * \dot{\mathbb{Q}}_2}$, for every $\alpha < \omega_1$, $\alpha \in C \iff |z \cap d^{(r)}_\alpha| < \aleph_0$. Let $\mathbb{Q} = \mathbb{Q}_1 * \dot{\mathbb{Q}}_2$ and notice that $\mathbb{Q} \in$ SSP in V. Hence, by RA(SSP), there is some $\dot{\mathbb{R}}$ with $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in$ SSP", giving that $H_{\aleph_1} \prec H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{\aleph_1}$. Clearly, ω_1 is preserved throughout. It is now easy to check that, in $H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{\aleph_1}$, we may express the statement "there is an r' coding a reshaped subset of ω_1 with $f_{r'} <^* f_r$". Thus, by elementarity, such a real must already exist in V, contradicting the $<^*$-minimality of f_r. $\square$

5. Concluding thoughts & questions

After having developed the formal mathematical side of the (unbounded) resurrection principles, some informal remarks and reflections are in order.

First of all, let us recall that the models of the various UR axioms that were obtained in Section 2 arose from standard forcing constructions, ones which had the advantage of exploiting the strength of the extendibility assumption, as opposed to the typical supercompactness one. We view these constructions as a natural "enhancement" of their original counterparts, where one takes one more step in the ladder of the large cardinal hierarchy, while employing the same underlying forcing machinery. Although, as it has hopefully become clear, their enhanced character makes all the difference in this note, these constructions are hardly exotic in spirit.

In parallel, and so far as the initial resurrection principles constitute a natural and robust framework for the study of the usual forcing axioms, the additional conceptual step which leads to their unbounded versions should not be –at least in retrospect– so surprising on the intuitive level either. Moreover, in light of Theorem 3.1, it indeed turns out that in the presence of a proper class of Woodin cardinals the axioms MM^{++} and UR(SSP) are actually equivalent. Recent work of Viale further shows that, again in the context of proper class many Woodin cardinals, UR(SSP) is also related to the axiom MM^{+++} which he introduced.[16]

We believe that the above reflections, together with the rest of the results highlighting their strength, consequences, and relations with the standard axioms, strongly indicate that the UR postulates arise as natural generalizations of the familiar and well-established forcing axioms in set theory; in this context, they seem to open a direction which is certainly worth investigating further.

As a related comment, we also feel that we should point out the following issue. The UR axioms, by their very formulation, seem to suggest intuitively that ω_2 is an extendible cardinal "in disguise"; more precisely, they imply that ω_2, which is –in most cases– the critical point of the produced generic embeddings, retains generically "shades of extendibility" that it might have had in some inner model. In other

[16] In fact, Viale argues that an appropriate strengthening of UR(SSP) is equivalent to MM^{+++}. Additionally, he shows that RA(Γ) makes the Σ_2-theory of $H_{\mathfrak{c}}$ invariant with respect to posets in Γ which force $\mathsf{BFA}_{<\mathfrak{c}}(\Gamma)$ (for more details, see § 5 in [28], and the final comments in [29]). Indeed, the reader is strongly encouraged to follow Viale's relevant work appearing in [28] and in [29].

words, these axioms seem to suggest that they were really obtained by some classical “forcing axiom construction" starting from an extendible cardinal, which was eventually collapsed to ω_2. Granted this intuition, it is then tempting to conjecture that such axioms are in fact equiconsistent with extendibility.[17] Nevertheless, the equivalence of MM^{++} with UR(SSP) in the presence of class many Woodin cardinals shows that the aforementioned intuition is misleading, at least for the class of stationary preserving posets: as stated in Corollary 3.3, a supercompact with a proper class of Woodin cardinals is an adequate bound.

Let us conclude with some open problems and general enquiries which have arisen along the way. The following list is certainly non-exhaustive.

Question 5.1. Can we separate MM (or even MM^{++}) from RA(SSP), by producing a model in which the former holds while the latter fails?

Question 5.2. Do the (unbounded) resurrection axioms enjoy (degrees of) indestructibility under appropriate forcings?

For the usual forcing axioms such as PFA and MM, there are various known indestructibility results (see, e.g., [18]).

Question 5.3. What is the exact consistency strength of the unbounded resurrection axioms? How about RA(SSP)?[18]

Finally, we cannot resist enquiries regarding the relation of the UR axioms with Woodin's $(*)$ axiom (cf. Definition 5.1 in [30]).

Question 5.4. Does UR(SSP) imply the $(*)$ axiom?

Although we do not intend to insinuate any unjustified optimism, a positive answer to the latter question would indeed be a remarkable result.

Acknowledgments. The author wishes to express his deep gratitude to David Asperó, who very generously contributed his valuable mathematical insights, comments, and suggestions to the present note. It was through Asperó's crucial intervention that the author was led to the formulation of the unbounded resurrection principles in the first place. Further gratitude goes to Joel D. Hamkins and to Thomas Johnstone for various discussions on themes related to the present material, as well as to the anonymous referee for many useful suggestions. Most of this work is part of the author's doctoral dissertation; the latter was composed under the supervision of Joan Bagaria, whose help and support are warmly acknowledged.

References

[1] Bagaria, J. *A characterization of Martin's Axiom in terms of absoluteness.* Journal of Symbolic Logic, Vol. 62 (2), pp. 366–372, 1997.

[2] Bagaria, J. *Bounded forcing axioms as principles of generic absoluteness.* Archive for Mathematical Logic, Vol. 39 (6), pp. 393–401, 2000.

[3] Bagaria, J. *$C^{(n)}$-cardinals.* Archive for Mathematical Logic, Vol. 51 (3–4), pp. 213–240, 2012.

[17] Thanks to David Asperó and to Ralf Schindler for their comments on this issue, many of which appear here verbatim.

[18] As far as the axiom RA(SSP) is concerned, it is worth mentioning that Ralf Schindler has pointed out to us that his arguments showing that BMM implies strong cardinals in inner models (cf. [22]) are applicable to this case as well. For the time being, we take his word for it and conveniently avoid delving into the world of inner model theory.

[4] Bagaria, J., Magidor, M. *On ω_1-strongly compact cardinals*. Israel Journal of Mathematics (to appear).

[5] Bagaria, J., Hamkins, J.D., Tsaprounis, K., Usuba, T. *Superstrong and other large cardinals are never Laver indestructible*. Preprint (2013) `http://arxiv.org/abs/1307.3486`

[6] Cohen, P.J. *The independence of the continuum hypothesis*. Proceedings of the National Academy of Sciences (U.S.A.), Vol. 50, pp. 1143–1148, 1963.

[7] Corazza, P. *Laver sequences for extendible and super-almost-huge cardinals*. Journal of Symbolic Logic, Vol. 64 (3), pp. 963–983, 1999.

[8] Cox, S. *PFA and ideals on ω_2 whose associated forcings are proper*. Notre Dame Journal of Formal Logic, Vol. 53 (3), pp. 397–412, 2012.

[9] Cummings, J., Foreman, M., Magidor, M. *Squares, scales and stationary reflection*. Journal of Mathematical Logic, Vol. 1 (1), pp. 35–98, 2001.

[10] Eisworth, T. *Successors of singular cardinals*. In Handbook of Set Theory, Eds. M. Foreman & A. Kanamori, pp. 1229–1350, Springer, 2010.

[11] Foreman, M. *Ideals and generic elementary embeddings*. In Handbook of Set Theory, Eds. M. Foreman & A. Kanamori, pp. 1229–1350, Springer, 2010.

[12] Foreman, M., Magidor, M. *A very weak square principle*. Journal of Symbolic Logic, Vol. 62 (1), pp. 175–196, 1997.

[13] Foreman, M., Magidor, M., Shelah, S. *Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I*. Annals of Mathematics, Vol. 127, pp. 1–47, 1988.

[14] Hamkins, J.D. *A simple maximality principle*. Journal of Symbolic Logic, Vol. 68 (2), pp. 527–550, 2003.

[15] Hamkins, J.D., Johnstone, T. *Resurrection axioms and uplifting cardinals*. Preprint (2013) `http://arxiv.org/abs/1307.3602`

[16] Jech, T. *Set Theory, The Third Millennium Edition*. Springer–Verlag, 2002.

[17] Kanamori, A. *The Higher Infinite*. Springer–Verlag, 1994.

[18] König, B. *Forcing indestructibility of set-theoretic axioms*. Journal of Symbolic Logic, Vol. 72 (1), pp. 349–360, 2007.

[19] Larson, P.B. *The Stationary Tower, Notes on a Course by W. Hugh Woodin*. University Lecture Series, Vol. 32, American Mathematical Society, 2004.

[20] Laver, R. *Making the supercompactness of κ indestructible under κ-directed closed forcing*. Israel Journal of Mathematics, Vol. 29, pp. 385–388, 1978.

[21] Schindler, R. *Semi-proper forcing, remarkable cardinals, and Bounded Martin's Maximum*. Mathematical Logic Quarterly, Vol. 50 (6), pp. 527–532, 2004.

[22] Schindler, R. *Bounded Martin's Maximum and strong cardinals*. In Set Theory, Centre de recerca Matemàtica, Barcelona, 2003–2004, Eds. J. Bagaria & S. Todorčević, pp. 401–406, Birkhäuser, 2006.

[23] Shelah, S. *Proper and Improper Forcing*. Springer–Verlag, 1998.

[24] Solovay, R., Tennenbaum, S. *Iterated Cohen extensions and Souslin's problem*. Annals of Mathematics, Vol. 94, pp. 201–245, 1971.

[25] Stavi, J., Väänänen, J. *Reflection principles for the continuum*. In Logic and Algebra, Ed. Yi Zhang, Contemporary Mathematics, Vol. 302, pp. 59–84, American Mathematical Society, 2002.

[26] Tsaprounis, K. *Elementary chains and $C^{(n)}$-cardinals*. Archive for Mathematical Logic (to appear).

[27] Tsaprounis, K. *Large cardinals and resurrection axioms*. Ph.D. dissertation, University of Barcelona, Spain, 2012.

[28] Viale, M. *Martin's maximum revisited*. Preprint (2013) available on the author's webpage.

[29] Viale, M. *Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms*. Preprint (2013) available on the author's webpage.

[30] Woodin, W.H. *The Axiom of Determinacy, Forcing Axioms, and the Non-Stationary Ideal*. Second Edition, De Gruyter Series in Logic and its Applications, 2010.

Department of Logic, History & Philosophy of Science, University of Barcelona, 08001 Barcelona, Spain

E-mail address: `kostas.tsap@gmail.com`