ULTRAHUGE CARDINALS

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ABSTRACT. In this note, we start with the notion of a superhuge cardinal and strengthen it by requiring that the elementary embeddings witnessing this property are, in addition, sufficiently superstrong above their target $j(\kappa)$. This modification leads to a new large cardinal which we call *ultrahuge*. Subsequently, we study the placement of ultrahugeness in the usual large cardinal hierarchy, while at the same time show that some standard techniques apply nicely in the context of ultrahuge cardinals as well.

1. INTRODUCTION

Superhuge cardinals (and their relatives) were introduced by Barbanel, Di Prisco, and Tan in 1984 (cf. [4]) and are placed near the highest layers of the large cardinal hierarchy, just below the so-called *rank-into-rank* cardinals which are the strongest axioms of infinity not known to be inconsistent with ZFC set theory.

During the years, large cardinals at the level of (super)hugeness have found several applications in various contexts. Recently, Viale has used superhuge cardinals in order to deduce the consistency of the forcing axiom MM^{+++} which he introduced, and which implies a strong form of generic absoluteness (see [21]). Moreover, Viale has underlined the connection between his work and the axioms of *unbounded resurrection* which were introduced by the author in [19].

In our earlier study of the unbounded resurrection axioms, the central large cardinal notion which was employed was that of extendibility. Indeed, we looked at extendible cardinals from a new perspective, characterizing them in terms of *class* elementary embeddings which are, at the same time, supercompact and sufficiently superstrong above their target $j(\kappa)$. In fact, such a characterization is also available for the corresponding $C^{(n)}$-version of extendibility as well (see [18]).

Motivated both by our study of extendibility and by Viale's recent work, we now wish to apply a similar idea to the notion of superhugeness: that is, blend it with the "sufficient" superstrongness requirement, as we did for supercompact cardinals. It turns out that this blend gives rise to a new large cardinal notion, which we call *ultrahugeness*, and which lies strictly between superhugeness and almost 2-hugeness. Thus, the main aim of the current note is to call further attention to the general intuitive idea of appending the additional superstrongness assumption to known large cardinals, a feature which seems to give noticeable flexibility with the respect to the set-theoretic techniques which can then go through smoothly.

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The structure of this note is as follows. The necessary preliminaries, as well as a brief overview of our earlier work, are given in Section 2. In Section 3, we introduce the notion of an ultrahuge cardinal, which is the central one of this note, and we then look at its placement in the usual large cardinal hierarchy.

In Section 4, we turn to the study of the $C^{(n)}$-version of ultrahugeness giving, at the same time, consistency upper bounds for $C^{(n)}$-superhugeness which improve on the previously known ones from [2]. We then move on to Section 5 where we show that ultrahuge cardinals carry their one, adequate, Laver functions. Finally, in Section 6, we establish some partial preservation of ultrahuge cardinals under the canonical class iteration which forces the GCH globally in the universe. We close the current note with some concluding remarks and questions in Section 7.

2. Preliminaries

2.1. **Notation.** Our notation and terminology are standard; we refer the reader to [12] or [15] for an account of all undefined set-theoretic notions, as well as for a comprehensive presentation of the theory of large cardinals. Adopting the notation of [2], and for every natural number n, we let $C^{(n)}$ denote the closed and unbounded proper class of ordinals α which are Σ_n-correct in V, that is, ordinals α such that V_α is a Σ_n-elementary substructure of V (denoted by $V_\alpha \prec_n V$). Note that the statement "$\alpha \in C^{(n)}$" is expressible by a Π_n-formula, for every $n \geqslant 1$.

Given any function f and any $A \subseteq dom(f)$, we write $f \restriction A$ for the restriction of f to A, and $f``A$ for the pointwise image of $A$ under $f$, i.e., $f``A = \{f(x) : x \in A\}$. We use the three-dot notation in order to indicate partial functions, that is, $f \vdots X \longrightarrow Y$ means that $dom(f) \subseteq X$, with the inclusion possibly being proper. If $\kappa \leqslant \lambda$ are (infinite) cardinals, we let $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$.

If $\mathbb{P}$ is a forcing poset and $p, q \in \mathbb{P}$, we write $p < q$ to mean than p is stronger than q; in addition, we denote the greatest element of a poset by $\mathbb{1}$. If κ, λ are regular cardinals, we let $\mathrm{Add}(\kappa, \lambda)$ denote the poset consisting of partial functions $p \vdots \lambda \times \kappa \longrightarrow 2$ with $|p| < \kappa$; the ordering is given by reversed inclusion. A poset $\mathbb{P}$ is called *weakly homogeneous* if for every $p, q \in \mathbb{P}$ there is an automorphism $\sigma_{p,q} : \mathbb{P} \longrightarrow \mathbb{P}$ such that $\sigma(p)$ and q are compatible. It is widely known that, for any regular cardinals κ and λ, $\mathrm{Add}(\kappa, \lambda)$ is weakly homogeneous.

If j is a non-trivial elementary embedding we write $\mathrm{cp}(j)$ for its critical point. Given an elementary embedding j with $\mathrm{cp}(j) = \kappa$, we let $j^{(n)}(\kappa)$ denote the n-th iterate as usual; i.e., $j^{(0)}(\kappa) = \kappa$ and, for every $n > 0$, $j^{(n)}(\kappa) = j(j^{(n-1)}(\kappa))$. Whenever we lift embeddings to forcing extensions we follow the standard practice and use the same letter j for the lifted version of the embedding.

Finally, we also recall some relevant definitions. For any $n \geqslant 1$, a cardinal κ is called *n-huge* if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$ and M closed under $j^{(n)}(\kappa)$-sequences. If the latter requirement is weakened to closure under $< j^{(n)}(\kappa)$-sequences, then κ is called *almost n-huge*. Moreover, we say that κ is *super n-huge* if, for every $\lambda > \kappa$, there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $\lambda < j(\kappa)$ and M closed under $j^{(n)}(\kappa)$-sequences. Super almost n-huge cardinals are defined accordingly. When $n = 1$, we just say that κ is huge, almost huge, superhuge, and super almost huge, respectively. For more details on such notions see [4] and [15].

2.2. **Extendible cardinals.** In order to motivate our study of ultrahuge cardinals, let us briefly review some related earlier work, mainly from [18] and [19], in the context of extendibility.

Recall that a cardinal κ is called λ-*extendible*, for some $\lambda > \kappa$, if there is some θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ such that $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$; κ is called *extendible* if it is λ-extendible for all $\lambda > \kappa$. That is, traditionally, extendibility is witnessed locally by set embeddings between rank initial segments of the universe. Nevertheless, and at the core of our main idea, we also have a characterization of extendibility in terms of class embeddings.

Definition 2.1 ([18]). A cardinal κ is called **jointly λ-supercompact and θ-superstrong**, for some $\lambda, \theta \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $V_{j(\theta)} \subseteq M$. In this case, we say that j *is jointly λ-supercompact and θ-superstrong for κ.*

We say that κ is jointly supercompact and θ-superstrong, for some fixed $\theta \geqslant \kappa$, if it is jointly λ-supercompact and θ-superstrong, for every $\lambda \geqslant \kappa$; moreover, we say that κ is jointly supercompact and superstrong if it is jointly λ-supercompact and λ-superstrong, for every $\lambda \geqslant \kappa$.

Notice that if κ is the least supercompact, then it is not jointly λ-supercompact and κ-superstrong, for any λ. In fact:

Theorem 2.2 ([18]). *A cardinal κ is extendible if and only if it is jointly supercompact and κ-superstrong if and only if it is jointly supercompact and superstrong.*

The previous theorem follows from Corollary 2.31 in [18] and its subsequent remarks; indeed, we furthermore showed in [18] that such a characterization is also available for the $C^{(n)}$-version of extendible cardinals, as this was introduced by Bagaria (cf. [2]).

It thus turns out that if one strengthens the notion of supercompactness by requiring that the various witnessing embeddings are, in addition, sufficiently superstrong above their target $j(\kappa)$, then one naturally arrives at the large cardinal notion of extendibility.

The previous characterizations of extendible cardinals have been particularly useful in some contexts. For instance, they have been employed in an essential way in [19] in order to (motivate the introduction and) derive the consistency of the *unbounded resurrection* axioms. This is accomplished by first establishing the existence of adequate Laver functions for extendible cardinals, as follows:[1]

Definition 2.3 ([19]). Let κ be an extendible cardinal. A function $\ell \,\vdots\, \kappa \longrightarrow V_\kappa$ is an **extendibility Laver function** for κ if for every cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$ there is an (extender) elementary embedding $j : V \longrightarrow M$ which is jointly λ-supercompact and λ-superstrong for κ, and such that $j(\ell)(\kappa) = x$.

Theorem 2.4 ([19]). *Every extendible cardinal carries an extendibility Laver function as above.*

Furthermore, recent unpublished work of Audrito and Viale makes additional use of the above characterization(s) of extendibility, this time in the context of *iterated* resurrection axioms (see [1]).

3. Ultrahuge cardinals

From our perspective, the combination of supercompactness with sufficient superstrongness above $j(\kappa)$ (as made precise in Definition 2.1) was indeed a fruitful one,

[1] Of course, it should be mentioned that Corazza had already obtained the existence of Laver functions for extendible cardinals (cf. [7]); nevertheless, in the current context and in the light of Theorem 2.2 above, we are interested in such functions in the presence of *class* elementary embeddings witnessing extendibility.

and although it led to the already familiar notion of extendibility, it nevertheless allowed for various old and new results to (re)emerge.

We now wish to take this idea and apply it further to large cardinals at the level of superhugeness. More precisely, we want to postulate the existence of elementary embeddings which, apart from being (almost) huge, are also sufficiently superstrong above their target, to any desired degree which is fixed beforehand. Let us underline the fact that this idea, by design, naturally leads to *global* large cardinal notions; this is the reason for focusing on superhuge as opposed to merely huge cardinals.[2]

Thus, we now give the following central definition of this note.

Definition 3.1. We say that a cardinal κ is λ-**ultrahuge**, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$. In such a case, we say that j is λ-*ultrahuge for* κ. As expected, we say that κ is **ultrahuge**, if it is λ-ultrahuge for all $\lambda \geqslant \kappa$.

A natural variant of the above definition is that of λ-**ultra almost hugeness**, where –as usual– the closure requirement ${}^{j(\kappa)}M \subseteq M$ is weakened to ${}^{<j(\kappa)}M \subseteq M$.[3]

Observe that the extra requirement of "sufficient superstrongness above the target $j(\kappa)$", which is expressed by the clause "$V_{j(\lambda)} \subseteq M$", is better tailored for the notion of 1-hugeness, i.e., for embeddings with closure under $j(\kappa)$-sequences. This is because, if $n > 1$, then every n-hugeness embedding of the form $j : V \longrightarrow M$ is such that $V_{j^{(n)}(\kappa)+1} \subseteq M$ anyway (where $j^{(n)}(\kappa) > j(\lambda)$), with this following from the closure of M under $j^{(n)}(\kappa)$-sequences.

Moreover, notice that the statement "κ is λ-ultra (almost) huge" can be expressed by a Σ_2-formula, since ultra (almost) huge embeddings can be captured by extenders (either of the Martin-Steel form, or ordinary –but quite long– ones; see the Appendix of [20] for related details). Consequently, the statement "κ is ultra (almost) huge" is Π_3-expressible and we note that, unless an inconsistency in these layers of the large cardinal hierarchy emerges, this complexity bound is optimal since every ultra (almost) huge cardinal is extendible and thus Σ_3-correct in the universe. Similar expressibility bounds are obtainable for super (almost) huge cardinals as well.

It is already clear that ultrahugeness directly implies superhugeness. Moreover:

Proposition 3.2. *If κ is an ultrahuge cardinal then there exists a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : \alpha$ is superhuge$\} \in \mathcal{U}$. In particular, the least superhuge cardinal is below the least ultrahuge, assuming both exist.*

Proof. Suppose that κ is ultrahuge and fix some inaccessible $\theta > \kappa$ and some $\theta+1$-ultrahuge embedding $j : V \longrightarrow M$, i.e., M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta+1$, ${}^{j(\kappa)}M \subseteq M$ and $V_{j(\theta)+1} \subseteq M$. Note that, in such a case, $j(\theta)$ is inaccessible. It is enough to argue that κ is superhuge in M, from which the conclusion follows. In turn, we just argue that κ is superhuge in $V_{j(\kappa)}$, since $j(\kappa)$ is Σ_3-correct in M.

For this, we employ an elementary chain construction, using the fact that j is $\theta + 1$-ultrahuge with $j(\kappa)$ and $j(\theta)$ inaccessibles.[4] Let us start by picking some

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[2]The latter are local notions, in the sense that they can be described by one single elementary embedding or, equivalently, by one single (appropriate) ultrafilter; see §24 in [15].

[3]Following Definition 2.1, one should perhaps call the relevent elementary embeddings *jointly super (almost) huge and λ-superstrong*, but this would be a rather long and inconvenient terminology. Assuming that the reader maintains in perspective the fact that each time we are referring to a single embedding witnessing *simultaneously* (almost) hugeness and sufficient superstrongness, we chose the more concise prefix "ultra" instead, with the hope that it is an adequate one.

[4]For more details on such constructions the reader is referred to [18] and [20], where various related examples can be found.

initial limit ordinal $\beta_0 \in (j(\kappa), j(\theta))$ and by letting

$$X_0 = \{j(f)(j``j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_0}\} \prec M.$$

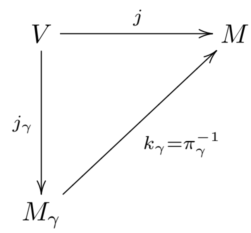
We also pick some $\gamma < j(\theta)$ with $\mathrm{cof}(\gamma) > j(\kappa)$, which will serve as the length of our constructed chain. Then, for any $\xi + 1 < \gamma$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\theta)) + \omega$ and

$$X_{\xi+1} = \{j(f)(j``j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_{\xi+1}}\} \prec M.$$

If $\xi < \gamma$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha \prec M$. Finally, we let $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is:

$$X_\gamma = \{j(f)(j``j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_\gamma}\} \prec M.$$

It is easy to see that the inaccessibility of $j(\theta)$ gives that $\beta_\gamma < j(\theta)$, where clearly $\mathrm{cof}(\beta_\gamma) = \mathrm{cof}(\gamma) > j(\kappa)$. We now consider the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$ and we then define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, producing a commutative diagram of elementary embeddings:



At this point, by standard arguments and the representation of the model M_γ, it is easy to check that j_γ is a θ-ultrahuge embedding for κ, with $\mathrm{cp}(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and

$$\mathrm{cp}(k_\gamma) = j_\gamma(\theta) = \sup(X_\gamma \cap j(\theta)) = \beta_\gamma.$$

Furthermore, again by the inaccessibility of $j(\theta)$, for every $\alpha < j(\theta)$ we have that $j_\gamma(\alpha) < j(\theta)$; hence, we may derive from the embedding j_γ a relevant (either Martin-Steel, or ordinary but long) extender E which witnesses its θ-ultrahugeness, and then it follows that E belongs to $V_{j(\theta)}$ and thus to M. Indeed, even the model $V_{j(\theta)}$ can faithfully verify that E is a θ-ultrahuge extender for κ and thus, we have that $M \models$ "κ is θ-ultrahuge".[5]

To conclude the proof, we observe that the extender E witnesses in M that, for every $\lambda \in (\kappa, j(\kappa))$, κ is huge with target past λ; in other words, and since $j(\kappa)$ is Σ_3-correct in M, we therefore have that $V_{j(\kappa)} \models$ "κ is superhuge", as desired. □

In the other direction, we can initially show the following.

Proposition 3.3. *If κ is super almost 2-huge then it is ultrahuge and, moreover, there exists a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : \alpha$ is ultrahuge$\} \in \mathcal{U}$. In particular, the least ultrahuge cardinal is below the least super almost 2-huge, assuming both exist.*

Proof. Suppose that κ is super almost 2-huge and, for some $\theta > \kappa$ with $\theta \in C^{(3)}$, fix some embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and ${}^{<j^{(2)}(\kappa)}M \subseteq M$. Note that $V_{j^{(2)}(\kappa)} \subseteq M$ and, additionally, $M \models \theta \in C^{(2)}$ because the latter is a Π_2-expressible statement and it thus reflects to $V_{j(\kappa)}$.

By the closure of M and the (easy) fact that $2^{j(\kappa)} < j^{(2)}(\kappa)$, we have that the restricted map $j \upharpoonright V_{j(\kappa)+1} : V_{j(\kappa)+1} \longrightarrow V^M_{j^{(2)}(\kappa)+1}$ belongs to M; indeed,

[5]Note that there is no (dangerous) contradiction here, since we started with a $\theta+1$-ultrahuge embedding, which is a stronger (marginally, but still stronger) assumption.

it witnesses in M the λ-ultrahugeness of κ, for every $\lambda \in (\kappa, j(\kappa)]$: to see this, we extract in M the (ordinary but long) $(\kappa, j^{(2)}(\kappa))$-extender E derived from the map $j \restriction V_{j(\kappa)+1}$, and then consider the corresponding extender embedding $j_E : M \longrightarrow M_E$. It is now straightforward to check that the latter witnesses in M the λ-ultrahugeness of κ, for every $\lambda \in (\kappa, j(\kappa)]$, as claimed.

Thus, on the one hand, for every $\lambda \in (\kappa, \theta)$, the λ-ultrahugeness of κ in M is reflected inside V_θ and, hence, $V_\theta \models$ "κ is ultrahuge". But the latter is correct in V, since $\theta \in C^{(3)}$; i.e., κ is indeed an ultrahuge cardinal.

On the other hand, we similarly get that κ is actually ultrahuge in M as well, since this is also witnessed inside $V_{j(\kappa)}$ and $j(\kappa)$ is Σ_3-correct in M. Consequently, a standard reflection argument gives that $\{\alpha < \kappa : \alpha \text{ is ultrahuge}\} \in \mathcal{U}$, where $\mathcal{U}$ is the usual normal measure on κ which is derived from the embedding j. □

Note that the previous result gives a direct implication, as well as an unbounded (indeed, stationary) set of $\alpha < \kappa$ which are ultrahuge in V. Nevertheless, we can do better consistency-wise.

Theorem 3.4. *If κ is an almost 2-huge cardinal then there exists a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : V_\kappa \models$ "α is ultrahuge"$\} \in \mathcal{U}$.*

Proof. Suppose that κ is almost 2-huge and let $j : V \longrightarrow M$ be an elementary embedding with M transitive, $\mathrm{cp}(j) = \kappa$ and ${}^{<j^{(2)}(\kappa)}M \subseteq M$. Clearly, $V_{j^{(2)}(\kappa)} \subseteq M$. Let us temporarily use the terminology "κ is (α, β, γ)-ultrahuge" to denote the fact that there is an embedding h which is huge for κ and, in addition, is such that $h(\kappa) = \alpha$, $h^{(2)}(\kappa) = \beta$ and $V_{h(\gamma)}$ is included in the target model; moreover, we shall say that "κ is (α, γ)-ultrahuge" if we drop from the previous statement any reference to the second iterate $h^{(2)}(\kappa)$ of the purported embedding h.

Now, as before, κ is $(j(\kappa), j^{(2)}(\kappa), j(\kappa))$-ultrahuge in M and, thus, if $\mathcal{U}$ is the usual normal measure on κ derived from j, we have that

$$S = \{\alpha < \kappa : \alpha \text{ is } (\kappa, j(\kappa), \kappa)\text{-ultrahuge}\} \in \mathcal{U}.$$

Fix $\alpha \in S$. Then, by the closure of M, the fact that α is $(\kappa, j(\kappa), \kappa)$-ultrahuge is true in M because it can be witnessed locally by some embedding of the form $h : V_{\kappa+1} \longrightarrow V_{j(\kappa)+1}$ with $\mathrm{cp}(h) = \alpha$, $h(\alpha) = \kappa$ and $h^{(2)}(\alpha) = j(\kappa)$ (from which an appropriate long extender may be derived, just as in the proof of Proposition 3.3). Therefore, we can now easily see that

$$S_\alpha = \{\xi < \kappa : V_\kappa \models \text{“}\alpha \text{ is } (\xi, \xi)\text{-ultrahuge”}\} \in \mathcal{U},$$

and so, by considering unboundedly many $\xi < \kappa$, we have that α is ultrahuge in V_κ. Therefore, it now follows that $\{\alpha < \kappa : V_\kappa \models$ "α is ultrahuge"$\} \in \mathcal{U}$, as desired. □

Consequently, if such notions are consistent, ultrahuge cardinals are strictly between superhuge and almost 2-huge cardinals in consistency strength. Moreover, the least superhuge is below the least ultrahuge, which in turn is below the least super almost 2-huge cardinal.

4. $C^{(n)}$-ULTRAHUGE CARDINALS

The so-called $C^{(n)}$-cardinals were introduced in [2] by Bagaria, who showed that such notions are closely related to the theme of reflection for the set-theoretic universe; in particular, Bagaria obtained level-by-level correspondence between $C^{(n)}$-extendible cardinals and *Vopěnka's Principle* (VP), with the latter being a sort of reflection principle which carries high consistency strength. Subsequently, the various $C^{(n)}$-cardinals were further studied by the author in [18].

In this section, we look at the $C^{(n)}$-version of ultrahugeness and give some of its basic properties. For more background details on $C^{(n)}$-superhuge and $C^{(n)}$-huge cardinals the reader is referred to [2].

As usual in the context of $C^{(n)}$-cardinals, the following is actually a schema of definitions, one for each meta-theoretic natural number $n \geqslant 1$.

Definition 4.1. We say that a cardinal κ is λ-$C^{(n)}$-**ultrahuge**, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{j(\kappa)}M \subseteq M$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) \in C^{(n)}$. Moreover, we say that κ is $C^{(n)}$-**ultrahuge**, if it is λ-$C^{(n)}$-ultrahuge for all $\lambda \geqslant \kappa$.

Note that, clearly, a cardinal κ is ultrahuge if and only if it is $C^{(1)}$-ultrahuge. By results in [2], and since any $C^{(n)}$-ultrahuge cardinal is clearly $C^{(n)}$-superhuge, we have that if κ is $C^{(n)}$-ultrahuge then it is $C^{(n)}$-extendible; moreover, $\kappa \in C^{(n+2)}$ and there is a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-extendible}\} \in \mathcal{U}$.

Additionally, it is not difficult to see that, for every $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-ultrahuge" is Σ_{n+1}-expressible via the use of (long) extenders; hence, for every $n \geqslant 1$, the statement "κ is $C^{(n)}$-ultrahuge" is Π_{n+2}-expressible.

In particular, for every $n \geqslant 1$, if κ is $C^{(n+1)}$-ultrahuge then there are unboundedly many $C^{(n)}$-ultrahuge cardinals below κ. Therefore, the $C^{(n)}$-ultrahuge cardinals form a strict hierarchy: for every $n \geqslant 1$, the least $C^{(n)}$-ultrahuge is below the least $C^{(n+1)}$-ultrahuge, assuming both exist.

Let us now turn to the placement of $C^{(n)}$-ultrahugeness in the usual large cardinal hierarchy. Initially, we obtain the following upper bound.

Proposition 4.2. *Fix $n \geqslant 1$. If $\kappa \in C^{(n+1)}$ is an almost 2-huge cardinal then there is a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : V_\kappa \models \text{"}\alpha \text{ is } C^{(n)}\text{-ultrahuge"}\} \in \mathcal{U}$.*

Proof. Suppose that $j : V \longrightarrow M$ is an elementary embedding with M transitive, $\mathrm{cp}(j) = \kappa$ and ${}^{<j^{(2)}(\kappa)}M \subseteq M$. Note that $V_{j^{(2)}(\kappa)} \subseteq M$ and $M \models j(\kappa) \in C^{(n+1)}$.

As before, the restricted map $j \restriction V_{j(\kappa)+1} : V_{j(\kappa)+1} \longrightarrow V^M_{j^{(2)}(\kappa)+1}$ belongs to M; moreover, it witnesses in M the λ-$C^{(n)}$-ultrahugeness of κ, for every $\lambda \in (\kappa, j(\kappa)]$. But since $j(\kappa)$ is Σ_{n+1}-correct in M, it follows that $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-ultrahuge". The conclusion now follows from a standard reflection argument, via the usual normal measure $\mathcal{U}$ on κ which is derived from the embedding j. □

However, we can obtain a better consistency upper bound for $C^{(n)}$-ultrahuge cardinals, by appropriately modifying the proof of Theorem 3.4.

Theorem 4.3. *Suppose that κ is almost 2-huge. Then, there is a normal measure $\mathcal{U}$ on κ such that*

$$\{\alpha < \kappa : \forall n \in \omega \, (V_\kappa \models \text{"}\alpha \textit{ is } C^{(n)}\textit{-ultrahuge"})\} \in \mathcal{U}.$$

Proof. Recall that in the proof of Theorem 3.4, for any fixed $\alpha \in S$, we had that

$$S_\alpha = \{\xi < \kappa : V_\kappa \models \text{"}\alpha \text{ is } (\xi, \xi)\text{-ultrahuge"}\} \in \mathcal{U}$$

and so, in particular, S_α is stationary in κ.

Now consider, for each $n \in \omega$, the club $C^{(n)}_\kappa \subseteq \kappa$ consisting exactly of those ordinals below κ which are Σ_n-correct in the sense of V_κ. Then, for any $n \in \omega$, we may intersect $C^{(n)}_\kappa$ with S_α in order to deduce that α is actually $C^{(n)}$-ultrahuge in V_κ. □

Since $C^{(n)}$-ultrahugeness implies $C^{(n)}$-superhugeness, the previous result improves the consistency upper bound for the $C^{(n)}$-superhuge cardinals, both for each individual n, and for all n simultaneously; previously, a known upper bound for each particular n was that of the existence of a $C^{(n)}$-2-huge cardinal, while for all n simultaneously the best consistency upper bound was at the level of rank-into-rank embeddings (see [2]).

In addition, we have now established that $C^{(n)}$-ultrahugeness gives, consistency-wise, a refinement of the large cardinal hierarchy between superhuge and almost 2-huge cardinals; indeed, the $C^{(n)}$-ultrahuge cardinals form a proper hierarchy which lies strictly between superhugeness and almost 2-hugeness, with the latter being a sufficient assumption for deriving the consistency of a cardinal which is, simultaneously, $C^{(n)}$-ultrahuge for every n. These bounds seem optimal, although such an issue remains open.

We conclude this section by stating the preservation of $C^{(n)}$-ultrahuge cardinals under small forcing.

Proposition 4.4. *Fix $n \geqslant 1$. Suppose that κ is $C^{(n)}$-ultrahuge and let $\mathbb{P}$ be a poset with $|\mathbb{P}| < \kappa$. Then, κ remains $C^{(n)}$-ultrahuge in $V^{\mathbb{P}}$.*

Proof. This follows from standard arguments, if we also employ Lemma 4.2 (*i*) from [18] which shows that Σ_n-correct ordinals are preserved by small forcing. □

5. Ultrahuge cardinals and Laver functions

In this section, we show that ultrahuge cardinals carry their own, adequate, Laver functions. This feature makes such cardinals slightly more appealing, at least in this respect, than (super)huge ones, since for the latter it is typically the case that one needs to assume slightly more than the cardinal at hand in order to obtain a desired Laver function; for instance, to get a hugeness Laver function one has to assume a super almost 2-huge cardinal (see Fact 13, and its subsequent remarks, in [8]).

We start by giving the official definition of the function we are aiming at.

Definition 5.1. Suppose that κ is ultrahuge. A function $\ell \colon \kappa \longrightarrow V_\kappa$ is called an **ultrahugeness Laver function** for κ if for every cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$ there is an (extender) elementary embedding $j : V \longrightarrow M$ which is λ-ultrahuge for κ, and such that $j(\ell)(\kappa) = x$.

We are now ready to show the following, in a way parallel to the proof of Theorem 1.7 in [19].

Theorem 5.2. *Every ultrahuge cardinal carries an ultrahugeness Laver function as above.*

Proof. We follow closely the proof of Theorem 1.7 in [19], adopting it appropriately in the current context. We fix an ultrahuge cardinal κ and some well-ordering $\lhd_\kappa$ of V_κ, and, towards a contradiction, we assume that there is no ultrahugeness Laver function for κ.

We recursively construct a function $\ell \colon \kappa \longrightarrow V_\kappa$ as follows. Given $\alpha < \kappa$ and $\ell \restriction \alpha$, we define $\ell(\alpha)$ only if $\ell``\alpha \subseteq V_\alpha$ and there exists $\lambda \geqslant \alpha$ and $x \in H_{\lambda^+}$ such that, for every (extender) embedding $j : V \longrightarrow M$ which is λ-ultrahuge for α, $j(\ell \restriction \alpha)(\alpha) \neq x$. In this case we let $\lambda_\alpha < \kappa$ be the least such cardinal $\lambda \geqslant \alpha$, and we let $\ell(\alpha)$ be the $\lhd_\kappa$-minimal witness $x \in H_{\lambda_\alpha^+}$. Otherwise, we leave ℓ undefined at α. This concludes the recursive construction of $\ell \colon \kappa \longrightarrow V_\kappa$.

By our assumption, there exists a least $\lambda^* \geqslant \kappa$ and some $x^* \in H_{\lambda^{*+}}$ such that every λ^*-ultrahuge (extender) embedding j fails to “anticipate" the set x^*, i.e., $j(\ell)(\kappa) \neq x^*$. Let $\psi(\lambda^*, x^*)$ be a fixed Π_2-statement asserting this fact (using κ, ℓ as parameters). Now fix some $\theta \in C^{(2)}$ with $\theta > \lambda^*$, some inaccessible $\overline{\theta} > \theta$, and an elementary embedding $j : V \longrightarrow M$ witnessing the $\overline{\theta}$-ultrahugeness of κ, with $j(\overline{\theta})$

inaccessible. It follows that, in M, $\theta \in C^{(2)}$ and λ^* is the least cardinal μ for which ψ holds for some $x \in H_{\mu^+}$; that is, M thinks that $\lambda^* = \lambda_\kappa$ in the above notation. Therefore, by elementarity and the recursive construction of ℓ, there is $y \in H_{\lambda^{*+}}$ such that $j(\ell)(\kappa) = y$. Essentially by definition, we have that $M \models \psi(\lambda^*, y)$. This will lead to the desired contradiction, once we find an appropriate factor embedding of j which is witnessed by some extender in M, and which anticipates the set y.

We now employ an elementary chain argument in order to obtain such a $\overline{\theta}$-ultrahuge factor embedding of j. We fix some initial limit ordinal $\beta_0 \in (j(\kappa), j(\overline{\theta}))$ and we pick some $\gamma < j(\overline{\theta})$ with $\mathrm{cof}(\gamma) > j(\kappa)$, which will serve as the length of our constructed chain. We then let

$$X_0 = \{j(f)(j``j(\kappa), x) : f \in V,\, f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_{\overline{\theta}} \longrightarrow V,\, x \in V_{\beta_0}\} \prec M.$$

For any $\xi + 1 < \gamma$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\overline{\theta})) + \omega$ and

$$X_{\xi+1} = \{j(f)(j``j(\kappa), x) : f \in V,\, f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_{\overline{\theta}} \longrightarrow V,\, x \in V_{\beta_{\xi+1}}\} \prec M.$$

If $\xi < \gamma$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha \prec M$. Finally, we let $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is:

$$X_\gamma = \{j(f)(j``j(\kappa), x) : f \in V,\, f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_{\overline{\theta}} \longrightarrow V,\, x \in V_{\beta_\gamma}\} \prec M.$$

The inaccessibility of $j(\overline{\theta})$ implies that $\beta_\gamma < j(\overline{\theta})$, where $\mathrm{cof}(\beta_\gamma) = \mathrm{cof}(\gamma) > j(\kappa)$. Let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and consider the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, which produces a commutative diagram of elementary embeddings as usual (with $k_\gamma = \pi_\gamma^{-1}$).

By standard arguments and the representation of M_γ, one can now check that j_γ is a $\overline{\theta}$-ultrahuge embedding for κ (and a factor of j) where, in fact, $\mathrm{cp}(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and

$$\mathrm{cp}(k_\gamma) = j_\gamma(\overline{\theta}) = \sup(X_\gamma \cap j(\overline{\theta})) = \beta_\gamma.$$

Furthermore, again by the inaccessibility of $j(\overline{\theta})$, for every $\alpha < j(\overline{\theta})$ we have that $j_\gamma(\alpha) < j(\overline{\theta})$; hence, the relevant (either Martin-Steel, or ordinary but long) extender E which is derived from j_γ and which witnesses its $\overline{\theta}$-ultrahugeness actually belongs to $V_{j(\overline{\theta})} \subseteq M$. Indeed, M certainly thinks that "E is λ^*-utrahuge for κ" and, moreover, it correctly computes the value $j_E(\ell)(\kappa)$ (which is equal to $j_\gamma(\ell)(\kappa)$).

To conclude, observe that κ, λ^*, $H_{\lambda^{*+}}$ and y all belong to V_{β_γ} and are, therefore, fixed by the collapse π_γ. Hence, $j_\gamma(\ell)(\kappa) = j_E(\ell)(\kappa) = y$. But the latter contradicts the fact that $M \models \psi(\lambda^*, y)$. □

Given the previous result one might be tempted to consider the possibility of making an ultrahuge cardinal *indestructible* under appropriate forcing notions, in the spirit of Laver (cf. [14]). Yet, recent results show that this is impossible; indeed, many of the popular posets which are used by set-theorists, such as $\mathrm{Add}(\kappa, 1)$, $\mathrm{Add}(\kappa, \kappa^{++})$, etc., will destroy the superstrongness (and even the Σ_3-extendibility) of the cardinal κ (see [3] for more details).

Having brought up again the machinery of forcing, one natural question is whether ultrahuge cardinals are preserved by other standard forcing constructions. Let us now look at one important example, in the next section.

6. Ultrahuge cardinals and the GCH

It is typically the case that after forcing globally the GCH many of the usual large cardinals are preserved. The first manifestation of this phenomenon was given by Jensen, who proved the preservation of measurable cardinals (cf. [13]). Subsequently, a similar result was proved by Menas for supercompacts (cf. [16]), by

Hamkins for I1 embeddings (cf. [11]), and by Friedman for n-superstrong cardinals (cf. [10]). Recently, the list was expanded further by Brooke-Taylor and Friedman who accounted for 1-extendible cardinals (cf. [6]), by Brooke-Taylor who considered Vopěnka's Principle (cf. [5]), and by the author who proved the preservation of (fully) extendible cardinals (cf. [17]).

Here, we appeal to known techniques and we look at the case of ultrahuge cardinals. Unfortunately, our results are not optimal in the sense that they start with a stronger assumption in order to conclude the preservation of ultrahugeness under the canonical GCH forcing. At any rate, our first step is to partially describe ultrahuge and super n-huge cardinals in terms of embeddings between the H_λ's.

Lemma 6.1. *Fix $n \geqslant 1$ and suppose that κ is super n-huge. Then, for every $\lambda > \kappa$, there is an elementary embedding $j : H_{j^{(n-1)}(\kappa)^+} \longrightarrow H_{j^{(n)}(\kappa)^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$.*

Proof. Let $n \geqslant 1$ and $\lambda > \kappa$ and fix an elementary embedding $h : V \longrightarrow M$ with M transitive, $\mathrm{cp}(h) = \kappa$, $h(\kappa) > \lambda$, and ${}^{h^{(n)}(\kappa)}M \subseteq M$. Let $\theta = h^{(n-1)}(\kappa)$ and $\mu = h(\theta)$, which are both inaccessible cardinals. Recall that $V_{\mu+1} \subseteq M$ and consider the restricted map:

$$h \restriction V_{\theta+1} : V_{\theta+1} \longrightarrow V_{\mu+1}.$$

Now, we can use standard techniques in order to code every element $x \in H_{\theta^+}$ by some binary relation $E_x \subseteq \theta \times \theta$, so that $E_x \in V_{\theta+1}$.[6] This leads to a definable translation of every first-order formula φ with parameters $x_i \in H_{\theta^+}$ to a formula φ^* with parameters the corresponding codes $E_{x_i} \in V_{\theta+1}$, such that

$$H_{\theta^+} \models \varphi(x_1, \ldots, x_n) \Longleftrightarrow V_{\theta+1} \models \varphi^*(E_{x_1}, \ldots, E_{x_n}).$$

A similar process can be done for H_{μ^+} and $V_{\mu+1}$, respectively. Then, using the elementarity of the restricted map $h \restriction V_{\theta+1}$, we may define the desired elementary embedding $j : H_{\theta^+} \longrightarrow H_{\mu^+}$ by sending every $x \in H_{\theta^+}$ to $\pi(max(h(E_x)))$, where π denotes the Mostowski collapse of $h(E_x)$, and where $max(h(E_x))$ denotes the unique ordinal in the union of the domain and range of $h(E_x)$ which is maximal with respect to $h(E_x)$. Finally, one easily checks that $\mathrm{cp}(j) = \kappa$ and $j \restriction (\theta + 1) = h \restriction (\theta + 1)$. □

A partial converse of the previous lemma is given below.

Lemma 6.2. *Fix $n \geqslant 1$ and suppose that, for every $\lambda > \kappa$, there is an elementary embedding $j : H_{j^{(n)}(\kappa)^+} \longrightarrow H_{j^{(n+1)}(\kappa)^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$. Then, κ is super n-huge.*

Proof. Given an elementary embedding $j : H_{j^{(n)}(\kappa)^+} \longrightarrow H_{j^{(n+1)}(\kappa)^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$, consider E the $(\kappa, j^{(n+1)}(\kappa))$-extender derived from j. Then, it is straightforward to check that the extender embedding $j_E : V \longrightarrow M_E$ is such that ${}^{j_E^{(n)}(\kappa)}M_E \subseteq M_E$ and $j_E(\kappa) = j(\kappa) > \lambda$. □

When $n = 1$, the proof of Lemma 6.2 (recall also the proof of Proposition 3.3) in fact gives:

Corollary 6.3. *Suppose that, for every $\lambda > \kappa$, there is an elementary embedding $j : H_{j(\kappa)^+} \longrightarrow H_{j^{(2)}(\kappa)^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$. Then, κ is ultrahuge.*

Let us now recall the following well-known forcing iteration.

[6] Essentially, we first fix some (any) bijection $f_x : |\mathrm{trcl}(\{x\})| \longrightarrow \mathrm{trcl}(\{x\})$ and then, for every $\alpha, \beta \in dom(f_x)$, we let $\langle \alpha, \beta \rangle \in E_x$ if and only if $f_x(\alpha) \in f_x(\beta)$. For more details on such coding arguments, as well as for a similar result in the case of extendibility, the reader is referred to Proposition 1.3 in [17].

Definition 6.4. The **canonical forcing** $\mathbb{P}$ **for global** GCH is the class-length reverse Easton iteration of $\langle\dot{\mathbb{Q}}_\alpha : \alpha \in \mathbf{ON}\rangle$, where $\mathbb{P}_0 = \{\mathbb{1}\}$ and, for each α, if α is an infinite cardinal in $V^{\mathbb{P}_\alpha}$, then $\dot{\mathbb{Q}}_\alpha$ is the canonical $\mathbb{P}_\alpha$-name for the poset $\mathrm{Add}(\alpha^+,1)^{V^{\mathbb{P}_\alpha}}$; otherwise, trivial forcing is done at that stage of the iteration. Finally, $\mathbb{P}$ is the direct limit of the $\mathbb{P}_\alpha$'s, for $\alpha \in \mathbf{ON}$.

It is known that the weak homogeneity of the individual posets $\mathrm{Add}(\alpha^+,1)^{V^{\mathbb{P}_\alpha}}$ transfers also to the class-length iteration and to its initial segments (see [9]). We now show the following, in a way similar to that of Theorem 2.2 in [17].

Theorem 6.5. *Suppose that κ is super 2-huge and let $\mathbb{P}$ be the canonical forcing for global GCH. Then, κ remains ultrahuge in $V^{\mathbb{P}}$.*

Proof. We follow closely the proof of Theorem 2.2 in [17], modifying it appropriately in the current context. Fix a super 2-huge cardinal κ and some $\lambda > \kappa$. Furthermore, by Lemma 6.1, fix an elementary embedding $j : H_{j(\kappa)^+} \longrightarrow H_{j^{(2)}(\kappa)^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$. Let $\theta = j(\kappa)$ and note that θ and $j(\theta)$ are inaccessibles.

Let G be $\mathbb{P}$-generic over V. Our aim is to lift this ground model embedding j to an embedding of the form $j : H^{V[G]}_{\theta^+} \longrightarrow H^{V[G]}_{j(\theta)^+}$ and then, using Corollary 6.3, to conclude that κ is ultrahuge in $V[G]$, as desired. To begin with, we factor the forcing iteration $\mathbb{P}$ as

$$\mathbb{P}_\kappa * \dot{\mathbb{P}}_{[\kappa,\theta)} * \dot{\mathbb{P}}_{[\theta,\infty)},$$

where each interval subscript indicates the (name for the) partial iteration which occurs at the relevant ordinal stages; similar notation will be used for the various projections of G, which are generic for the corresponding partial iterations of $\mathbb{P}$.

Initially, we may easily lift through the forcing $\mathbb{P}_\kappa$ in order to get

$$j : H_{\theta^+}[G_\kappa] \longrightarrow H_{j(\theta)^+}[G_\theta],$$

where $\mathbb{P}_\theta = j(\mathbb{P}_\kappa) \in H_{j(\theta)^+}$ and $j``G_\kappa = G_\kappa \subseteq G_\theta$.

The next step is to lift further through the forcing $\mathbb{P}_{[\kappa,\theta)} = (\dot{\mathbb{P}}_{[\kappa,\theta)})_{G_\kappa}$. For this, we shall need to verify the lifting criterion $j``G_{[\kappa,\theta)} \subseteq G_{[\theta,j(\theta))}$.

Notice that $\mathbb{P}_{[\kappa,\theta)}$ has size θ both in $H_{\theta^+}[G_\kappa]$ and in $H_{j(\theta)^+}[G_\theta]$, and also $j``\theta \in H_{j(\theta)^+}[G_\theta]$. In addition, observe that $G_{[\kappa,\theta)}$ is part of the filter $G_\theta$. Therefore, it follows that $j``G_{[\kappa,\theta)}$ belongs to $H_{j(\theta)^+}[G_\theta]$ and, furthermore, it is a directed subset of size θ of $\mathbb{P}_{[\theta,j(\theta))}$, with the latter being $\leqslant \theta$-directed closed in $H_{j(\theta)^+}[G_\theta]$. Consequently, there exists a lower bound for $j``G_{[\kappa,\theta)}$, i.e., there exists some $r \in \mathbb{P}_{[\theta,j(\theta))}$ such that $r \leqslant j``G_{[\kappa,\theta)}$. Note, however, that this r may not belong to the filter $G_{[\theta,j(\theta))}$. To overcome this problem, we now produce an appropriate filter G^* which is $\mathbb{P}_{[\theta,j(\theta))}$-generic over $H_{j(\theta)^+}[G_\theta]$ and which, in addition, is such that $r \in G^*$.

By the weak homogeneity of $\mathbb{P}_{[\theta,j(\theta))}$ in the model $H_{j(\theta)^+}[G_\theta]$, the set of conditions t for which there is an automorphism $e : \mathbb{P}_{[\theta,j(\theta))} \longrightarrow \mathbb{P}_{[\theta,j(\theta))}$ such that $e(t) \leqslant r$ is dense. Thus, by genericity of $G_{[\theta,j(\theta))}$, there is such a $t \in G_{[\theta,j(\theta))}$; then, by standard forcing facts, if G^* is the filter generated by $e``G_{[\theta,j(\theta))}$, then G^* is a $\mathbb{P}_{[\theta,j(\theta))}$-generic filter over $H_{j(\theta)^+}[G_\theta]$ with $r \in G^*$ and, moreover,

$$H_{j(\theta)^+}[G_{j(\theta)}] = H_{j(\theta)^+}[G_\theta][G^*].$$

It follows that we may further lift j in order to obtain the embedding

$$j : H_{\theta^+}[G_\theta] \longrightarrow H_{j(\theta)^+}[G_{j(\theta)}].$$

Finally, by standard coding arguments and the fact that (the rest of the) iteration $\dot{\mathbb{P}}_{[\theta,\infty)}$ is forced to be $\leqslant \theta$-directed closed, we can easily see that

$$H^{V[G]}_{\theta^+} = H^{V[G_\theta]}_{\theta^+} = H_{\theta^+}[G_\theta],$$

and similarly $H^{V[G]}_{j(\theta)^+} = H_{j(\theta)^+}[G_{j(\theta)}]$ as well. Hence, the currently lifted embedding is indeed of the desired form $j : H^{V[G]}_{\theta^+} \longrightarrow H^{V[G]}_{j(\theta)^+}$, and this concludes the proof. □

Question 6.6. Can we optimize the previous theorem by directly proving that every ultrahuge cardinal is preserved by the canonical forcing for global GCH? Moreover, what about a similar preservation of super n-huge cardinals, for $n \geqslant 1$?

7. Final remarks

Let us conclude by giving a few general remarks and thoughts for further study. First of all, both our earlier work on extendible cardinals (cf. [19]) and the current note seem to suggest that the elementary chain method ties nicely with embeddings which are sufficiently superstrong above their target. For example, this feature has been exploited in Section 3 towards establishing consistency bounds, and also in Section 5 in order to obtain adequate Laver functions. In this respect, and given the generality and flexibility of the elementary chain method, it is worth looking at other (global) large cardinal notions and try to fortify them with the additional superstrongness assumption.

Furthermore, and as we have already pointed out, the initial application of our methods in the context of extendibility lead to the introduction (and consistency) of the unbounded resurrection axioms. In light of Viale's recent work on the axiom MM^{+++}, one should perhaps observe that if we start with an ultrahuge cardinal κ then the standard iteration of length κ will produce a model in which both MM^{+++} and UR(SSP) hold, where the latter is the unbounded resurrection axiom for stationary preserving posets. Indeed, since the ultrahugeness assumption is already stronger than what is necessary for the consistency of MM^{+++}, it is not unlikely that more conclusions can be drawn from it regarding the properties that the aforementioned model satisfies.[7]

In a more general setting, several questions can be asked regarding the theory of $C^{(n)}$-ultrahuge cardinals. For instance:

Question 7.1. What other forcing constructions preserve the $C^{(n)}$-ultrahugeness of a given κ?

Question 7.2. Fix $n \geqslant 1$ and suppose that κ is $C^{(n)}$-ultrahuge. Can we force to destroy its $C^{(n)}$-ultrahugeness while preserving its ultrahugeness?

Conceivably, one might need a class forcing in order to tackle this latter problem. A similar question has been left unanswered in the case of $C^{(n)}$-supercompact cardinals (see [18]).

Finally, let us repeat once more that, from our perspective, the blend of a given large cardinal property with the sufficient superstrongness assumption seems to be a rather appealing and fruitful one, and it has certainly not been fully explored so far. Thus, we expect that it will lead to several new set-theoretic results and, possibly, to more interesting applications in various contexts in the future.

References

[1] G. Audrito and M. Viale, Absoluteness via resurrection. Preprint (2014). http://www2.dm.unito.it/paginepersonali/viale/
[2] J. Bagaria, $C^{(n)}$-cardinals. Arch. Math. Logic **51** (3–4), 213–240 (2012).

[7] It is worth mentioning that Viale has also commented on the fact that, in the presence of class many Woodin cardinals, MM^{+++} is actually equivalent to an appropriate strengthening of UR(SSP); see [21] for more details.

[3] J. Bagaria, J.D. Hamkins, K. Tsaprounis and T. Usuba, Superstrong and other large cardinals are never Laver indestructible. Arch. Math. Logic (to appear). Preprint (2014). http://arxiv.org/abs/1307.3486

[4] J.B. Barbanel, C.A. Di Prisco and I.B. Tan, Many-times huge and superhuge cardinals. J. Symbolic Logic **49** (1), 112–122 (1984).

[5] A. Brooke-Taylor, Indestructibility of Vopěnka's Principle. Arch. Math. Logic **50** (5), 515–529 (2011).

[6] A. Brooke-Taylor and S.D. Friedman, Large cardinals and gap-1 morasses. Ann. Pure Appl. Logic **159** (1–2), 71–99 (2009).

[7] P. Corazza, Laver sequences for extendible and super-almost-huge cardinals. J. Symbolic Logic **64** (3), 963–983 (1999).

[8] S. Cox, PFA and ideals on ω_2 whose associated forcings are proper. Notre Dame J. Formal Logic **53** (3), 397–412 (2012).

[9] N. Dobrinen and S.D. Friedman, Homogeneous iteration and measure one covering relative to HOD. Arch. Math. Logic **47** (7–8), 711–718 (2008).

[10] S.D. Friedman, Large cardinals and L-like universes. In: Set Theory: Recent Trends and Applications (A. Andretta ed.), Quaderni di Matematica, vol. 17, pp. 93–110. Seconda Università di Napoli (2005).

[11] J.D. Hamkins, Fragile measurability. J. Symbolic Logic **59** (1), 262–282 (1994).

[12] T. Jech, Set Theory, The Third Millennium Edition. Springer–Verlag, Berlin (2002).

[13] R.B. Jensen, Measurable cardinals and the GCH. In: Axiomatic Set Theory (T. Jech ed.), Proceedings of Symposia in Pure Mathematics, vol. 13 (II), pp. 175–178. American Mathematical Society (1974).

[14] R. Laver, Making the supercompactness of κ indestructible under κ-directed closed forcing. Israel J. Mathematics **29**, 385–388 (1978).

[15] A. Kanamori, The higher infinite. Springer–Verlag, Berlin (1994).

[16] T.K. Menas, Consistency results concerning supercompactness. Trans. Amer. Math. Soc. **223**, 61–91 (1976).

[17] K. Tsaprounis, On extendible cardinals and the GCH. Arch. Math. Logic **52** (5–6), 593–602 (2013).

[18] K. Tsaprounis, Elementary chains and $C^{(n)}$-cardinals. Arch. Math. Logic **53** (1–2), 89–118 (2014).

[19] K. Tsaprounis, On resurrection axioms. J. Symbolic Logic (to appear).

[20] K. Tsaprounis, K. Large cardinals and resurrection axioms. Doctoral dissertation, University of Barcelona, Spain, 2012.

[21] M. Viale, Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms. Preprint (2014). http://www2.dm.unito.it/paginepersonali/viale/

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